Orthogonal polynomials,

Integrable systems and

non-intersecting Brownian motions

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Overview:

1. Bi-orthogonal polynomials and the 2-component KP hierarchy

2. Orthogonal polynomials with regard to several weights and the n-component KP hierarchy

Example 1: n non-intersecting Brownian motions (in general)

Example 2: Airy process (edge-rescaled Brownian motion)

Example 3: n non-intersecting Brownian motions leaving from 0 forced to go to $-a$ and a

Example 4: Pearcey Process (gap-rescaled Brownian motion)

1. Bi-orthogonal polynomials

and

the 2-component KP hierarchy

Inner-product

$$\langle f \mid g \rangle := \iint_{\mathbb{R}^2} f(x)g(y)\rho(x,y)dxdy.$$

$$\Downarrow$$

$$\langle f \mid g \rangle_{t,s} := \iint_{\mathbb{R}^2} f(x)g(y)\rho(x,y)e^{\sum_1^\infty (t_i y^i - s_i x^i)}dxdy.$$

(Deformed weight)

Bi-orthogonal polynomials (monic):

$$\left\langle p_n^{(2)} e^{-\sum_1^\infty s_i x^i} \,\middle|\, p_m^{(1)} e^{\sum_1^\infty t_i y^i} \right\rangle = \iint_{\mathbb{R}^2} p_n^{(2)}(x)p_m^{(1)}(y)\rho(x,y)e^{\sum_1^\infty (t_i y^i - s_i x^i)}dxdy$$

$$= \delta_{nm}h_n.$$

Moment matrix

$$\tau_n(t,s) := \det\left(\left\langle x^k e^{-\sum_1^\infty s_i x^i} \,\middle|\, y^\ell e^{\sum_1^\infty t_i y^i} \right\rangle\right)_{0\le k,\ell \le n-1}.$$

Theorem *(Adler-PvM, CPAM '97)*: $\tau_n(t,s)$ satisfies*:

$$\begin{aligned}
z^n \frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} &= p_n^{(1)}(z) \\
z^n \frac{\tau_n(t,s+[z^{-1}])}{\tau_n(t,s)} &= p_n^{(2)}(z) \\
z^{-n-1} \frac{\tau_{n+1}(t+[z^{-1}],s)}{\tau_n(t,s)} &= \iint_{\mathbb{R}^2} \frac{p_n^{(2)}(x)}{z-y} e^{\sum_1^\infty (t_i y^i - s_i x^i)} \rho(x,y)\,dx\,dy \\
z^{-n-1} \frac{\tau_{n+1}(t,s-[z^{-1}])}{\tau_n(t,s)} &= \iint_{\mathbb{R}^2} \frac{p_n^{(1)}(y)}{z-x} e^{\sum_1^\infty (t_i y^i - s_i x^i)} \rho(x,y)\,dx\,dy.
\end{aligned}$$

Bilinear equations:

$$\oint_{z=\infty} \tau_{n-1}(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')e^{\sum_1^\infty (t_i-t_i')z^i} z^{n-m-2}dz$$

$$= \oint_{z=\infty} \tau_n(t,s-[z^{-1}])\tau_m(t',s'+[z^{-1}])e^{\sum_1^\infty (s_i-s_i')z^i} z^{m-n}dz,$$

(2-component KP hierarchy)

(Ueno-Takesaki '84)

$^*[\alpha] := (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots)$ for $\alpha \in \mathbb{C}$

Hirota symbol between functions $f = f(t_1, t_2, \ldots)$ and $g = g(t_1, t_2, \ldots)$

$$p(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, ...) f \circ g := p(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, ...) f(t+y) g(t-y) \Big|_{y=0}.$$

Computing the residues: ($\mathbf{S}_j(t)$:= elementary Schur polynomials)

$$\begin{aligned}\mathbf{S}_j\left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots\right)\tau_{n+1} \circ \tau_{n-1} &= -\tau_n^2 \frac{\partial^2}{\partial s_1 \partial t_{j+1}} \log \tau_n \\ \mathbf{S}_j\left(\frac{\partial}{\partial s_1}, \frac{1}{2}\frac{\partial}{\partial s_2}, \ldots\right)\tau_{n-1} \circ \tau_{n+1} &= -\tau_n^2 \frac{\partial^2}{\partial t_1 \partial s_{j+1}} \log \tau_n\end{aligned}$$

Combining several PDE's yields:

$$\left\{\frac{\partial^2 \log \tau_n}{\partial t_1 \partial s_2}, \frac{\partial^2 \log \tau_n}{\partial t_1 \partial s_1}\right\}_{t_1} + \left\{\frac{\partial^2 \log \tau_n}{\partial s_1 \partial t_2}, \frac{\partial^2 \log \tau_n}{\partial t_1 \partial s_1}\right\}_{s_1} = 0.$$

