Special solutions of the Painleve equations: P1 to P5 Part 2

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The Painleve equations and Monodromy problem

-1. List of Painleve equations (P1 is Australia)

$$
\begin{aligned}
\text{P1)} \quad y'' &= 6y^2 + t, \\
\text{P2)} \quad y'' &= 2y^3 + ty + \alpha, \\
\text{P4)} \quad y'' &= \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{P3)} \quad y'' &= \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \\
\text{P5)} \quad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) \\
&\qquad + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1},
\end{aligned}
$$

$$
\text{P6)} \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y'
$$
$$
+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right].
$$

0. Review

There exist two kind of special solutions

- algebraic solutions trans.deg $(K[q,p]:K)=0$

  Puiseux or Laurent expansion, Residue theorem.

- Riccati solutions trans.deg $(K[q,p]:K)=1$

  Differential Galois theory (condition (J))

[**Takano**] The initial value space is a finite union of $\mathbb{C}^2$.
On each affine chart, there exist a unique polynomial Hamiltonian.

· $q \equiv 0, \infty$ may be a rational.
· Take a canonical coordinate (q,p). (Polynomial system)
· Bäcklund transformation plays a key role.
· Use the parameters of affine root systems.

* Takano's work is one of motivation of Sakai's theory

0.1 Algebraic solutions of Painlevé equations

Algebraic solution = **center type** + **Riccati type**

center type	parameter	Algebraic
P1	none	none
P2	(0)	$y = 0$
P34	$(1/4)$	$y = t/2$
P4	$(0, -2/9)$	$y = -2t/3$
$P3'(D_8^{(1)})$	$(8h, -8h, 0, 0)$	$y = -\sqrt{t}$
$P3'(D_7^{(1)})$	$(0, -2, 2, 0)$	$y = t^{1/3}$
$P3'(D_6^{(1)})$	$(a, -a, 4, -4)$	$y = -\sqrt{t}$
deg-P5	$(h^2/2, -8, -2, 0)$	$y = 1 + 2\sqrt{t}/h$
P5	$(a, -a, 0, \delta)$	$y = -1$

	Riccati	Algebraic
P1	none	none
P2	$(-1/2)$: Airy	none
P34	(1):Airy	none
P4	$(1-s,-2s^2)$: Hermite-Weber	$(0,-2)$: $y=-2t$
$\mathrm{P3}(D_8^{(1)})$	none	none
$\mathrm{P3}(D_7^{(1)})$	none	none
$\mathrm{P3}(D_6^{(1)})$	$(4h, 4(h+1), 4, -4)$: Bessel	none
deg-P5	$(\alpha, 0, \gamma, 0)$: Bessel	none
P5	$\left(\frac{(\kappa_0+s)^2}{2}, -\frac{\kappa_0^2}{2}, -(s+1), -\frac{1}{2}\right)$: Kummer	$\left(\frac{(s+1)^2}{2}, -\frac{1}{2}, -(s+1), -\frac{1}{2}\right)$: $y=1+t/(s+1)$

Theorem All of classical solutions of P1 to P5 are equivalent to the above solutions up to the Bäcklund transformations.

1. P4 case

Symmetric form:

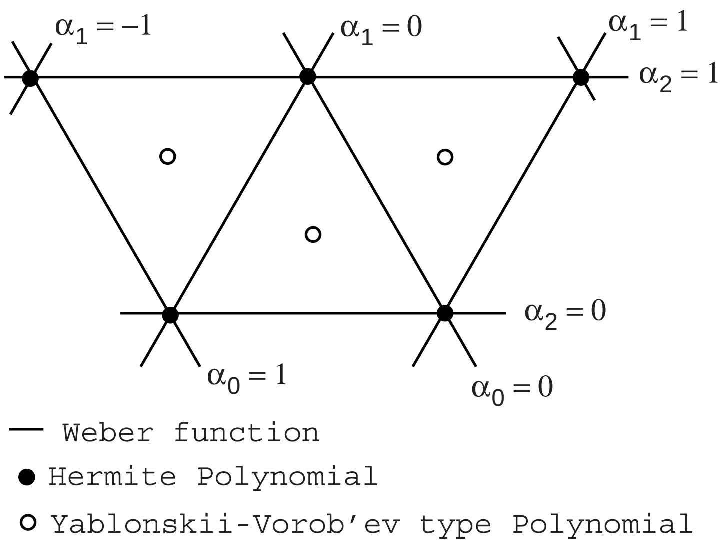
$$f_0' = f_0(f_1 - f_2) + \alpha_0,$$
$$f_1' = f_1(f_2 - f_0) + \alpha_1,$$
$$f_2' = f_2(f_0 - f_1) + \alpha_2,$$

$$f_0 + f_1 + f_2 = t, \qquad \alpha_0 + \alpha_1 + \alpha_2 = 1$$

· It is sympathetic to the Bäcklund transformations.

