On Middle Convolution for Fuchsian Systems

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Painlevé Equations and Monodromy Problems: An Introduction

Abstract

In the talk I explain the algorithm of Dettweiler and Reiter who generalized the Katz middle convolution functor.

Middle convolution is an operation for Fuchsian systems of differential equations which preserves rigidity (and, hence, the number of accessory parameters) but changes the rank and monodromy group.

In the simplest case of the sixth Painlevé equation which describes monodromy preserving deformations of the rank 2 Fuchsian system with four singularities on the projective line the algorithm is applied to derive the Okamoto birational transformation. Next I discuss the invariance of deformation equation under middle convolution.

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Katz Theory

$t_1, t_2, \ldots, t_{p+1} \in \mathbb{CP}^1$: distinct points

$x_0 \in X$, $X = \mathbb{CP}^1 \setminus \{t_1, t_2, \ldots, t_{p+1}\}$

$\pi_1(X, x_0)$: fundamental group

$$\pi_1(X, x_0) = \langle \gamma_1, \gamma_2, \ldots, \gamma_{p+1} \mid \gamma_1 \gamma_2 \cdots \gamma_{p+1} = 1 \rangle.$$

A rank n local system on X is determined by a representation

$$\pi_1(X, x_0) \longrightarrow \mathrm{GL}(n, \mathbb{C}),$$

and hence by the tuple $(A_1, A_2, \ldots, A_{p+1})$ of matrices in $\mathrm{GL}(n, \mathbb{C})$ such that

$$A_1 A_2 \cdots A_{p+1} = I.$$

A local system given by $(A_1, A_2, \ldots, A_{p+1})$ is said to be physically rigid, if, for any tuple $(B_1, B_2, \ldots, B_{p+1})$ of matrices in $\mathrm{GL}(n, \mathbb{C})$ satisfying $B_1 B_2 \cdots B_{p+1} = I$ and $B_j = C_j A_j C_j^{-1}$ $(1 \leq j \leq p+1)$, there exists $C \in \mathrm{GL}(n, \mathbb{C})$ such that simultaneously $B_j = C A_j C^{-1}$ for $1 \leq j \leq p+1$.

The index of rigidity

$$\iota = (2 - (p+1))n^2 + \sum_{j=1}^{p+1} \dim Z(A_j),$$

where $Z(A_j)$ denotes the centralizer of A_j.

It is known that ι is even, $\iota \leq 2$ for any irreducible tuple, and in this case $\iota = 2$ means physically rigid.

The number $2 - \iota$ can be regarded as the dimension of the moduli space of local systems with prescribed local monodromies, i.e., number of accessory parameters.

Katz: any irreducible and rigid system on $\mathbb{P}^1 - \{t_1, \ldots, t_k\}$ is obtained from rank one system by finite number of middle convolutions and multiplications.

Middle convolution preserves the index of rigidity and irreducubility, but changes rank and monodromy group.

Middle convolution is related to the Riemann-Liouville integral along Pochhammer contour: $\int_\Delta \Phi(u) Y(u) (x-u)^\lambda \, du$

Dettweiler and Reiter: algebraic analogue (multiplicative version for monodromy matrices, additive version for Fuchsian systems).

Though the algorithm was introduced for rigid systems (no accessary parameters), it can be applied to any Fuchsian system and is useful for getting the symmetries of the system.

Algorithm of Dettweiler and Reiter

Middle convolution

(1) Original Fuchsian system

$$\frac{dY(x)}{dx} = \sum_{i=1}^{r} \frac{A_i}{x - a_i} Y(x), \ A_k \in \mathbb{C}^{n \times n}$$

(2) Convolution

$$\frac{dV(x)}{dx} = \sum_{i=1}^{r} \frac{B_i}{x - a_i} V(x), \ B_k \in \mathbb{C}^{rn \times rn}$$

(3) Middle convolution (irreducible part of the above system)

$$\frac{dZ(x)}{dx} = \sum_{i=1}^{r} \frac{\widetilde{B}_i}{x - a_i} Z(x), \ \widetilde{B}_k \in \mathbb{C}^{m \times m}$$

m depends on n, r, and a parameter $\mu \in \mathbb{C}$.