Proof:

$$\tau_n(t,s)\tau_{n+1}(t',s')$$

$$\iint_{\mathbb{R}^2} dxdy\ p^{(2)}_{n+1}(t',s';x)p^{(1)}_n(t,s;y)e^{\sum_1^\infty (t_i y^i - s'_i x^i)}\rho(x,y)\ \Bigg|_{\substack{t\mapsto t-a\\ t'\mapsto t'+a\\ s'=s}}$$

$$= \left(\sum_{j=0}^{\infty} -2a_{j+1}\mathbf{S}_j\left(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\dots\right)\tau_{n+2}\circ\tau_n + O(a^2)\right)$$

$$= \left(\sum_{j=0}^{\infty} 2a_{j+1}\tau^2_{n+1}\frac{\partial^2}{\partial s_1 \partial t_{j+1}}\log\tau_{n+1} + O(a^2)\right)$$

Using the following idea:

Space $\mathcal{H} := \text{span}\{z^i,\ i \in \mathbb{Z}\}$ equipped with two inner products:

(1)

$$\langle f, g \rangle = \int_{\mathbb{R}} f(z) g(z)\, dz,$$

(2) residue pairing about $z = \infty$, between $f = \sum_{i \geq 0} a_i z^i \in \mathcal{H}^+$ and $g = \sum_{j \in \mathbb{Z}} b_j z^{-j-1} \in \mathcal{H}$:

$$\langle f, g \rangle_\infty = \oint_{z=\infty} f(z) g(z) \frac{dz}{2\pi i} = \sum_{i \geq 0} a_i b_i,$$

Lemma

$$\langle f, g \rangle = \int_{\mathbb{R}} f(z) g(z)\, dz = \left\langle f, \int_{\mathbb{R}} \frac{g(u)}{z-u} du \right\rangle_\infty .$$

2. Orthogonal polynomials with regard to several weights

and

the n-component KP hierarchy

Aptekarev '98, Van Assche-Coussement '01, Geronimo-Kuijlaars-Van Assche '01, Bleher-Kuijlaars '04, Daems-Kuijlaars '05, Adler-PvM '99, Adler-PvM-Vanhaecke '06

Two sets of weights:

$$\psi_1, \ldots, \psi_q \quad \text{and} \quad \varphi_1, \ldots, \varphi_p,$$

Deform each weight with its own set of times:

$$\psi_k^{-s}(x) := \psi_k(x) e^{-\sum_1^\infty s_{ki} x^i} \qquad \text{and} \qquad \varphi_k^{t}(y) := \varphi_k(y) e^{\sum_1^\infty t_{ki} y^i},$$

depending on parameters

$$s_k = (s_{k1}, s_{k2}, \ldots) \text{ for } 1 \le k \le q \qquad t_k = (t_{k1}, t_{k2}, \ldots) \text{ for } 1 \le k \le p$$

$$\left(\sum_1^q m_i = \sum_1^p n_i\right)$$

$$\tau_{m_1,\ldots,m_q;n_1,\ldots,n_p}(s_1, \ldots, s_q; t_1, \ldots, t_p) :=$$

$$\det \begin{pmatrix} \left(\left\langle x^k \psi_1^{-s_1}(x) \,\middle|\, y^\ell \varphi_1^{t_1}(y) \right\rangle\right)_{\substack{0 \le k < m_1 \\ 0 \le \ell < n_1}} & \cdots & \left(\left\langle x^k \psi_1^{-s_1}(x) \,\middle|\, y^\ell \varphi_p^{t_p}(y) \right\rangle\right)_{\substack{0 \le k < m_1 \\ 0 \le \ell < n_p}} \\ \vdots & & \vdots \\ \left(\left\langle x^k \psi_q^{-s_q}(x) \,\middle|\, y^\ell \varphi_1^{t_1}(y) \right\rangle\right)_{\substack{0 \le k < m_q \\ 0 \le \ell < n_1}} & \cdots & \left(\left\langle x^k \psi_q^{-s_q}(x) \,\middle|\, y^\ell \varphi_p^{t_p}(y) \right\rangle\right)_{\substack{0 \le k < m_q \\ 0 \le \ell < n_p}} \end{pmatrix}.$$

I.