x	α_0	α_1	α_2	f_0	f_1	f_2
$s_0(x)$	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	f_0	$f_1 - \frac{\alpha_0}{f_0}$	$f_2 + \frac{\alpha_0}{f_0}$
$s_1(x)$	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	$f_0 + \frac{\alpha_1}{f_1}$	f_1	$f_2 - \frac{\alpha_1}{f_1}$
$s_2(x)$	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$f_0 - \frac{\alpha_2}{f_2}$	$f_1 + \frac{\alpha_2}{f_2}$	f_2
$\pi(x)$	α_1	α_2	α_0	f_1	f_2	f_0

Line	Riccati solutions $\mathbb{P}^1(\mathbb{C})$
Vertex	Two Riccati solutions intersect: A_2 The intersection is the Hermite polynomial.
Center	A Rational solution



Explicit form

Line	$\alpha_0 = 0$	$f_0 = 0;\ \ f_1' = f_1(t - f_1) + \alpha_1$
Vertex	$\alpha_0 = \alpha_1 = 0$	$f_0 = f_1 = 0;\ f_2 = t.$
Center	$\alpha_0 = \alpha_1 = \alpha_2 = 1/3$	$f_0 = f_1 = f_2 = t/3.$

Proof.

Riccati: Condition (J).

Rational: Residue calculus

Symmetric form is sympathetic to initial value space.

If we take (f_0, f_1), (f_1, f_2), (f_2, f_0), they are canonical coordinates of the initial value space.

The initial value space of P4 is a four union of $\mathbb{C}^2$.

1.1 A_4 Painleve equation.

$$f_i' = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \qquad (i = 0, 1, 2, 3, 4)$$

$$f_0 + f_1 + f_2 + f_3 + f_4 = t,$$

Bäcklund transformation

$$\begin{aligned} s_i(\alpha_i) &= -\alpha_i, \ s_i(\alpha_j) = \alpha_j + \alpha_i \, (j = i \pm 1), \ s_i(\alpha_j) = \alpha_j \, (j \neq i, i \pm 1) \\ s_i(f_i) &= f_i, \ s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \, (j = i \pm 1), \ s_i(f_j) = f_j \, (j \neq i, i \pm 1) \end{aligned}$$

Rational solution (Matsuda, [12])

(Edge)	$\alpha_i \in \mathbb{Z}$.
(P4)	$\begin{cases} \pm\frac{1}{3}(1,1,1,0,0) \mod \mathbb{Z} \\ \pm\frac{1}{3}(1,-1,-1,1,0) \mod \mathbb{Z}. \end{cases}$
(Center)	$\begin{cases} \frac{j}{5}(1,1,1,1,1) \mod \mathbb{Z} \\ \frac{j}{5}(1,2,1,3,3) \mod \mathbb{Z} \end{cases}$

* The initial value space of A_4 Painleve (4-dim) by Tahara [20].

2. P5 case A_3

$$tH_V = p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1)$$

$$\mathcal{H}_V : \begin{cases} tq' = q(2pq - 2p + tq - t - \alpha_1 - \alpha_3) + \alpha_1, \\ tp' = -p(2pq - p + 2tq - t - \alpha_1 - \alpha_3) - \alpha_2 t, \end{cases}$$

$y = 1 - 1/q$ satisfies P5 for

$$\alpha = \frac{{\alpha_1}^2}{2}, \quad \beta = -\frac{{\alpha_3}^2}{2}, \quad \gamma = \alpha_0 - \alpha_2, \quad \delta = -\frac{1}{2},$$

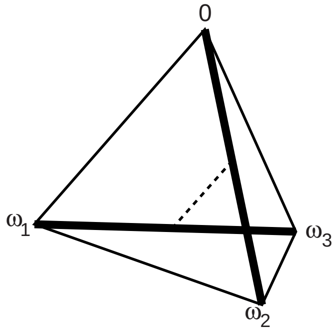
$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$$