Addition

The addition with parameters $(\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{C}$ is defined as a map

$$(A_1, A_2, \ldots, A_p) \mapsto (A_1 + \alpha_1 Id_n, A_2 + \alpha_2 Id_n, \ldots, A_p + \alpha_p Id_n).$$

For the Fuchsian system the result of the addition is the system

$$\frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j + \alpha_j Id_n}{x - t_j} \right) Y$$

Middle convolution: details

Let $\mathbf{A} = (A_1, \ldots, A_r)$, $A_k \in \mathbb{C}^{n\times n}$ and

$$\frac{dY}{dx} = \sum_{i=1}^{r} \frac{A_i}{x-a_i} Y$$

be a Fuchsian system. For $\mu \in \mathbb{C}$ one defines the convolution matrices $\mathbf{B} = mc_\mu(\mathbf{A}) = (B_1, \ldots, B_r)$ as follows:

$$B_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_1 & \ldots & A_{k-1} & A_k + \mu Id_n & A_{k+1} & \ldots & A_r \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{C}^{nr\times nr}$$

such that B_k is zero outside the k-th block row.

There are the following invariant subspaces of the column vector space $\mathbb{C}^{nr}$:

$$\mathcal{L}_k = (0, \ldots, 0, \mathsf{Ker}(A_k), 0, \ldots, 0)^{tr}, \ k = 1, \ldots, r,$$

and

$$\mathcal{K} = \bigcap_{k=1}^{r} \mathsf{Ker}(B_k) = \mathsf{Ker}(B_1 + \ldots + B_r).$$

Let $\mathcal{L} = \bigoplus_{k=1}^{r} \mathcal{L}_k$ and fix an isomorphism between $\mathbb{C}^{nr}/(\mathcal{K} + \mathcal{L})$ and $\mathbb{C}^m$ for some m. The matrices $C = mc_\mu(A) := (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in \mathbb{C}^{m\times m}$, where $\widetilde{B}_k$ is induced by the action of B_k on $\mathbb{C}^m \simeq \mathbb{C}^{nr}/(\mathcal{K} + \mathcal{L})$ are called the additive version of the middle convolution of $\mathbf{A}$ with parameter μ.

Algorithm in case of the sixth Painlevé equation

The sixth Painlevé equation (P_{VI}) is given by

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2- \\ &- \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}+ \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\right. \\ &+ \left.\delta\frac{t(t-1)}{(y-t)^2}\right), \end{aligned}$$

$\alpha,\ \beta,\ \gamma,\ \delta$ being arbitrary parameters.

The Painlevé property implies that every solution may be analytically continued to a meromorphic function on the universal cover of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

Start with Fuchsian system $(n = 2,\ r = 3)$, deformations of which lead to P_{VI}

$$\downarrow$$

apply middle convolution with parameter equal to one of the eigenvalues of matrix at infinity

$$\downarrow$$

get a new system with $m = 2$

$$\downarrow$$

find explicitly Okamoto transformation for the solutions of the sixth Painlevé equation.

Explicitly,

$$y_1(t) = mc_{k_1}(y(t)) =$$
$$y - \frac{(\theta_0+\theta_1-\theta_\infty+\theta_t)(t-y)(y-1)y}{(\theta_0+\theta_t-1+t(\theta_0+\theta_1))y-(\theta_0+\theta_1+\theta_t-1)y^2-t(\theta_0+(t-1)y')}$$

and parameters

$$\begin{aligned}\alpha_1 &= \frac{1}{8}(\theta_0+\theta_1+\theta_\infty+\theta_t-2)^2,\ \beta_1 = -\frac{1}{8}(\theta_0-\theta_1+\theta_\infty-\theta_t)^2,\\ \gamma_1 &= \frac{1}{8}(-\theta_0+\theta_1+\theta_\infty-\theta_t)^2,\ \delta_1 = \frac{1}{2}(1-(\theta_0+\theta_1-\theta_\infty-\theta_t)^2/4),\end{aligned}$$

which coincides with Okamoto's birational transformation.

(..., K.Okamoto ('87),
R. Conte ('01): singular manifold method,
M. Mazzocco: Laplace transform to irregular system and easy gauge transformation,
M. Noumi, Y. Yamada ('03): symmetric form,
K. Iwasaki ('03): Riemann-Hilbert correspondence,
D.Novikov ('06): integral transformation, Boalch, ...)