$$z^{n_\ell}\frac{\tau_{mn}(t_\ell-\left[z^{-1}\right])}{\tau_{mn}} := Q^{(\ell\ell)}(z)=z^{n_\ell}+\dots$$

$$\pm z^{n_\alpha-1}\frac{\tau_{m,n+e_\ell-e_\alpha}(t_\alpha-\left[z^{-1}\right])}{\tau_{mn}} = Q^{(\ell\alpha)}(z)=c_\alpha z^{n_\alpha-1}+\dots,\quad \text{for } \alpha\neq\ell$$

polynomials ($\sum n_\alpha$ coeff), satisfying $\sum m_\alpha$ orthogonality conditions

$$\left\langle x^j\psi_\alpha^{-s}(x)\;\middle|\;\sum_{i=1}^{p}Q^{(\ell i)}(y)\varphi_i^t(y)\right\rangle=0\quad\text{for}\quad\begin{cases}1\leq\alpha\leq q\\0\leq j\leq m_\alpha-1.\end{cases}$$

I.

$$z^{n_\ell}\frac{\tau_{mn}(t_\ell - \left[z^{-1}\right])}{\tau_{mn}} := Q^{(\ell\ell)}(z) = z^{n_\ell} + \dots$$

$$\pm z^{n_\alpha - 1}\frac{\tau_{m,n+e_\ell - e_\alpha}(t_\alpha - \left[z^{-1}\right])}{\tau_{mn}} = Q^{(\ell\alpha)}(z) = c_\alpha z^{n_\alpha - 1} + \dots, \quad \text{for } \alpha \neq \ell$$

polynomials ($\sum n_\alpha$ coeff), satisfying $\sum m_\alpha$ orthogonality conditions

$$\left\langle x^j \psi_\alpha^{-s}(x) \,\middle|\, \sum_{i=1}^{p} Q^{(\ell i)}(y)\varphi_i^t(y) \right\rangle = 0 \quad \text{for} \quad \left\{ \begin{array}{l} 1 \leq \alpha \leq q \\ 0 \leq j \leq m_\alpha - 1. \end{array} \right.$$

II.

$$\pm\, z^{m_\alpha - 1}\,\frac{\tau_{m-e_\alpha, n-e_\ell}(s_\alpha + [z^{-1}])}{\tau_{mn}} = P^{(\ell\alpha)}(z) \quad \text{of degree} < m_\alpha$$

polynomials ($\sum m_\alpha$ coeff), satisfying $\sum n_\alpha$ orthogonality conditions

$$\left\{ \begin{array}{l} \left\langle \displaystyle\sum_{i=1}^{q} P^{(\ell i)}(x)\psi_i^{-s}(x) \,\middle|\, y^j \varphi_\alpha^t(y) \right\rangle = 0 \quad \text{for} \left\{ \begin{array}{l} 1 \leq \alpha \leq p,\ 0 \leq j \leq n_\alpha - 1 \\ \text{except } \alpha = \ell,\ j = n_\ell - 1 \end{array} \right. \\ \\ \left\langle \displaystyle\sum_{i=1}^{q} P^{(\ell i)}(x)\psi_i^{-s}(x) \,\middle|\, y^{n_\ell - 1} \varphi_\ell^t(y) \right\rangle = 1 \end{array} \right.$$

III. Cauchy transforms of the polynomials obtained in II:

$$z^{-n_\ell}\frac{\tau_{mn}(t_\ell+\left[z^{-1}\right])}{\tau_{mn}} := \left\langle \sum_{i=1}^{q} P^{(\ell i)}(x)\psi_i^{-s}(x) \,\middle|\, \frac{\varphi_\ell^t(y)}{z-y} \right\rangle$$

$$\varepsilon_{\alpha\ell}(n)z^{-n_\ell-1}\frac{\tau_{m,n+e_\ell-e_\alpha}(t_\ell+\left[z^{-1}\right])}{\tau_{mn}} = \left\langle \sum_{i=1}^{q} P^{(\alpha i)}(x)\psi_i^{-s}(x) \,\middle|\, \frac{\varphi_\ell^t(y)}{z-y} \right\rangle$$

IV. Cauchy transforms of the polynomials obtained in I:

$$\varepsilon_{\alpha\ell}(m,n)\,z^{-m_\alpha-1}\,\frac{\tau_{m+e_\alpha,n+e_\ell}(s_\alpha-[z^{-1}])}{\tau_{mn}} = \left\langle \frac{\psi_\alpha^{-s}(x)}{z-x} \,\middle|\, \sum_{i=1}^{p} Q^{(\ell i)}(y)\varphi_i^t(y) \right\rangle .$$

satisfies $\underline{p+q\text{-}\mathbf{KP\ hierarchy}}$

$$\sum_{\beta=1}^{p}\oint_{\infty}\tau_{m,n-e_\beta}(t_\beta-[z^{-1}])\tau_{m',n'+e_\beta}(t'_\beta+[z^{-1}])e^{\sum_1^\infty(t_{\beta i}-t'_{\beta i})z^i}z^{n_\beta-n'_\beta-2}\,dz=$$
$$\pm\sum_{\alpha=1}^{q}\oint_{\infty}\tau_{m+e_\alpha,n}(s_\alpha-[z^{-1}])\tau_{m'-e_\alpha,n'}(s'_\alpha+[z^{-1}])e^{\sum_1^\infty(s_{\alpha i}-s'_{\alpha i})z^i}\,z^{m'_\alpha-m_\alpha-2}\,dz,$$

where $\sum m'_\alpha=\sum n'_\alpha+1$ and $\sum m_\alpha=\sum n_\alpha+1$

Example 1:

N non-intersecting

Brownian motions

Dyson '62, Grabiner '99, Johansson '01, Bleher-Kuijlaars '04, Adler-PvM '05, Kuijlaars-Daems '05, Tracy-Widom '05, Adler-PvM-Vanhaecke '06

The transition probability for Brownian motion $x(t)$ in $\mathbb{R}$ is given by

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$$

The transition probability for N non-intersecting Brownian motions $x_1(t), \dots, x_N(t)$ in $\mathbb{R}$ is given by Karlin-McGregor:

$$\int_{E^N} \det[p(t,\alpha_i,x_j)]_{1\le i,j\le N} \det[p(1-t,x_i,\beta_j)]_{1\le i,j\le N} \prod_{i=1}^{N} dx_i$$

Set

$$a := (\overbrace{a_1,a_1,\dots,a_1}^{m_1}, \overbrace{a_2,a_2,\dots,a_2}^{m_2}, \dots, \overbrace{a_q,a_q,\dots,a_q}^{m_q}) \in \mathbb{R}^N$$

$$b := (\overbrace{b_1,b_1,\dots,b_1}^{n_1}, \overbrace{b_2,b_2,\dots,b_2}^{n_2}, \dots, \overbrace{b_p,b_p,\dots,b_p}^{n_p}) \in \mathbb{R}^N$$

$$(0 < t < 1)$$

$$\mathbb{P}\left(\text{all } x_i(t) \in E \,\middle|\, \begin{array}{c} (x_1(0), \ldots, x_N(0)) = a \\ (x_1(1), \ldots, x_N(1)) = b \end{array}\right)$$

$$= \lim_{\substack{(\alpha_1, \ldots, \alpha_N) \to a \\ (\beta_1, \ldots, \beta_N) \to b}} \frac{1}{Z_N} \int_{E^N} \det[p(t, \alpha_i, x_j)]_{1 \le i,j \le N} \det[p(1-t, x_i, \beta_j)]_{1 \le i,j \le N} \prod_{i=1}^{N} dx_i$$

$$= \frac{N!}{Z'_N} \det\left(\left(\int_{\tilde{E}} dy\ e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde{a}_\alpha + \tilde{b}_\beta) y}\right)_{\substack{0 \le i < m_\alpha \\ 0 \le j < n_\beta}}\right)_{\substack{1 \le \alpha \le q \\ 1 \le \beta \le p}}$$

where

$$\tilde{E} = \frac{E}{\sqrt{t(1-t)}}, \quad \tilde{a}_i = \sqrt{\frac{1-t}{t}} a_i, \quad \tilde{b}_i = \sqrt{\frac{t}{1-t}} b_i$$

Proof:

$$\det(A_{ik})_{1 \le i,k \le n} \det(B_{ik})_{1 \le i,k \le n} = \sum_{\sigma \in S_n} \det\left(A_{i,\sigma(j)}\ B_{j,\sigma(j)}\right)_{1 \le i,j \le n}.$$

Adding extra-time parameters:

$$\det\left(\left(\int_{\tilde{E}} dy\ e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde{a}_\alpha+\tilde{b}_\beta)y}\right)_{\substack{0\le i<m_\alpha\\ 0\le j<n_\beta}}\right)_{\substack{1\le\alpha\le q\\ 1\le\beta\le p}}$$

$$\Downarrow$$

$$\tau_{m_1,\dots,m_q;n_1,\dots,n_p}(t_1,\dots,t_p;s_1,\dots,s_q)$$

$$= \ \det\left(\left(\int_{\tilde{E}} dy\ e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde{a}_\alpha+\tilde{b}_\beta)y+\sum_1^\infty (t_{\beta,k}-s_{\alpha,k})y^k}\right)_{\substack{0\le i<m_\alpha\\ 0\le j<n_\beta}}\right)_{\substack{1\le\alpha\le q\\ 1\le\beta\le p}}$$

satisfies $p+q$-KP hierarchy

EXAMPLE 2: AIRY PROCESS

The AIRY PROCESS:

$$A(t) = \lim_{n\to\infty} \sqrt{2} n^{1/6} \left(\lambda_n (n^{-1/3} t) - \sqrt{2n} \right).$$

"Motion of the right most particle in the edge-rescaled non-intersecting Brownian motions"

(Prähofer-Spohn '02, Johansson '03)

Properties:

(i) Stationary process, with TW distribution: (q solution of PII, behaving like Airy function at ∞)

$$\mathbb{P}(A(t) \le u) = F_2(u) := \exp\left(- \int_u^\infty (\alpha - u) q^2(\alpha) d\alpha \right).$$

(ii) Joint probability ($t := t_2 - t_1 > 0$)

$$g(t;x,y) := \log \mathbb{P}\left(A(t_1) < \frac{y+x}{2}, A(t_2) < \frac{y-x}{2}\right),$$

satisfies the 3rd order 2nd degree PDE:

$$2t\frac{\partial^3 g}{\partial t \partial x \partial y} = \left(t^2 \frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2}\right) + 8\left\{\frac{\partial^2 g}{\partial x \partial y}, \frac{\partial^2 g}{\partial y^2}\right\}_y,$$

with initial condition

$$\lim_{t \searrow 0} g(t;x,y) = \log F_2\left(\min(\frac{y+x}{2}, \frac{y-x}{2})\right).$$

(iii) Covariance: ($t := t_2 - t_1$)

$$E(A(t_2)A(t_1)) - E(A(t_2))E(A(t_1)) = \frac{1}{t^2} + \frac{2}{t^4}\iint_{\mathbb{R}^2} \Phi(u,v)du\ dv + \ldots$$

(Adler-PvM, '98 and '03)

where

$$\Phi(u,v) := F_2(u)F_2(v) \times \left(\begin{array}{l} \frac{1}{4}\left(\int_u^\infty q^2\right)^2 \left(\int_v^\infty q^2\right)^2 \\ \\ +q^2(u)\left(\frac{q^2(v)}{4} + \frac{1}{2}\int_v^\infty q^2\right) \\ \\ +\int_u^\infty q^2 \int_v^\infty \left(\begin{array}{c} 2(v-\alpha)q^2(\alpha) \\ +q'(\alpha)^2 - q^4(\alpha) \end{array} \right) d\alpha \end{array} \right).$$

where $q(\alpha)$ is the solution of Painlevé II behaving like Airy function at ∞.

Example 3: n non-intersecting

Brownian motions

leaving from 0

forced to go to $-a$ and a

Pastur '72, Brézin-Hikami '96-98, Zinn-Justin '97-98, Johansson '01, Bleher-Kuijlaars '04, Tracy-Widom '04, Okounkov-Reshetikhin '05, Adler-PvM '05

1. Non-intersecting Brownian motions $\Longrightarrow$ Gaussian Hermitian matrices coupled in a chain with external source $(n = n_1 + n_2)$

$x_1(\tau) < x_2(\tau) < \ldots < x_n(\tau)$ at times: $0 < \tau_1 < \tau_2 < \ldots < \tau_m < 1$

$$\mathbb{P}_0^{\pm a}\left(\left\{\begin{array}{c}\text{all } x_i(\tau_1) \in \tilde{E}_1\\ \vdots \\ \text{all } x_i(\tau_m) \in \tilde{E}_m\end{array}\right\}\middle|\begin{array}{c}\text{all } x_j(0) = 0\\ n_1 \text{ left paths end up at } -a \text{ at time } \tau = 1,\\ n_2 \text{ right paths end up at } +a \text{ at time } \tau = 1\end{array}\right)$$

1. Non-intersecting Brownian motions $\Longrightarrow$ Gaussian Hermitian matrices coupled in a chain with external source $(n = n_1 + n_2)$

$$\mathbb{P}_0^{\pm a}\left(\left\{\begin{array}{c}\text{all } x_i(\tau_1)\in\tilde{E}_1\\ \vdots\\ \text{all } x_i(\tau_m)\in\tilde{E}_m\end{array}\right\}\middle|\begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ left paths end up at } -a \text{ at time } \tau=1,\\ n_2 \text{ right paths end up at } +a \text{ at time } \tau=1\end{array}\right)$$

$$= \frac{1}{Z_n}\int_{\prod_{\ell=1}^m \mathcal{H}_n(E_\ell)} dM_1 \dots dM_m$$

$$e^{-\frac{1}{2}\operatorname{Tr}(M_1^2+\dots+M_m^2-2c_1M_1M_2-\dots-2c_{m-1}M_{m-1}M_m-2AM_m)}$$

$$\mathbb{P}_0^{\pm a}\left(\left\{\begin{array}{c}\text{all } x_i(\tau_1)\in\tilde{E}_1\\ \vdots\\ \text{all } x_i(\tau_m)\in\tilde{E}_m\end{array}\right\}\middle|\begin{array}{c}\text{all } x_j(0)=0\\ n_1 \text{ left paths end up at } -a \text{ at time } \tau=1,\\ n_2 \text{ right paths end up at } +a \text{ at time } \tau=1\end{array}\right)$$