Backlund transformation

x	α_0	α_1	α_2	α_3	q	p	t
$s_0(x)$	$-\alpha_0$	$\alpha_1 + \alpha_0$	α_2	$\alpha_3 + \alpha_0$	$q + \frac{\alpha_0}{p+t}$	p	t
$s_1(x)$	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	q	$p - \frac{\alpha_1}{q}$	t
$s_2(x)$	α_0	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$q + \frac{\alpha_2}{p}$	p	t
$s_3(x)$	$\alpha_0 + \alpha_3$	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	q	$p - \frac{\alpha_3}{q-1}$	t
$\pi(x)$	α_1	α_2	α_3	α_0	$-\frac{p}{t}$	$t(q-1)$	t
$\sigma(x)$	α_0	α_3	α_2	α_1	$1 - q$	$-p$	$-t$

Face	Riccati solutions $\mathbb{P}^1(\mathbb{C})$
Thin Line	Two Riccati solutions intersect: A_2 The intersection is the Legendre polynomial
Fat Line	Two Riccati solutions do NOT intersect: $A_1 \oplus A_1$ Reduce to P_{III} by folding transformation
Vertex	Three Riccati solutions intersect: A_3
Dotted line	Fixed points of $\pi^2 : q \to 1 - q$, A Rational solution

Figure 1: $[2 : \sqrt{3} : \sqrt{3}]$- Tetrahedron



3. τ-series [15]

$$H_{II} = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q$$

$$(\mathcal{H}_{II}) : \quad \begin{cases} q' = -q^2 + p - \frac{t}{2}, \\ p' = 2pq + \alpha + \frac{1}{2}. \end{cases}$$

We set $\kappa = \alpha + 1/2 = \alpha_1$.

$$\begin{aligned} q(\kappa - 1) &= -q(\kappa) + \frac{\kappa - 1}{p(\kappa) - 2q(\kappa)^2 - t} \\ p(\kappa - 1) &= -p(\kappa) + 2q(\kappa)^2 + t, \end{aligned}$$

or $q(\kappa) = H_{II}(\kappa - 1) - H_{II}(\kappa)$. For (q, q'):

$$q(\kappa - 1) = -q(\kappa) + \frac{\kappa - 1}{q'(\kappa) - q(\kappa)^2 - t/2}$$

3.1 Toda equation (Okamoto's talk)

Definition of τ-function:

$$\frac{d}{dt}\log\tau(\kappa) = H_{\mathrm{II}}(\kappa)$$

Toda equation

$$\frac{d^2}{dt^2}\log\tau(\kappa) = \frac{\tau(\kappa+1)\tau(\kappa-1)}{\tau(\kappa)^2}$$

Proof.

1)

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{1}{2}q, \qquad \frac{d^2H}{dt^2} = -pq - \frac{1}{2}\kappa$$

$$\Longrightarrow q = \frac{1}{2H'}(H'' + \kappa/2)$$

2)

$$\frac{d\underline{H}}{dt} = \frac{dH}{dt} + q' = \frac{1}{2}p - q^2 - \frac{1}{2}t$$

$$\frac{d^2\underline{H}}{dt^2} = -qp + 2q^3 + tq + \frac{1}{2}(\kappa - 1)$$

$$\Longrightarrow q = -\frac{1}{2\underline{H}'}(\underline{H}'' + (1-\kappa)/2)$$

3) By (1) and (2)

$$q(\kappa) - q(\kappa+1) = \frac{d}{dt}\log H' = \frac{d}{dt}\log\frac{d^2}{dt^2}\log\tau(\kappa).$$

4)

$$q(\kappa) = H_{\mathrm{II}}(\kappa-1) - H_{\mathrm{II}}(\kappa) = \frac{d}{dt}\log\frac{\tau(\kappa-1)}{\tau(\kappa)}$$

$$\Longrightarrow q(\kappa) - q(\kappa+1) = \frac{d}{dt}\log\frac{\tau(\kappa+1)\tau(\kappa-1)}{\tau(\kappa)^2}$$

5) By (3) and (4)

$$\frac{d^2}{dt^2}\log\tau(\kappa) = c(\kappa)\frac{\tau(\kappa+1)\tau(\kappa-1)}{\tau(\kappa)^2}$$

