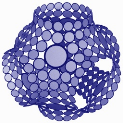
Linear and nonlinear theories of discrete analytic functions. Integrable structure

Alexander Bobenko

Technical University Berlin

Painlevé Equations and Monodromy Problems,
Cambridge, September 18, 2006



DFG Research Unit 565 “Polyhedral Surfaces”

Discrete Differential Geometry

- **Aim:** Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.
- Intelligent discretizations lead to:
  - interesting geometric objects in discrete geometry
  - new methods (difference equations)
  - deep understanding of smooth theory (unifies surfaces and their transformations)
  - solution of problems in differential geometry (Weil's problem: convex surfaces from convex metrics; Alexandrov's solution with polyhedra)
  - represent smooth shape by a discrete shape with just few elements; best approximation (Applications)

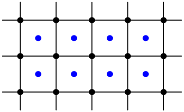
This talk

Based on a joint work with Ch. Mercat and Yu. Suris

- Discrete complex analysis, discrete holomorphic
- Integrability (geometric definition)
- Isomonodromic Green's function
- Linear and nonlinear theories (circle patterns)
- Discrete Painlevé equations

Harmonic and holomorphic on the square lattice

[Ferrand ’44, Duffin ’56]



conjugate harmonic

Cauchy-Riemann

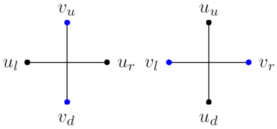
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

discrete Cauchy-Riemann

$$u_r - u_l = v_u - v_d$$

$$u_u - u_d = v_u - v_d$$

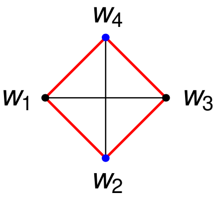


holomorphic

$w = u + iv$

$$\frac{\partial w}{\partial y} = i\frac{\partial w}{\partial x}$$

$$w = \begin{cases} \bullet\ u \\ \bullet\ iv \end{cases}$$

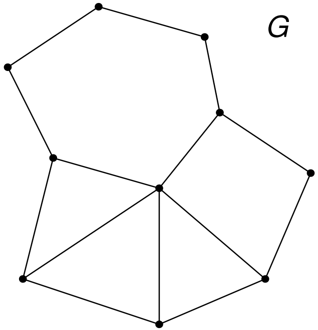


$$w_4 - w_2 = i(w_3 - w_1)$$

Quad-graph

f harmonic on a graph $G = (V, E)$, $\Delta f = 0$

$$\Delta f(x_0) = \sum_{x_k \sim x_0} \nu(x_0, x_k)(f(x_k) - f(x_0)), \quad \nu : E \to \mathbb{R}_+$$



G cell decomposition of $\mathbb{C}$

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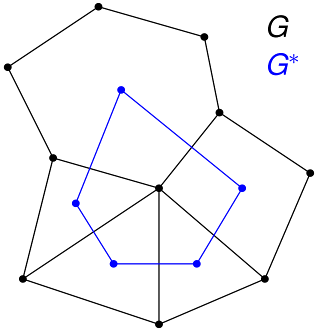

G cell decomposition of $\mathbb{C}$
G^* dual

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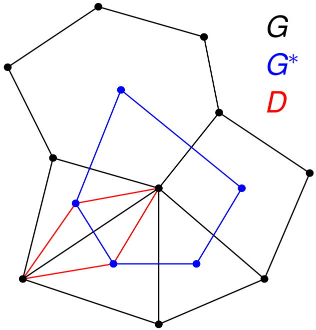


G cell decomposition of $\mathbb{C}$
G^* dual
D double - quad-graph

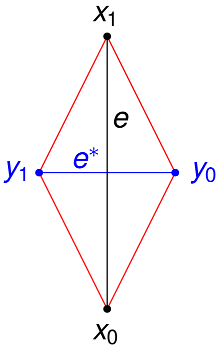
Harmonic and holomorphic on a graph

[Mercat '01]

$f : V(D) = V(G) \cup V(G^*) \to \mathbb{C}$
discrete holomorphic if it satisfies
discrete Cauchy-Riemann equations

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = i\nu(x_0, x_1) = -\frac{1}{i\nu(y_0, y_1)}$$

