A real pole-free solution of the 4th order analogue of the Painlevé I equation and critical edge points in random matrix ensembles

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Cambridge, 2006, September 15

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1. The P_I^2 equation

The second member of the Painlevé I hierarchy (or P_I^2 equation) is the following differential equation:

$$x = Ty - \left(\frac{1}{6}y^3 + \frac{1}{24}(y_x^2 + 2yy_{xx}) + \frac{1}{240}y_{xxxx}\right).$$

Theorem:
There is a real solution $y(x,T)$ of the P_I^2 equation which has no poles for real x and T. Moreover, for $T \in \mathbb{R}$ bounded, this solution has asymptotics

$$y(x;T) = \mp 6^{1/3}|x|^{1/3} + \mathcal{O}(x^{-1/3}) \qquad \text{as } x \to \pm\infty.$$

$\longrightarrow$ the existence part of the theorem was conjectured by Dubrovin

$\longrightarrow$ asymptotics of y were suggested by Kapaev for $T = 0$

1. The P_I^2 equation

Background of Dubrovin's conjecture

- Dubrovin considered Hamiltonian perturbations of hyperbolic equations of the form $u_t + a(u)u_x = 0$.

  $\longrightarrow$ Example: small dispersion limit of the KdV equation

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0$$

- In regions where the unperturbed solution has bounded x-derivative, solutions of the perturbed equation behave nicely, $u_\epsilon = u_0 + \mathcal{O}(\epsilon)$.

- Near a critical point (x_0, t_0) where $u_x(x_0; t_0) = \infty$, the error term is of larger order

  $\longrightarrow$ gradient catastrophe, dispersive shock waves

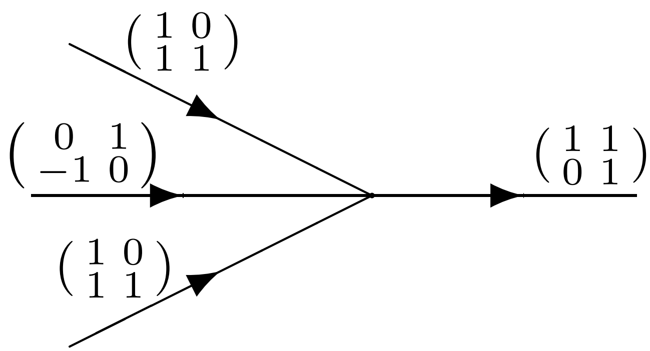
  $\longrightarrow$ The behavior of the perturbed solution u_ϵ is conjectured to be as follows for (x, t) close to (x_0, t_0):

$$u_\epsilon(x,t) = u_0(x,t) + c\epsilon^{2/7} y\left(\epsilon^{-6/7} f(x,t); \epsilon^{-4/7} g(t)\right) + \mathcal{O}(\epsilon^{4/7}),$$

  where y is a real pole-free solution P_I^2.

2. RH problem for the P_I^2 equation

We consider the following RH problem, depending on parameters x and T:



(a) $\Phi : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is analytic,

(b) $\Phi_+(z) = \Phi_-(z)v(z)$ for $z \in \Gamma$, with v as indicated in the figure,

(c) Φ has the following behavior at infinity,

$$\Phi(z) = (I + \mathbf{A_1}/z + \mathcal{O}(1/z^2))z^{-\frac{1}{4}\sigma_3} N e^{-\theta(z;x,T)\sigma_3}, \qquad \text{as } z \to \infty,$$

where

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{1}{4}\pi i \sigma_3}, \qquad \theta(z;x,T) = \frac{1}{105} z^{7/2} - \frac{1}{3}\mathbf{T} z^{3/2} + \mathbf{x} z^{1/2}.$$

2. RH problem for the P_I^2 equation

- **Claim 1:** The RH problem for Φ is solvable for real x and T.
- **Claim 2:** If we define y by

$$y(x,T) = 2A_{1,11} - A_{1,12}^2,$$

y is real and solves the P_I^2 equation.

Claim 1 and Claim 2 imply together the existence of a real solution of the P_I^2 equation with no poles on the real line.