Q.E.D

3.2 Yablonskii-Vorob'ev polynomials

By the Backlund transformation

κ	q	p
$1/2$	0	$t/2$
$-1/2$	$1/t$	$t/2$
$-3/2$	$\dfrac{2(t^2-2)}{4t+t^4}$	$\dfrac{t^3+4}{2t^2}$

$$H(-1/2) = -\frac{1}{8}t^2, \qquad \tau(-1/2) = \exp\left(-\frac{1}{24}t^3\right)$$

We set

$$\tau(-1/2-m) = P_m \exp\left(-\frac{1}{24}t^3\right)$$

P_m is called the Yablonskii-Vorob'ev polynomial

$$q(-1/2-m) = \frac{d}{dt}\log\frac{P_{m+1}}{P_m}$$

$$P_m P_m^{(4)} - 4P'_m P_m^{(3)} + 3\left(P''_m\right)^2 - t\left(P_m P''_m - \left(P'_m\right)^2\right) - P_m P'_m = 0.$$

By the Toda equation

$$P_{m+1}P_{m-1} = tP_m^2 - 4\left(P_m P''_m - \left(P'_m\right)^2\right)$$

$$\deg P_m = \frac{1}{2}m(m+1), \qquad P_{-m-1} = P_m$$

$$\begin{aligned}
P_{-1} &= 1 \\
P_0 &= 1 \\
P_1 &= t \\
P_2 &= t^3 + 4 \\
P_3 &= t^6 + 20t^7 - 80 \\
P_4 &= t^{10} + 60t^7 + 11200t
\end{aligned}$$

3.3 Schur polynomials (Kajiwara-Ohta [6])

$t = (t_1, t_2, t_3, \ldots)$. We define polynomials $p_k(t)$ by

$$\sum_{k=0}^{\infty} p_k(t) z^k = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right).$$

Then

$$p_n(t) = \sum_{k_1+2k_2+k_3+\cdots=n} \frac{t_1^{k_1}}{k_1!}\frac{t_2^{k_2}}{k_2!}\frac{t_3^{k_3}}{k_3!}\cdots$$

Let $Y = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is a partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$.

$$S_Y = \det \begin{pmatrix} p_{\lambda_1} & p_{\lambda_2-1} & \cdots & p_{\lambda_N-N+1} \\ p_{\lambda_1+1} & p_{\lambda_2} & \cdots & p_{\lambda_N-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\lambda_1+N-1} & p_{\lambda_2+N-2} & \cdots & p_{\lambda_N} \end{pmatrix}$$

Rational solutions of KP hierarchy is completely solved by the Schur polynomial. [18]

[Theorem] (Kajiwara-Ohta)

Let $Y = (N, N-1, \cdots, 1)$.

$$\tau_N(t) = S_Y(t, 0, -\frac{4}{3}, 0, 0, 0, \cdots).$$

Then P2($N+1$) is solved by

$$q = \frac{d}{dt} \log \frac{\tau_{N+1}}{\tau_N}.$$

Remark.

$\tau_N(t)$ is a rational solution of the KdV equation.

KO theorem is related to the similarity reduction from KdV to P2.

There exist similar determinant formula for other Painleve.

4.1 Richard Fuchs

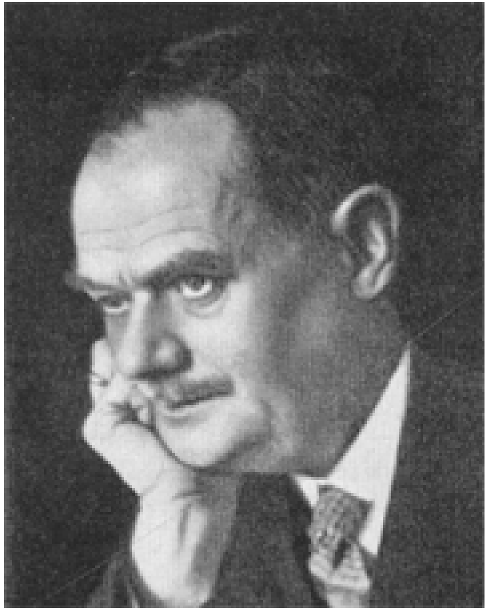


Figure 2: Richard Fuchs (1873 – 1945) from `www.ilr.tu-berlin.de`

(This picture is used in J. Phys.:A R. Fuchs' memorial issue)

R. Fuchs' Three Papers on Math. Ann.

"Sur quelques équations différentielles linéarires du second order", *C. R. Acad. Sci. Paris* **141** (1905), 555–558.

"Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen", *Math. Ann.* **63** (1906), 301–321.

"Ueber lineare homogene Differentialgleichungen zweiter Ordnung mit vier wesentlich singulären Stellen", *Nachr. Kgl. Ges. Wiss. G̈ttingen, math.-phys.*, (1910) 146–153.

"Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen", *Math. Ann.* **70** (1911), 525–549.

"Über die analytische Natur der Lösungen von Differentialgleichungen zweiter Ordnung mit festen kritischen Punkten", *Nachr. Kgl. Ges. Wiss.*, (1914) 14–22.

"Über die analytische Natur der Lösungen von Differentialgleichungen zweiter Ordnung mit festen kritischen Punkten", *Math. Ann.* **75** (1914), 469–496.

4.2 Picard's solutions

> "J'indiquerai, en terminant, une équation différentielle curieuse, à laquelle conduit la théorie des fonctions elliptiques, et dont l'intégrale générale a ses points critiques fixes." [16]

[Theorem] Let k is an elliptic modulus. For any integral constants a, b,

$$w = \frac{1}{\mathrm{sn}^2(a\omega + b\omega')}$$

satisifies P6(0,0,0,1/2) for $t = k^2$.

Proof. $y = a\omega + b\omega'$ satisfy the Picard-Fuchs equation

$$k(1-k^2)\frac{d^2y}{dk^2} + (1-3k^2)\frac{dy}{dk} - ky = 0.$$

We set

$$\Omega(u,k) = \int_0^u \frac{dx}{\Delta(x)}, \qquad \Delta(x) = \sqrt{(1-x^2)(1-k^2x^2)}.$$

We think u as a constant. Ω satisfies

$$k(1-k^2)\frac{\partial^2\Omega}{\partial k^2}+(1-3k^2)\frac{\partial\Omega}{\partial k}-k\Omega+\frac{ku(1-u^2)}{(1-k^2u^2)\Delta(u)}=0.$$

Put $u=\mathrm{sn}(a\omega+b\omega')$. Then $\Omega=y$. And

$$\frac{dy}{dk}=\frac{\partial\Omega}{\partial k}+\frac{1}{\Delta(u)}\frac{du}{dk},$$

$$\frac{d^2y}{dk^2}=\frac{\partial^2\Omega}{\partial k^2}+\frac{1}{\Delta(u)}\frac{du^2}{dk^2}-\frac{u(2k^2u^2-1-k^2)}{\Delta(u)(1-u^2)(1-k^2u^2)}\left(\frac{du}{dk}\right)^2+2\frac{ku^2}{\Delta(u)(1-k^2u^2)}\frac{du}{dk}.$$

We erase Ω. Then $u(t=k^2)$ satisfy

$$\begin{aligned}\frac{d^2u}{dt^2} &- \frac{u(2tu^2-1-t)}{(1-u^2)(1-tu^2)}\left(\frac{du}{dt}\right)^2\\ &+ \left(\frac{u^2-1}{(1-t)(1-tu^2)}+\frac{1}{t}\right)\frac{du}{dt}-\frac{u(1-u^2)}{t(1-t)(1-tu^2)}=0\end{aligned}$$

Put $u=w^{-1/2}$. Then w satisfy P6. QED

4.3 R. Fuchs' missing paper

Picard's solution can be written as

$$y = \wp(a\omega_1(t) + b\omega_2(t); \omega_1(t), \omega_2(t)) + \frac{t+1}{3} \qquad (P)$$

$$\omega_j(t) = \int_{\gamma_j} \frac{ds}{\sqrt{s(s-1)(s-t)}}$$

The linear monodromy of the Picard solution:

$$M_j = \begin{pmatrix} -1 + a_j b_j & -b_j^2 \\ a_j^2 & -1 - a_j b_j \end{pmatrix} \qquad j = 0, 1, t, \infty$$

where

$$a_0 b_1 - a_1 b_0 = 2\cos(a-b)\pi, \quad a_0 b_\infty - a_\infty b_0 = 2\sin a\pi$$

$$a_0 b_t - a_t b_0 = 2\cos b\pi, \quad a_1 b_\infty - a_\infty b_1 = -2\sin(2a-b)\pi$$

$$a_1 b_t - a_t b_1 = -2\cos a\pi, \quad a_t b_\infty - a_\infty b_t = -2\sin b\pi$$

Remark. This is the first example that the linear monodromy of a Painlevé function is exactly determined. Mazzocco found again [13].