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$$=\frac{1}{Z_n}\det\left(\begin{array}{c}(\mu_{ij}^{+})_{1\le i\le n_1,\ 1\le j\le n_1+n_2}\\ \\ (\mu_{ij}^{-})_{1\le i\le n_2,\ 1\le j\le n_1+n_2}\end{array}\right)$$

with

$$\mu_{ij}^{\pm} = \int\dots\int_{\prod_1^m E_i} x_1^i x_m^j e^{-\frac{1}{2}\sum_1^m x_i^2+\sum_1^{m-1} c_i x_i x_{i+1}\pm\alpha x_m}\prod_{\ell=1}^m dx_\ell$$

where

$$A := \begin{pmatrix} \alpha & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & \alpha & & & \\ & & & -\alpha & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & -\alpha \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \\ \\ \\ \updownarrow n_2 \end{matrix} \quad \text{and} \quad \alpha = a\sqrt{\frac{2(\tau_m - \tau_{m-1})}{(1-\tau_m)(1-\tau_{m-1})}}$$

and

$$E_\ell = \tilde{E}_\ell\sqrt{\frac{2(\tau_{\ell+1} - \tau_{\ell-1})}{(\tau_{\ell+1} - \tau_\ell)(\tau_\ell - \tau_{\ell-1})}}\ , \ c_j = \sqrt{\frac{(\tau_{j+2} - \tau_{j+1})(\tau_j - \tau_{j-1})}{(\tau_{j+2} - \tau_j)(\tau_{j+1} - \tau_{j-1})}}$$

$$(\text{for } 1 \le j \le m-1)$$

Add extra-time parameters:

$$\mu_{ij}^{\pm}(t,s,u) = \int \ldots \int_{\prod_1^m \tilde{E}_i} \prod_{\ell=1}^{m} dx_\ell$$

$$x_1^i x_m^j e^{-\frac{1}{2}\sum_1^m x_i^2 + \sum_1^{m-1} c_i x_i x_{i+1} \pm \alpha x_m + \sum_{k=1}^{\infty}\left(t_k x_1^k - \binom{s_k}{u_k} x_m^k\right)}$$

Then

$$\tau_{n_1 n_2} := \det \begin{pmatrix} (\mu_{ij}^{+}(t,s,u))_{1\le i\le n_1,\ 1\le j\le n_1+n_2} \\ \\ (\mu_{ij}^{-}(t,s,u))_{1\le i\le n_2,\ 1\le j\le n_1+n_2} \end{pmatrix}$$

satisfies

(i) 3-component KP $(p+q=2+1=3)$

(ii) Virasoro constraints

$\Longrightarrow$

(ii) Virasoro equations

Lemma

$$\mathcal{E}(x,y) := \prod_{k=1}^{n} e^{\sum_1^\infty t_i x_k^i + \sum_{i,j\geq 1} c_{ij} x_k^i y_k^j + \sum_1^\infty s_i y_k^i}$$

$$\frac{\partial}{\partial \varepsilon} \mathcal{E}(x+\varepsilon x^{k+1}, y)\Big|_{\varepsilon=0}$$

$$= \begin{cases} \left(\sum_{i\geq 2} i t_i \frac{\partial}{\partial t_{i-1}} + n t_1 + \sum_{i\geq 2, j\geq 1} i c_{ij} \frac{\partial}{\partial c_{i-1,j}} + \sum_{j\geq 1} c_{1j} \frac{\partial}{\partial s_j} \right) \mathcal{E}(x,y), & k=-1 \\ \left(\sum_{i\geq 1} i t_i \frac{\partial}{\partial t_i} + \sum_{i,j\geq 1} i c_{ij} \frac{\partial}{\partial c_{ij}} \right) \mathcal{E}(x,y), & k=0 \end{cases}$$

Combining 3-component KP with Virasoro:

Theorem *(Adler-PvM '06)*

$$\log \mathbb{P}_n(\alpha, c_1, \ldots, c_m; E_1, \ldots, E_m) \;=\; \log \frac{1}{Z_n} \int_{\prod_{\ell=1}^m \mathcal{H}_n(E_\ell)} dM_1 \ldots dM_m$$
$$e^{-\frac{1}{2}\operatorname{Tr}(M_1^2+\ldots+M_m^2-2c_1M_1M_2-\ldots-2c_{m-1}M_{m-1}M_m-2AM_m)}$$

satisfies fourth-order PDE in α and $b_1^{(\ell)}, \ldots, b_{2r}^{(\ell)}$, with quartic non-linearity:

$$\left(F^+\mathcal{C}_1G^- + F^-\mathcal{C}_1G^+\right)\left(F^+\mathcal{C}_1F^- - F^-\mathcal{C}_1F^+\right)$$
$$-\left(F^+G^- + F^-G^+\right)\left(F^+\mathcal{C}_1^2F^- - F^-\mathcal{C}_1^2F^+\right) = 0,$$