2a. Relation between RH problem and P_I^2 equation

- Suppose the RH problem is solvable and consider the function $\Psi(z) = \left(\begin{smallmatrix} 1 & 0 \\ A_{1,12} & 1 \end{smallmatrix}\right) \Phi(z)$.
- Using the fact that Ψ has constant jump matrices (condition (b) of the RH problem), we find that $U = \Psi_z \Psi^{-1}$ and $W = \Psi_x \Psi^{-1}$ are entire functions.
- Using the asymptotic expansion of Φ for $z \to \infty$ (condition (c) of the RH problem), calculations yield

$$U = \frac{1}{240} \begin{pmatrix} az + g & 8z^2 + 8yz + b \\ 8z^3 - 8yz^2 + cz + d & -az - g \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & 1 \\ z + (h_x - y) & 0 \end{pmatrix},$$

with $y = 2A_{1,11} - A_{1,12}^2$, and a, b, c, d, g, h (which are unknown for now) depend on x and T, but not on z.

2a. Relation between RH problem and P_I^2 equation

- The compatibility condition between $\Psi_z = U\Psi$ and $\Psi_x = W\Psi$ (i.e. $\Psi_{zx} = \Psi_{xz}$) leads to

$$U_x - W_z + UW - WU = 0.$$

- Inserting the formulae for U and V gives us the following information:

    explicit formulae for a, b, c, d, g in terms of y, y_x, y_{xx}, and y_{xxxx},

    $h_x = y$,

    y satisfies the P_I^2 equation.

$\longrightarrow$ What remains to prove is the solvability of the RH problem

2b. Solvability of the RH problem

Consider a RH problem of the form (our original RH problem can be transformed to this form),

(a) Φ is analytic in $\mathbb{C} \setminus \Gamma$,

(b) $\Phi_+(z) = \Phi_-(z)v(z)$ as $z \in \Gamma$,

(c) $\Phi(z) = I + \mathcal{O}(1/z)$ as $z \to \infty$.

The related **homogeneous** RH problem consists of the same conditions (a) and (b), with condition (c) replaced by

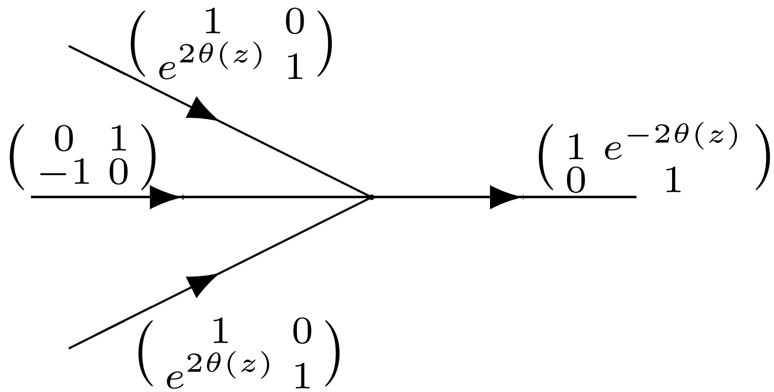
$$\Phi(z) = \mathcal{O}(1/z) \qquad \text{as } z \to \infty.$$

Under certain conditions on the jump matrix v, the following result holds (Fokas, Zhou):

The (non-homogeneous) RH problem is solvable if and only if the homogeneous RH problem has only the trivial solution $\Phi \equiv 0$.

2b. Solvability of the RH problem

In order to prove the solvability of the RH problem for Φ, it is sufficient to prove that the following homogeneous RH problem has only the solution $\Phi_0 \equiv 0$ (**vanishing lemma**).



(a) Φ_0 is analytic in $\mathbb{C} \setminus \Gamma$.

(b) $\Phi_{0,+}(z) = \Phi_{0,-}(z)v(z)$ for $z \in \Gamma$, with v as indicated in the figure

(c) Φ_0 has the following behavior at infinity,

$$\Phi_0(z) = \mathcal{O}(1/z)z^{-\frac{1}{4}\sigma_3}N, \qquad \text{as } z \to \infty.$$

2b. Solvability of the RH problem

The proof that $\Phi_0 \equiv 0$ goes along the following steps (cfr. Deift, Kriecherbauer, McLaughlin, Venakides, Zhou 1999):

- transformation to a RH problem on the real line,
- using symmetries in the jump matrices ($v + v^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$), the matrix RH problem can be reduced to a scalar RH problem,
- the vanishing of the scalar RH problem can be proven using complex analysis techniques.