How did R. Fuchs calculate the linear monodromy?

1. Expand the solution

$$f\left(\frac{k}{n}\omega_1 + \frac{l}{n}\omega_2\right) = c_0 t^{2l/n} + c_1 t^{2l/n} + \cdots \quad (l/n < 1/2)$$

2. Take the limit $t \to 0$. Then the linearized equations becomes the hypergeometric equations.

3. Expand at $t = 1$. Take the limit $t \to 1$.

· R. Fuchs' method is generalized by Jimbo. [5]

· **Kaneko** calculated the linear monodromy for holomorphic solutions around the fixed point [7] by Jimbo's method.

· Kaneko-O. also calculated the linear monodromy for holomorphic solutions around the fixed point for P5. [9]

5. Monodromy solvablity

It is very difficult to calculate the monodromy data of linear equations. The Riemann-Hilbert correspondence is highly transcendental.

Schlesinger, Garnier tried to solve the Riemann-Hilbert problem by isomonodromic deformation. [19] [2]

Riemann's problem:

Find a linear equation for given monodromy.

L. Fuchs' problem:

Study the nonlinear equation for isomonodromic deformation.

Study the FAMILY of linear equations!

5.2 "Special solutions" – Beyond Umemura

Q. When can we calculate the linear monodromy exactly?

A. Not yet.

[Example]

- The Boutruoux solutions for P1 (Duits) [10]
- Ablowitz-Segur's solution for P2 [4]
- Clarkson-McLeod solution for P4 [3]
- Symmetric solutions for P1, P2, P4, P6 [11] [7]
- Picard's solutions for P6 [16]
- Jimbo's boundary behavior for P6 [5]
- holomorphic at fixed singularity for P3, P5, P6 [9] [8]

5.3 Symmetric solutions for P1, P2, P34, P4

$$\begin{array}{ll} \text{P1} & y'' = 6y^2 + t, \\ \text{P2}(\alpha) & y'' = 2y^3 + ty + \alpha, \\ \text{P34}(a) & y'' = \dfrac{(y')^2}{2y} + 2y^2 - ty - \dfrac{a}{2y}, \\ \text{P4}(\alpha,\beta) & y'' = \dfrac{1}{2y}y'^2 + \dfrac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \dfrac{\beta}{y}, \end{array}$$

have a **simple symmetry**:

$$\begin{array}{llll} \text{P1} & y \to \zeta^3 y, & t \to \zeta t, & (\zeta^5 = 1) \\ \text{P2} & y \to \omega y, & t \to \omega^2 t, & (\omega^3 = 1) \\ \text{P34} & y \to \omega y, & t \to \omega t, & (\omega^3 = 1) \\ \text{P4} & y \to -y, & t \to -t, & \end{array}$$

Kitaev: P1, P34 [11] Kaneko: P2, P4 [7]

- **The simple symmetries** do not change parameters.
- **Symmetric solution** = invariant under the symmetry
  - P1: 2 symmetric solutions
  - P2, 34: 3 symmetric solutions
  - P4: 4 symmetric solutions

- P2, P34, P4: symmetric solutions are

  equivalent up to the Bäcklund transformations
- Symmetric solutions **exist for any parameters**
- Symmetric solutions are not **classical** in general.
- For special parameters, symmetric solutions become classical.