Iwasaki ('03)

$$\frac{d^2 f}{dz^2} - u_1(z)\frac{df}{dz} + u_2(z) f = 0$$

Fuchsian differential equation on $\mathbb{P}^1 - \{t_1, t_2, t_3, t_4, q, \infty\}$. Fix $t = \frac{(t_1-t_3)(t_2-t_4)}{(t_1-t_2)(t_3-t_4)}$.

$z = t_i$: a regular singularity with local exponents λ_i and 0,

$z = q$: apparent with exponents 2 and 0,

$z = \infty$: removable by $f = z^{-\lambda_0} g$.

Fuchsian relation: $2\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1.$

$\Lambda = \{\lambda = (\lambda_0, \dots, \lambda_4) \in \mathbb{C}^5 | 2\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1\}$: parameter space, $W(D_4^{(1)}) =< \sigma_0, \dots, \sigma_4 >$, σ_i are reflections on Λ across $\lambda_i = 0$.

$G =< s_0, \dots, s_4 >$ is a nonlinear representation of $W(D_4^{(1)})$.
$s_i : \; P_{VI}(\lambda) \to P_{VI}(\sigma_i(\lambda))$.

$s_i, \; i \in \{1, \dots, 4\}$ can be obtained from $f = (z - t_i)^{\lambda_i} h$. What about s_0?

In general, Bäcklund transformations s_i are transformations which are identity on the moduli of monodromy representations through the Riemann-Hilbert correspondence.

Let $\sigma \in W(D_4^{(1)})$, $\lambda = (\lambda_0, \ldots, \lambda_4)$,

$\mathcal{E}_t(\lambda)$: set of all Fuchsian equations with prescribed local exponents λ at singular points $t_1, \ldots, t_4$,

$\mathcal{R}_t(a(\lambda))$: moduli of monodromy representations up to conjugacy,
$R_t(a(\lambda)) \cong \mathcal{S}(\theta(\lambda))$ (certain affine cubic surface),

RH ba a Riemann-Hilbert correspondence

$$\begin{array}{ccc} \mathcal{E}_t(\lambda) & \xrightarrow{\ s\ } & \mathcal{E}_t(\sigma(\lambda)) \\ {\scriptstyle RH}\downarrow & & \downarrow{\scriptstyle RH} \\ \mathcal{S}(\theta(\lambda)) & \xrightarrow[id]{} & \mathcal{S}(\theta(\lambda)) \end{array}$$

Next we consider the middle convolution with parameter λ, where λ is a complex number different from k_1, k_2 and 0. The result is the system

$$\frac{dY}{dx} = \left(\frac{G_0}{x} + \frac{G_1}{x-1} + \frac{G_2}{x-t}\right) Y$$

with $G_i \in GL(3, \mathbb{C})$.

This system is related to systems studied by Harnad, Mazzocco and Boalch by means of the Laplace transformation.

It is useful for considering algebraic solutions in the theory of the sixth Painlevé equation.

By combining middle convolutions and additions we can get systems with
$G_i \in GL(k, \mathbb{C}),\ k = 4, 5, 6, \ldots$

Explicitely,

$$G_0 = \begin{pmatrix} \lambda + \theta_0 & \dfrac{(z_0+\theta_0)u_1 z_1}{u_0 z_0} - z_1 & \dfrac{(z_0+\theta_0)u_2 z_2}{u_0 z_0} - z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{(z_1+\theta_1)u_0 z_0}{u_1 z_1} - z_0 & \lambda + \theta_1 & \dfrac{(z_1+\theta_1)u_2 z_2}{u_1 z_1} - z_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{(z_2+\theta_2)u_0 z_0}{u_2 z_2} - z_0 & \dfrac{(z_2+\theta_2)u_1 z_1}{u_2 z_2} - z_1 & \lambda + \theta_2 \end{pmatrix}.$$