2c. Asymptotics of y

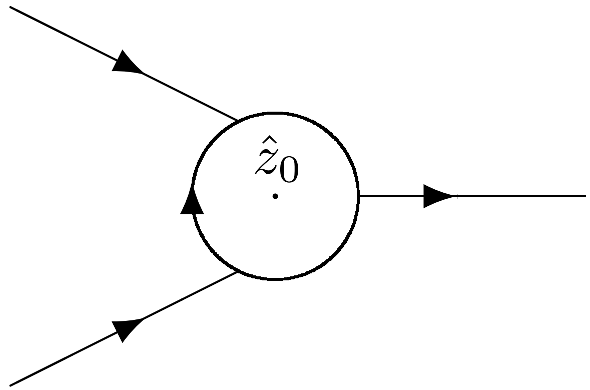
- The asymptotics of $y(x;T)$ for $x \to \pm\infty$ can be derived from asymptotics for the RH solution Φ.
- In order to find asymptotics for Φ, we apply the Deift/Zhou steepest descent method on the RH problem for P_I^2.

  series of transformations of the RH problem:

$$\Phi \longrightarrow Y \longrightarrow S \longrightarrow R$$

  those transformations involve
  - rescaling,
  - contour deformation,
  - construction of parametrices.

2c. Asymptotics of y



This results in a RH problem for R:

(a) R is analytic in $\mathbb{C} \setminus \Gamma_R$,

(b) For $z \in \Gamma_R$, $R_+(z) = R_-(z)(I + \mathcal{O}(x^{-1}))$,

(c) As $z \to \infty$, $R(z) = I + \mathcal{O}(1/z)$.

It follows that $R(z) = I + \mathcal{O}(x^{-1})$, uniformly in z. Reversing the transformations

$$R \longrightarrow S \longrightarrow Y \longrightarrow \Phi$$

leads to asymptotics for Φ and for y.

3. Random matrices and orthogonal polynomials

Consider a unitary random matrix ensemble with probability measure

$$Z_n^{-1} e^{-n \operatorname{Tr} V(M)}\, dM$$

on the space of $n \times n$ Hermitian matrices, for some real analytic potential V with enough growth at infinity.

- The limiting mean density of eigenvalues depends on V, and is supported on a finite union of intervals,
- The eigenvalues follow a determinantal point process with correlation kernel

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x) p_k(y),$$

where the polynomials p_k of degree k are orthonormal with respect to the weight $e^{-nV(x)}$.

3. Random matrices and orthogonal polynomials

- Local scaling limits

$$\lim_{n\to\infty} \frac{1}{cn^\alpha} K_n\left(x^* + \frac{u}{cn^\alpha}, x^* + \frac{v}{cn^\alpha}\right)$$

of the kernel near a point x^* show universal behavior, depending on the scaling regime of x^*, but not explicitly on V or x^*:

$\longrightarrow$ bulk scaling: sine kernel (Deift et al.)

$\longrightarrow$ regular edge scaling (square root vanishing of the limiting mean eigenvalue density): Airy kernel (Deift et al.),

$\longrightarrow$ singular endpoint (where the density vanishes like a power $5/2$): kernel built out of the function Φ associated with the real pole-free solution of P_I^2.

3. Random matrices and orthogonal polynomials

Monic orthogonal polynomials $\pi_{k,t}$ with respect to the weight $e^{-nV_{s,t}(x)}$ (with $V_{s,t} = V_0 + sV_1 + tV_2$) satisfy a three-term recurrence relation

$$xp_k(x) = a_{k+1}p_{k+1}(x) + b_k p_k(x) + a_k p_{k-1}(x).$$

If V_0 is such that there is a critical edge point (as in the random matrix model), asymptotics of a_n and b_n are as follows,

$$a_n = \frac{b-a}{4} + \frac{1}{2c} y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7}),$$
$$b_n = \frac{b+a}{2} + \frac{1}{c} y(c_1 n^{6/7} s, c_2 n^{4/7} t) n^{-2/7} + \mathcal{O}(n^{-3/7})$$

in the double scaling limit where we let $n \to \infty$ and $s, t \to 0$ in such a way that $|n^{6/7} s| \leq M$, $|n^{4/7} t| \leq M$.