> **The symmetric solutions are a generalization of classical solutions.**

Theorem 1. *1) For P1, we have two symmetric solutions*

$$y = \frac{1}{6}t^3 + \frac{1}{336}t^8 + \frac{1}{26208}t^{13} + \frac{95}{224550144}t^{18} + \cdots,$$
$$y = t^{-2} - \frac{1}{6}t^3 + \frac{1}{264}t^8 - \frac{1}{19008}t^{13} + \cdots.$$

2) For P2(α), we have three symmetric solutions

$$\begin{aligned} y &= \frac{\alpha}{2}t^2 + \frac{\alpha}{40}t^5 + \frac{10\alpha^3+\alpha}{2240}t^8 + \cdots, \\ y &= t^{-1} - \frac{\alpha+1}{4}t^3 + \frac{(\alpha+1)(3\alpha+1)}{112}t^5 + \cdots, \\ y &= -t^{-1} - \frac{\alpha-1}{4}t^3 - \frac{(\alpha-1)(3\alpha-1)}{112}t^5 + \cdots. \end{aligned}$$

2) For P34(a^2), we have three symmetric solutions

$$\begin{aligned} y &= at + \frac{a(2a-1)}{8}t^4 + \frac{a(2a-1)(10a-3)}{560}t^7 + \cdots, \\ y &= -at + \frac{a(2a+1)}{8}t^4 - + \frac{a(2a+1)(10a+3)}{560}t^7 + \cdots, \\ y &= \frac{2}{t^2} + \frac{t}{2} - \frac{4a^2-9}{224}t^4 - \frac{4a^2-9}{5600}t^7 + \cdots. \end{aligned}$$

4) For P4($\alpha, -8\theta_0^2$), we have four symmetric solutions

$$\begin{aligned} y &= \pm 4\theta_0 \left(t - \frac{2\alpha}{3}t^3 + \frac{2}{15}(\alpha^2 + 12\theta_0^2 \pm \theta_0 + 1)t^5 + \cdots \right), \\ y &= \pm t^{-1} + \frac{2}{3}(\pm\alpha - 2)t^3 \mp \frac{2}{45}(-7\alpha^2 \pm 16\alpha + 36\theta_0^2 - 4)t^5 + \cdots. \end{aligned}$$

5.3 Riccati solution is monodromy solvable

$$\text{(P6)} \begin{cases} \dfrac{dA_1}{dt} = \dfrac{[A_3, A_1]}{t}, \\ \dfrac{dA_2}{dt} = \dfrac{[A_3, A_2]}{t-1}. \end{cases}$$

$$A_j = \begin{pmatrix} a_j & b_j \\ 0 & d_j \end{pmatrix}, \quad A_4 = -(A_1 + A_2 + A_3) : \ \text{diagonal}$$

$\Longrightarrow$ a_j and d_j are independent of t $(e_j = a_j - d_j)$

$$\begin{aligned} \frac{db_1}{dt} &= \frac{1}{t}\left(e_3 b_1 - e_1 b_3\right), \\ \frac{db_2}{dt} &= \frac{1}{t-1}\left(e_3 b_2 - e_2 b_3\right), \\ b_3 &= -(b_1 + b_2) \end{aligned}$$

$$t(t-1)\frac{d^2 b_1}{dt^2} - \left((e_1 + e_2 + 2e_3 - 1)t - e_1 - e_3 + 1\right)\frac{db_1}{dt} + e_3(e_1 + e_2 + e_3)b_1 = 0.$$

Riccati = monodromy is reducible

5.4 Algebraic solution and R. Fuchs' problem

R. Fuchs' Problem (1910) Let $y(t)$ be an algebraic solution $y(t)$ of a Painleve equation. Find a suitable transformation $x = x(z,t)$ such that the corresponding linear differential equation

$$\frac{d^2v}{dz^2} = Q(t, y(t), y'(t); z)v$$

is changed to the form without the deformation parameter t:

$$\frac{d^2u}{dx^2} = \tilde{Q}(x)u.$$

Here $v = \sqrt{dz/dx}\, u$.

Theorem 2. **(R. Fuchs' observations)** *The n-th divied points of the Picard solutions are algebraic solutions (Picard-R. Fuchs' solution).*

For the third, fourth divied points of the Picard solutions, the linearization reduced to the Gauss hypergeometric equation.