where ($\mathcal{A}_1^{\pm}$ and $\mathcal{C}_1$ are gradients; $\mathcal{A}_2$ and $\mathcal{C}_2$ are Euler operators)

$$F^{\pm} \;:=\; \mathcal{A}_1^{\pm}\mathcal{C}_1 \log \mathbb{P}_n + n_{\left\{ {1 \atop 2} \right\}} J_{1m},$$
$$G^{\pm} \;:=\; \left\{ \left(\mathcal{A}_2\mathcal{C}_1 \pm J_{1m}\frac{\partial}{\partial\alpha}\right) \log \mathbb{P}_n \mp k^{\pm} \;,\; F^{\pm} \right\}_{\mathcal{C}_1}$$
$$+ \left\{ (\mathcal{C}_2 \pm 2\alpha J_{1m}\mathcal{C}_1)\, \mathcal{A}_1^{\pm} \log \mathbb{P}_n \;,\; F^{\pm} \right\}_{\mathcal{A}_1^{\pm}}$$

$$E_\ell^c := \bigcup_{i=1}^{r} [b_{2i-1}^{(\ell)}, b_{2i}^{(\ell)}] \subset \mathbb{R} \text{ and } J^{-1} := \begin{pmatrix} -1 & & c_1 & & & & \\ & & & \ddots & & \mathbf{O} & \\ c_1 & & -1 & & \ddots & & \\ & \ddots & & \ddots & & \ddots & \\ & & \ddots & & -1 & & c_{m-1} \\ & \mathbf{O} & & \ddots & & & \\ & & & & c_{m-1} & & -1 \end{pmatrix}$$

$$\mathcal{A}_1^{\pm} := -\frac{1}{2}\left(\sum_{\ell=1}^{m} J_{m\ell} \sum_{i=1}^{2r} \frac{\partial}{\partial b_i^{(\ell)}} \pm \frac{\partial}{\partial \alpha}\right)$$

$$\mathcal{C}_1 := \sum_{j=1}^{m} J_{1\ell} \sum_{i=1}^{2r} \frac{\partial}{\partial b_i^{(\ell)}}$$

$$\mathcal{A}_2 := \frac{1}{2}\left(\sum_{i=1}^{2r} b_i^{(m)} \frac{\partial}{\partial b_i^{(m)}} - \alpha \frac{\partial}{\partial \alpha} - c_{m-1}\frac{\partial}{\partial c_{m-1}}\right)$$

$$\mathcal{C}_2 := -\sum_{i=1}^{2r} b_i^{(1)} \frac{\partial}{\partial b_i^{(1)}} + c_1 \frac{\partial}{\partial c_1}$$

$$k^{\pm} = \mp J_{1m}\left(2n_{\left\{ {1 \atop 2} \right\}}\alpha J_{mm} - \frac{n_1 n_2}{\alpha}\right)$$

Example 4. Pearcey Process:

n **non-intersecting (for** $n \to \infty$**)**

Brownian motions

leaving from 0

forced to reach $-\sqrt{\frac{n}{2}}$ **and** $\sqrt{\frac{n}{2}}$

“Gap rescaling”

Pastur ’72, Brézin-Hikami ’96-98, Zinn-Justin ’97-98, Johansson ’01, Bleher-Kuijlaars ’04, Tracy-Widom ’04, Okounkov-Reshetikhin ’05, Adler-PvM ’05

The Pearcey process $\mathcal{P}(t)$

$$\mathbb{P}\begin{pmatrix} \mathcal{P}(t_1)\cap E_1=\emptyset \\ \vdots \\ \mathcal{P}(t_m)\cap E_m=\emptyset \end{pmatrix} := \lim_{z\to 0}\mathbb{P}_0^{\pm 1/z^2}\left.\begin{pmatrix} \text{all } x_j\left(\frac{1+t_1z^2}{2}\right)\notin zE_1 & \\ \vdots & ;\ 1\le j\le n \\ \text{all } x_j\left(\frac{1+t_mz^2}{2}\right)\notin zE_m & \end{pmatrix}\right|_{n=\frac{2}{z^4}}.$$

$$\mathbb{P}(\mathcal{P}(t)\cap E=\emptyset)=\det\left(I-K_t\chi_E\right) \qquad \text{(Fredholm determinant)}$$

Given $E_\ell := \bigcup_{i=1}^{r}[x_{2i-1}^{(\ell)}, x_{2i}^{(\ell)}] \subset \mathbb{R}$ and times $t_1 < t_2 < \ldots < t_m$ define