$\nu(e) = 1/\nu(e^*)$

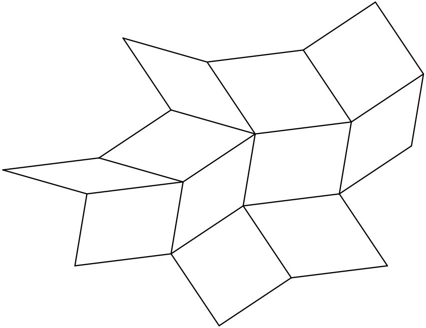


- $f : V(D) \to \mathbb{C}$ discrete holomorphic
  $\Rightarrow f|_{V(G)}, f|_{V(G^*)}$ discrete harmonic
- $f : V(G) \to \mathbb{C}$ discrete harmonic
  $\Rightarrow$ there exists unique (up to additive constant) extension to discrete holomorphic $f : V(D) \to \mathbb{C}$

Applications in computer graphics [Gu, Yau '05]

Rhombic quad-graphs

- Quad-graph = quadrilateral cell decomposition
- Rhombic quad-graph = there exists a rhombic representation in $\mathbb{R}^2$
- combinatorial characterization [Kenyon, Schlenker '04]
  - no strip crosses itself or periodic
  - strips cross at most once



Quasicristallic embedding of a rhombic quad-graph = finite number of slopes $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{C}$

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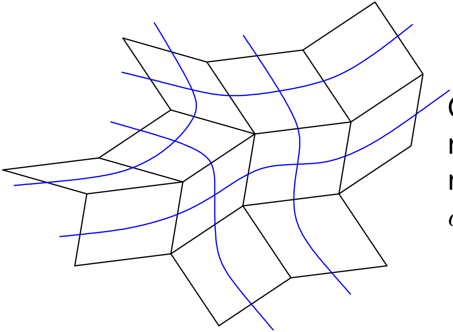

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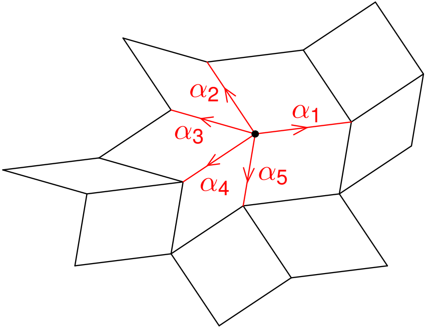


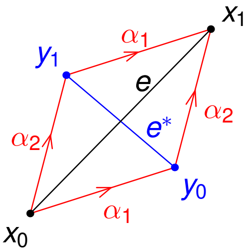
Quasicristallic embedding of a rhombic quad-graph = finite number of slopes $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{C}$

Green's function

D quasicristallic embedding of a rhombic quad-graph

- labelling $\alpha : E(D) \to \mathbb{C},\ |\alpha| = 1$
- weights $\nu(e) = \tan\dfrac{\phi}{2},\ \nu(e^*) = \cot\dfrac{\phi}{2}$

$$\frac{f(y_1) - f(y_0)}{f(x_1) - f(x_0)} = i \tan\frac{\phi}{2}$$



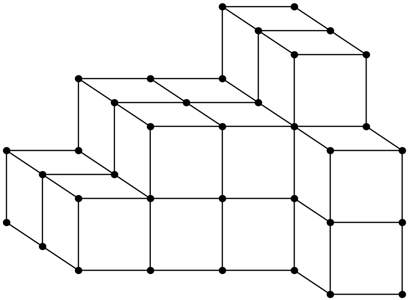
Green's function

$$\Delta g_{x_0}(x) = \delta_{xx_0}, \quad g_{x_0}(x) \to \log|x - x_0|, x \to \infty$$

Problem - compute explicitly

Method. Results

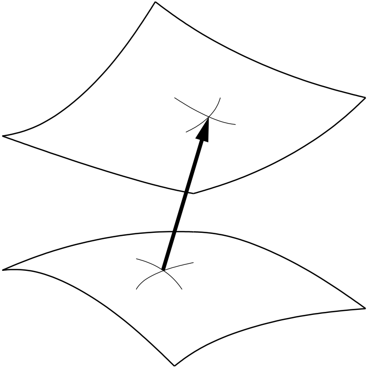
- lift rhombic quad-graph D to $\Omega_D \subset \mathbb{Z}^d$, d= number of different slopes $P : V(D) \to \mathbb{Z}^d$
- extend holomorphic functions from Ω_D to $\mathbb{Z}^d$