[fourth divied points] $y(t) = \wp(\omega_1/4) = -\sqrt{t}$,

$$\tilde{Q}(x) = -\frac{1}{4}\frac{1}{x^2} - \frac{1}{4}\frac{1}{(x-1)^2} + \frac{5}{16}\frac{1}{x(x-1)}$$

$$x = \frac{z(1+\lambda)^2}{(z+\lambda)^2}$$

[third divied points] For $y(t) = \wp(\omega_1/3)$, y satisfies

$$3y^4 - 4(t+1)y^3 + 6ty^2t - t^2 = 0.$$

$$\tilde{Q}(x) = -\frac{2}{9}\frac{1}{x^2} - \frac{3}{16}\frac{1}{(x-1)^2} + \frac{23}{144}\frac{1}{x(x-1)}$$

$$x = -\frac{1}{4}\frac{\psi^3(\lambda)}{t^4(t-1)^4}\frac{G_2^3(z)}{\psi(z)},$$

$$G_2(z) = 9z^2 + 2z(2\lambda - 4 - 4t) + 3\lambda^2 - 4\lambda(1+t) + 6t$$

$$\psi(z) = z(z-1)(z-t)$$

R. Fuchs' problem is true for P1-P5. The linear equation reduces to Confluent Hypergeometric. [14]

Example: D_7**-alg** $(2, 3/2)$

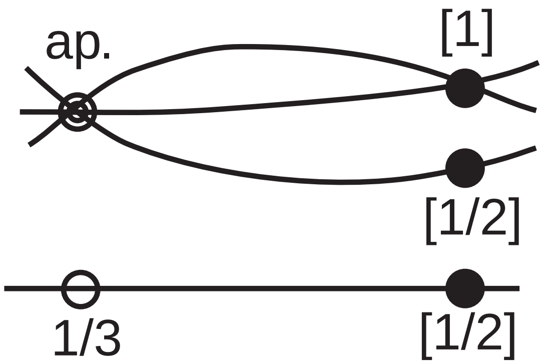


Figure 3: D_7-alg $(3|2+1)$ from $DW_{1/6}$

Remark. Andreev-Kitaev studied the P6 case. [1]

6. Summary

Riemann-Hilbert problem (Fokas)

For given $G(z)$ on Γ, find

$$Y_{\pm}(z) \in GL(n, \mathcal{O}(\Gamma_{\pm}))$$

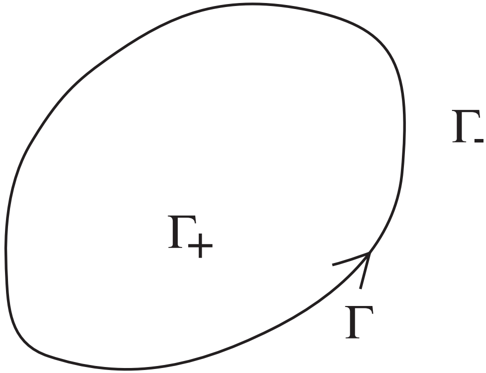
such that

$$Y_{+}(z) = Y_{-}(z)G(z)$$

$Y_{-} \sim \mathrm{Id}$ at $z = \infty$

Painleve = Non-abelian hypergeomtric

Riemann-Hilbert = Non-abelian hyperfunction ([17])



Riemann-Hilbert problem

Rank 1.

Find a good secion of the line bundle = Abelian Integral

Rank ≥ 2**.**

Find a good secion of the vector bundle =

Hyperbelian Integral (Weil)

Hyperbelian Integral go back to Zeta-Fuchsian series by Poincare.

∞. Dream

H. Poincare

Les Mathématiques constituent un continent solidement agencé, dont tous les pays sont bien reliés les uns aux autres; l'oeuvre de Paul Painlevé est une île originale et splendide dans l'océan voisin.

Now the Painlevé equation is one of the most important country in the mathematical Continent.

Appendix

In this appendix, we list up reference on special solutions. Since there are many references, we list up partially.

[Rational solutions]

P1. There are no good reference. Show Umemura's survey [21]

P2. Vorob'ev [24]. But Yablonskii [23] is difficult to obtain. See Murata [25].

P4. Murata [25].

P3. For D_6, Gromak [27] and Murata [28]. For D_7, Gromak [29]. D_8 reduces D_6. See [26].

P5. Kitaev, A. V., Law, C. K. and McLeod [30]

P6. Mazzocco [31] and Yuan-Li [32].

[Riccati solutions]

P1. See Umemura's survey [21].

P2. Umemura-Watanabe [33]. See Umemura's survey [22]

P3. For D_6, Umemura-Watanabe [34].
For D_7 and D_8, Ohyama-Kawamuko-Sakai-Okamoto [26].

P4. Umemura-Watanabe [33].

P5. Watanabe [35]

P6. Watanabe [36]

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