$$\bar{G}_0 = \begin{pmatrix} z_0+\theta_0+a_0+\lambda & -u_0 z_0 & z_1+\theta_1 & z_2+\theta_2 \\ \frac{z_0+\theta_0}{u_0} & a_0-z_0+\lambda & \frac{z_1+\theta_1}{u_1} & \frac{z_2+\theta_2}{u_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{G}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z_0+\theta_0+a_0-\frac{(z_0+\theta_0)u_1 z_1}{u_0(z_1+\theta_1)} & -u_0 z_0-\frac{(a_0-z_0)u_1 z_1}{z_1+\theta_1} & \theta_1+\lambda & z_2+\theta_2-\frac{(z_2+\theta_2)u_1 z_1}{u_2(z_1+\theta_1)} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{G}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z_0+\theta_0+a_0-\frac{(z_0+\theta_0)u_2 z_2}{u_0(z_2+\theta_2)} & -u_0 z_0-\frac{(a_0-z_0)u_2 z_2}{z_2+\theta_2} & z_1+\theta_1-\frac{(z_1+\theta_1)u_2 z_2}{u_1(z_2+\theta_2)} & \theta_2+\lambda \end{pmatrix}.$$

Monodromy preserving deformations

Aim: show that isomonodromic family is invariant under middle convolution.

$$\frac{dY}{dx} = \left(\sum_{j=1}^{p} \frac{A_j}{x - t_j} \right) Y,$$

$$A_{p+1} = -(A_1 + A_2 + \cdots + A_p).$$

Under certain assumptions on eigenvalues of A_j, the Schlesinger system govern monodromy preserving deformations of the Fuchsian system above:

$$\begin{cases} \dfrac{\partial A_i}{\partial t_i} = -\sum\limits_{k \neq i} \dfrac{[A_i, A_k]}{t_i - t_k}, & (Schlesinger\ system) \\ \dfrac{\partial A_j}{\partial t_i} = \dfrac{[A_i, A_j]}{t_i - t_j} & (j \neq i)\ . \end{cases}$$

Example [Boalch, Hitchin].

For $G_i \in GL(3,\mathbb{C}),\ i=1,2,3,$ define
$x=\mathrm{tr}(G_1G_3),\ y=\mathrm{tr}(G_2G_3),\ \ f(x,y)=\mathrm{tr}(G_1[G_2,G_3]).$

Then the Schlesinger system becomes

$$\frac{dx}{dt}=\frac{f(x,y)}{t-1},\quad \frac{dy}{dt}=-\frac{f(x,y)}{t}$$

and one can show that $f(x,y)=f_H(x-a,x-b)$.

Generalization:

$$\begin{cases} \dfrac{\partial}{\partial t_i}\mathrm{tr}(A_iA_j) = -\displaystyle\sum_{k\neq i,j}\frac{\mathrm{tr}([A_i, A_k]A_j)}{t_i - t_k}, \\ \dfrac{\partial}{\partial t_i}\mathrm{tr}(A_jA_k) = \dfrac{\mathrm{tr}([A_i, A_j]A_k)}{t_i - t_j} + \dfrac{\mathrm{tr}(A_j[A_i, A_k])}{t_i - t_k}. \end{cases} \tag{1}$$

Theorem. If for $j = 1, 2, \ldots, p+1$, there is no integral difference between any two distinct eigenvalues of A_j and the Jordan canonical form of A_j is independent of $t_1, t_2, \ldots, t_p$, then the systems (1) for the Fuchsian systems obtained by addition and middle convolution with parameters independent of $t_1, t_2, \ldots, t_p$ coincide with the system (1) for the initial Fuchsian system.

For P_{VI}

$$
\begin{array}{ccc}
(A_0, A_1, A_2) \in (GL(2,\mathbb{C}))^3 & \xrightarrow{mc_{\kappa_1}} & (\bar{A}_0, \bar{A}_1, \bar{A}_2) \in (GL(2,\mathbb{C}))^3 \\
\downarrow & & \downarrow \\
(p = Tr(A_0A_2), q = Tr(A_1A_2)) & \xrightarrow{shift} & (\bar{p} = Tr(\bar{A}_0\bar{A}_2), \bar{q} = Tr(\bar{A}_1\bar{A}_2)) \\
\updownarrow & & \updownarrow \\
y(t) & \xrightarrow{BTr} & \bar{y}(t)
\end{array}
$$

Thank you very much for your attention !