- integrable Laplacians = Laplacians on rhombic quad-graphs
- discrete flat connection (Lax representation)
- isomonodromic solutions $\Rightarrow$ Green's function
- comes from linearization of a nonlinear integrable theory (circle patterns)

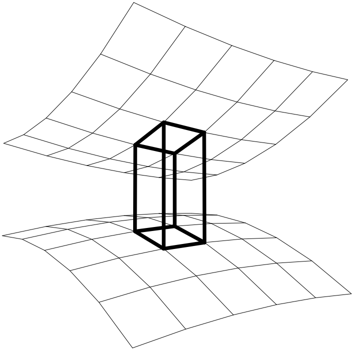
Surfaces and transformations

Classical theory of (special classes of) surfaces (constant curvature, isothermic, etc.)



special transformations (Bianchi, B"acklund, Darboux)

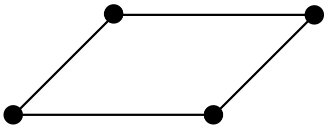
General and special Quad-surfaces



discrete $\rightarrow$ symmetric

Basic idea

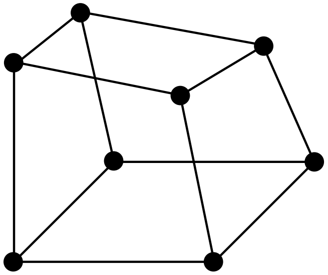
Do not distinguish discrete surfaces and their transformations.
Discrete master theory.



Example - planar quadrilaterals as discrete conjugate systems.
Multidimensional Q-nets [Doliwa, Santini '97].

Basic idea

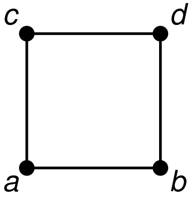
Do not distinguish discrete surfaces and their transformations. Discrete master theory.



Example - planar quadrilaterals as discrete conjugate systems. Multidimensional Q-nets [Doliwa, Santini '97].

Integrability as Consistency

- Equation



$$f(a, b, c, d) = 0$$

- Consistency

Integrability as Consistency

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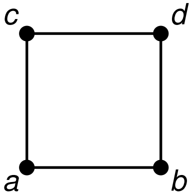

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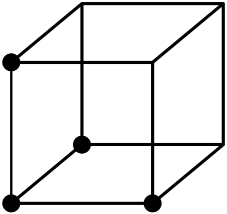

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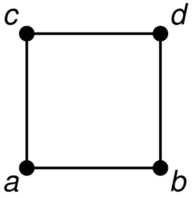

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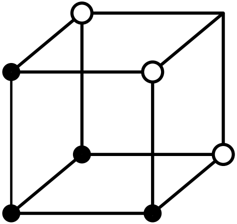

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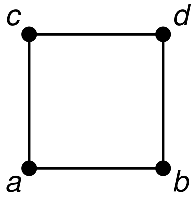

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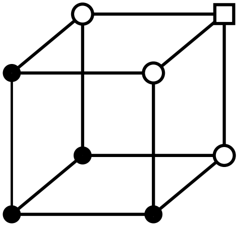
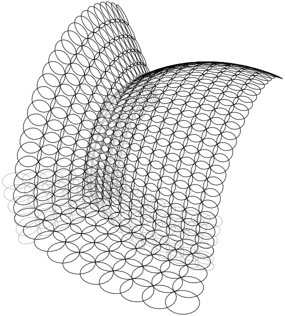
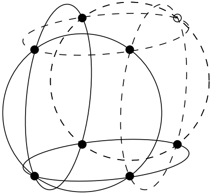


Circular nets

Martin, de Pont, Sharrock ['86], Nutbourne ['86], B. ['96], Cieslinski, Doliwa, Santini ['97], Konopelchenko, Schief ['98], Akhmetishin, Krichever, Volvovski ['99], ...