- "*space*" operator $\mathcal{X}_k := \sum_{\ell} \sum_{i=1}^{2r} (x_i^{(\ell)})^{k+1} \frac{\partial}{\partial x_i^{(\ell)}}$
- "*time*" operator $\mathcal{T}_k = \sum_{\ell} t_\ell^{k+1} \frac{\partial}{\partial t_\ell}$
- "*space-time*" operator $\tilde{\mathcal{X}} = \sum_{\ell} t_\ell \sum_{i=1}^{2} \frac{\partial}{\partial x_i^{(\ell)}}$

Given $E_\ell := \bigcup_{i=1}^r [x_{2i-1}^{(\ell)}, x_{2i}^{(\ell)}] \subset \mathbb{R}$ and times $t_1 < t_2 < \ldots < t_m$ define

- "*space*" operator $\mathcal{X}_k := \sum_\ell \sum_{i=1}^{2r} (x_i^{(\ell)})^{k+1} \frac{\partial}{\partial x_i^{(\ell)}}$
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- "*space-time*" operator $\tilde{\mathcal{X}} = \sum_\ell t_\ell \sum_{i=1}^{2} \frac{\partial}{\partial x_i^{(\ell)}}$

Theorem (Adler-vM '06) $\mathbb{Q} := \log \mathbb{P} \begin{pmatrix} \mathcal{P}(t_1) \cap E_1 = \emptyset \\ \vdots \\ \mathcal{P}(t_m) \cap E_m = \emptyset \end{pmatrix}$ satisfies :

$$0 = \left\{ \mathcal{X}_{-1}^2 \mathcal{T}_{-1} \mathbb{Q}, \right.$$

$$\left. \frac{1}{8} \left\{ \mathcal{X}_{-1} \mathcal{T}_{-1} \mathbb{Q} \ , \ \mathcal{X}_{-1}^2 \mathbb{Q} \right\}_{\mathcal{X}_{-1}} + (\mathcal{X}_0 + 2\mathcal{T}_0 - 2)\, \mathcal{X}_{-1}^2 \mathbb{Q} - 4(\tilde{\mathcal{X}} \mathcal{X}_{-1} - \mathcal{T}_{-1}^2) \mathcal{T}_{-1} \mathbb{Q} \right\}_{\mathcal{X}_{-1}}$$

For $m = 1$:

$$\mathbb{Q}(t;x) = \log \mathbb{P}\left(\mathcal{P}(t) \cap E = \emptyset\right) = \det\left(I - K_t \chi_E\right)$$

satisfies

$$\left\{\mathcal{X}_{-1}^2 \frac{\partial \mathbb{Q}}{\partial t}\ ,\ \frac{1}{8}\left\{\mathcal{X}_{-1}\frac{\partial \mathbb{Q}}{\partial t}\ ,\ \mathcal{X}_{-1}^2\mathbb{Q}\right\}_{\mathcal{X}_{-1}} + (\mathcal{X}_0 - 2)\mathcal{X}_{-1}^2\mathbb{Q} + 4\frac{\partial^3 \mathbb{Q}}{\partial t^3}\right\}_{\mathcal{X}_{-1}} = 0.$$

Fredholm determinant

$$\mathbb{P}(\mathcal{P}(t) \cap E = \emptyset) = \det\left(I - K_t \chi_E\right),$$

where $K_t(x,y)$ is the Pearcey kernel, defined as follows:

$$\begin{aligned} K_t(x,y) &:= \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - tp(x)q(y)}{x-y} \\ &= \int_0^\infty p(x+z)q(y+z)dz, \end{aligned}$$

where (note $\omega = e^{i\pi/4}$)

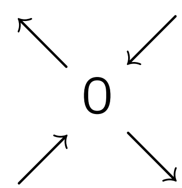
$$p(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{u^4}{4}-\frac{tu^2}{2}-iux}du$$

$$q(y) := \frac{1}{2\pi i}\int_X e^{\frac{u^4}{4}-\frac{tu^2}{2}+uy}du = \mathrm{Im}\left[\frac{\omega}{\pi}\int_0^{\infty} due^{-\frac{u^4}{4}-\frac{it}{2}u^2}(e^{\omega uy}-e^{-\omega uy})\right]$$

satisfy the differential equations

$$p''' - tp' - xp = 0 \text{ and } q''' - tq' + yq = 0.$$

The contour X:



Open questions:

(1) Find initial or "final" conditions for this equation; i.e., for $t = -\infty$ or $+\infty$.

(2) How does the Markov property and, in particular, the Chapman-Kolmogorov equations for the transition probabilities, reflect itself in these equations?

(3) Can one extract large-time asymptotics for the Pearcey process from the non-linear equation ?

(4) Does this PDE contain some hidden Painlevé equation? Can one take the limit of the equation, when $a \to 0$, and obtain Painlevé IV or Painlevé II ?