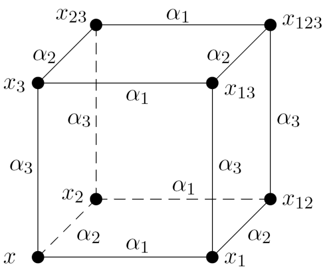
three "coordinate nets" of a discrete orthogonal coordinate system



elementary cube
→ Miquel theorem

Rhombic = Integrable

- Discrete Cauchy-Riemann equations (dCR) on a quad-graph integrable (3D consistent) iff the weights come from parallelogram immersions of the quad-graph
  - weight $i\nu$ = ratio of diagonals
  - $\nu \in \mathbb{R}$ iff rhombi



- Rhombic dCR
$$\frac{x_{12} - x}{x_1 - x_2} = \frac{\alpha_2 + \alpha_1}{\alpha_1 - \alpha_2}$$

- Lax representation (affine transformation $x_3 \mapsto x_{13}, \lambda = \alpha_3$)

$$L(x_1, x, \alpha; \lambda) = \begin{pmatrix} \lambda + \alpha & -2\alpha(x + x_1) \\ 0 & \lambda - \alpha \end{pmatrix}$$

Extension to $\mathbb{Z}^d$

Discrete holomorphic $f : \mathbb{Z}^d \to \mathbb{C}$:

- $\dfrac{f(n+e_i+e_k)-f(n)}{f(n+e_i)-f(n+e_k)} = \dfrac{\alpha_i+\alpha_k}{\alpha_i-\alpha_k}$ on each square
- specified by its values on the axes
- discrete exponential $e(me_k;\lambda) = \left(\dfrac{\lambda+\alpha_k}{\lambda-\alpha_k}\right)^m$
- discrete logarithm

$$f(me_k) = \begin{cases} 2(1+\frac{1}{3}+\frac{1}{5}+\dots\frac{1}{m-1}) & m \text{ even} \\ \log \alpha_k & m \text{ odd} \end{cases}$$

Discrete Log as Green's function

- Discrete Green's function on a quasicristallic quad-graph D is the real part (i.e. restriction to G) of the discrete logarithm function $g : \mathbb{Z}^d \to \mathbb{C}$
- Integral representation for discrete logarithm $g : \mathbb{Z}^d \to \mathbb{C}$ (Integral representation for discrete Green's function on D [Kenyon '02])

$$g(n) = \frac{1}{2\pi} \int_\gamma \frac{\log \lambda}{2\lambda} e(n, \lambda) d\lambda$$

γ loops around $\alpha_1, \dots, \alpha_d \in \mathbb{C}$

- Isomonodromic

Isomonodromic discrete Log

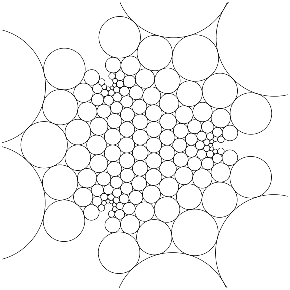
$\psi(n + e_k, \lambda) = L_k(n, \lambda)\psi(n, \lambda)$, $\quad L_k(n, \lambda)$ Lax matrices

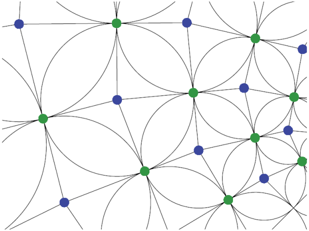
- $A(n, \lambda) = \dfrac{d\psi(n, \lambda)}{d\lambda}\psi^{-1}(n, \lambda)$
- isomonodromic $\Leftrightarrow$ $A(n, \lambda)$ meromorphic in λ, poles (position and order) independent of $n \in \mathbb{Z}$
- $A(n, \lambda) = \dfrac{\mathcal{A}(n)}{\lambda} + \sum_{k=1}^{d} \left(\dfrac{\mathcal{B}^k(n)}{\lambda + \alpha_k} + \dfrac{\mathcal{C}^k(n)}{\lambda - \alpha_k} \right)$
- Constraint [Nijhoff, Ramani, Grammaticos, Ohta '01] for $\mathbb{Z}^2$ $\sum_{k=1}^{d} n_k(f(n + e_k) - f(n - e_k)) = 1 - (-1)^{n_1 + \ldots n_g}$
- discrete logarithm

$$f(me_k) = \begin{cases} 2(1 + \frac{1}{3} + \frac{1}{5} + \ldots \frac{1}{m-1}) & m \text{ even} \\ \log \alpha_k & m \text{ odd} \end{cases}$$

Nonlinear Theory. Circle Patterns

Circle packings - discrete analogs of conformal maps [Thurston '85]





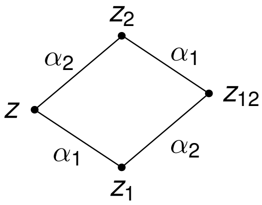
Conformal maps can be discretized as orthogonal circle patterns [Schramm '97]

$f : \mathbb{C} \to \mathbb{C}$ is a conformal map if $f_x \perp f_y$ and $|f_x| = |f_y|$

Nonlinear Theory. Equations

Cross-ratio equation

$$\frac{(z_1 - z)(z_2 - z_{12})}{(z_{12} - z_1)(z - z_2)} = \frac{\alpha_1^2}{\alpha_2^2}$$



- equivalent $z_1 - z = \alpha_1 w w_1$ to Hirota equation

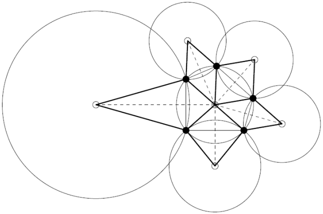
$$\alpha_1(ww_1 - w_2 w_{12}) = \alpha_2(ww_2 - w_1 w_{12})$$

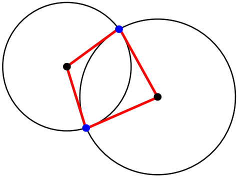
- cross-ratio and Hirota equations are 3D-consistent
- Lax representation (Möbius transformation $w_3 \mapsto w_{13}$)

$$L(w_1, w, \alpha; \lambda) = \begin{pmatrix} 1 & -\alpha w_1 \\ -\lambda\alpha/w & w_1/w \end{pmatrix}$$

Integrable Circle Patterns

Circle patterns: combinatorial data G and intersection angles

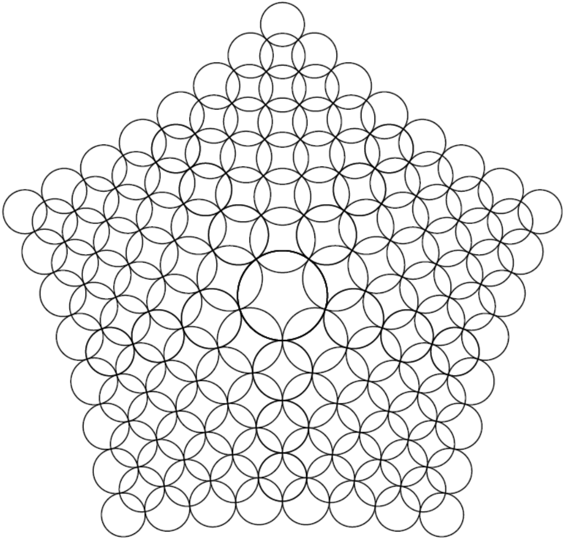




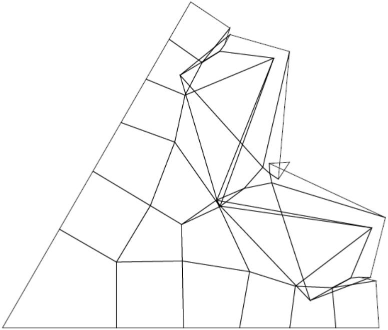
Circle patterns from z and w:
z - centers and intersection points of circles
$w(\bullet) \in \mathbb{R}_+$ - radii, $w({\color{blue}\bullet}) \in S^1$ - rotation angles

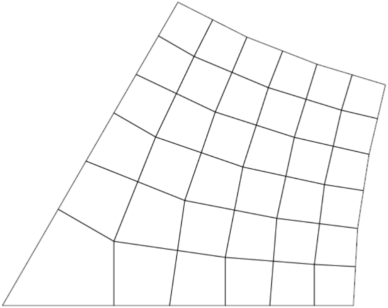
- $|\alpha| = 1$ rhombic realization $w \equiv 1$
- Combinatorial data and intersection angles belong to an integrable circle pattern iff they admit an isoradial realization. $\Rightarrow$ rhombic immersion of the double D.

Z^a circle pattern



Nontrivial. Unstable





Z^a circle pattern. Construction

$\psi(n+e_k,\lambda) = L_k(n,\lambda)\psi(n,\lambda), \quad L_k(n,\lambda)$ Lax matrices

- $A(n,\lambda) = \frac{d\psi(n,\lambda)}{d\lambda}\psi^{-1}(n,\lambda)$, isomonodromic
- $A(n,\lambda) = \frac{\mathcal{A}(n)}{\lambda} + \sum_{k=1}^{d} \frac{\mathcal{B}^k(n)}{\lambda - \alpha_k^{-2}}$
- Constraint [Nijhoff, Ramani, Grammaticos, Ohta '01] for $\mathbb{Z}^2$
  $\sum_{k=1}^{d} n_k \frac{w(n+e_k) - w(n-e_k)}{w(n+e_k) + w(n-e_k)} = \frac{a-1}{2}(1-(-1)^{n_1+\ldots n_g})$
- discrete z^a:
  $w(0) = 1, w(e_k) = \alpha_k^{a-1} \Leftrightarrow z(0) = 0, z(e_k) = \alpha_k^a$
  Asymptotics on the coordinate axes: $n \to \infty$
  $w(2ne_k) = n^{a-1}(1 + O(1/n))$,
  $z(ne_k) = (n\alpha_k)^a(1 + O(1/n))$

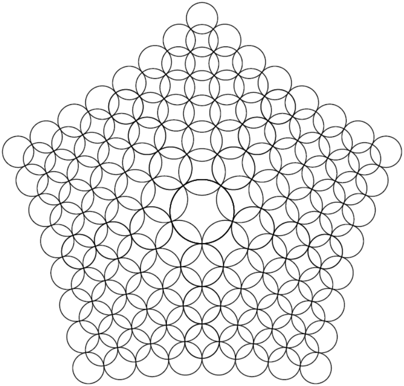
Z^a circle pattern. Analysis

Coordinate planes in $\mathbb{Z}^d$ are immersed (neighboring quads do not overlap) [Agafonov, B. '00]
Equation for rotation angles $x_n = w_{2n-1}$ on the diagonal of the ij-coordinate plane $q = \alpha_j/\alpha_i \in S^1$

$$(n+1)(x_n^2-1)\left(\frac{x_{n+1}+x_n/q}{q+x_nx_{n+1}}\right)-n(1-\frac{x_n^2}{q^2})\left(\frac{x_{n-1}+x_nq}{q+x_nx_{n-1}}\right)=ax_n\frac{q^2-1}{2q^2}$$

- immersed iff unitary solution with $0 < \arg x_n < \arg q$
- uniqueness of the solution
- discrete Painlevé II equation [Nijhoff, Ramani, Grammaticos, Ohta '01]

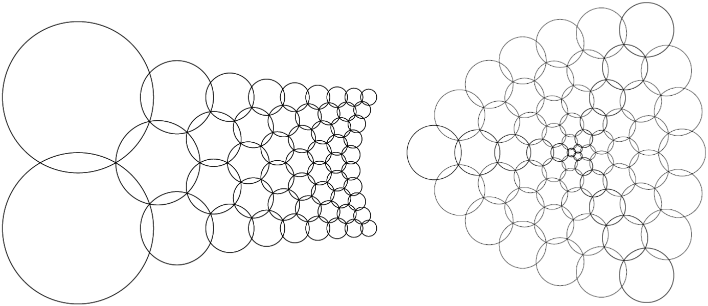
Z^a circle pattern. Theorems



- embedded [Agafonov '03]
- uniqueness for $a = 4/n$ follows from [He '99]
- uniqueness for arbitrary a [Bücking '06]

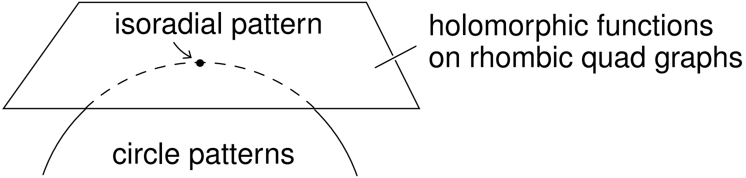
Non-orthogonal circle patterns

- Hexagonal [B., Hoffmann '03]



- Quasicristallic circle patterns from circle patterns with $\mathbb{Z}^N$ combinatorics

Linearization



- one-parameter family ϵ of circle patterns, $\epsilon = 0$ isoradial, w_ϵ solution of the Hirota equations. $f = w_\epsilon^{-1} \frac{dw_\epsilon}{d\epsilon}_{|\epsilon=0}$ satisfies dCR equations
- Discrete Green's function (discrete log of the linear theory) is the $\frac{d}{da}$-derivative of the circle pattern z^a at $a = 1$.

$$f(n) = \frac{d}{da} w_a(n)_{|a=1}$$