Universality of Painlevé functions in random matrix models

ARNO KUIJLAARS

Katholieke Universiteit Leuven, Belgium

References:

Tom Claeys and A.K.

Universality of the double scaling limit in random matrix models

math-ph/0501074, Comm. Pure Appl. Math. 59 (2006), 1573–1603.

Tom Claeys, A.K., and Maarten Vanlessen

Multi-critical unitary random matrix ensembles and the general Painlevé II equation

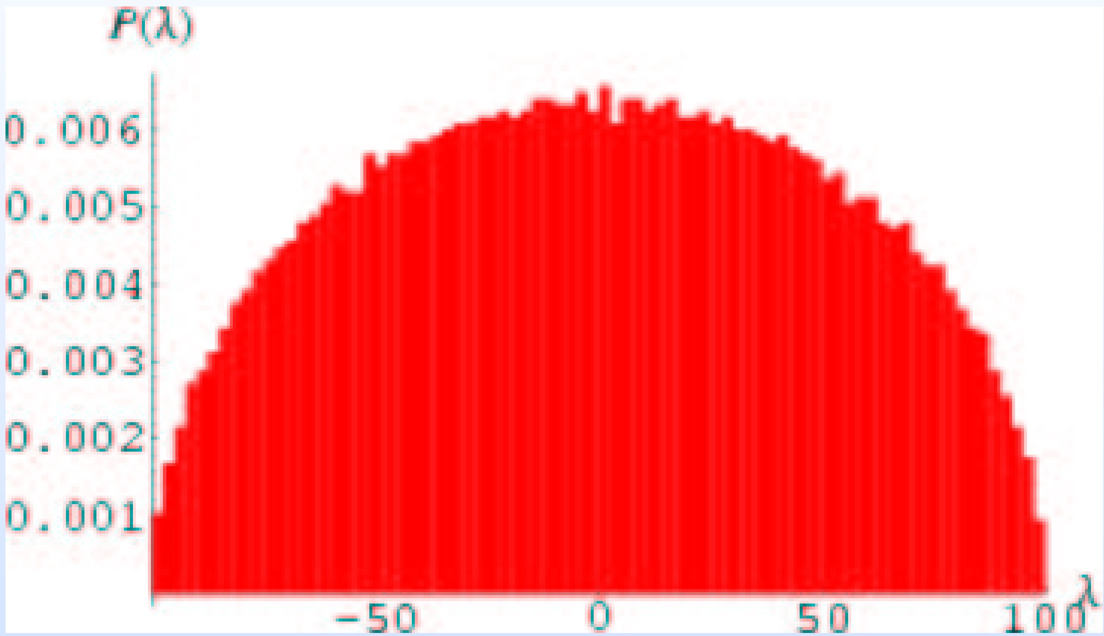
math-ph/0508062, to appear in Annals of Mathematics.

Random matrix ensembles

- Set of matrices with a probability measure on it.
  - Main interest in distribution of eigenvalues, especially as the size of the matrices tends to infinity.
- Simplest ensembles are Gaussian ensembles.
- Matrix entries have normal distribution with mean zero. The entries are independent up to the constraints that are imposed by the symmetry class.
  - Gaussian Unitary Ensemble GUE: complex Hermitian matrices
  - Gaussian Orthogonal Ensemble GOE: real symmetric matrices
  - Gaussian Symplectic Ensemble GSE: self-dual quaternionic matrices
- Where are the eigenvalues?

Wigner's semi-circle law

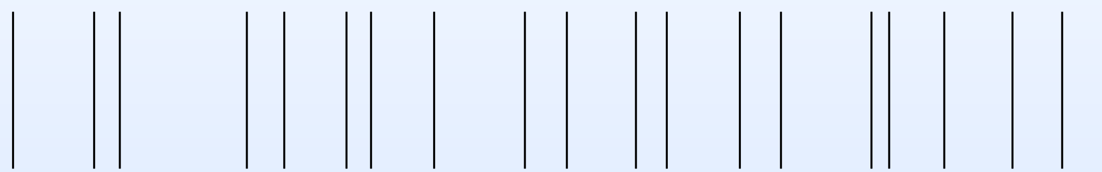
- Histogram of eigenvalues of large Gaussian matrix, size $10^4 \times 10^4$



- Eigenvalues are distributed according to a semi-circle.
- This is special for Gaussian ensembles. Other random matrix ensembles have other limiting mean density of eigenvalues.

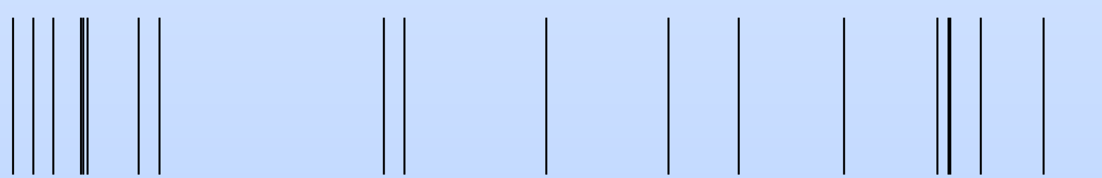
Universality 1: Local eigenvalue statistics

- **Global statistics of eigenvalues depend on the particular random matrix ensemble in contrast to local statistics. Distances between consecutive eigenvalues show regular behavior.**
- **Rescale eigenvalues around a certain value so that mean distance is one.**



plot shows only a few rescaled eigenvalues of a very large GUE matrix

- **Same behavior is seen in energy spectra in quantum physics and for the zeros of the Riemann zeta function on the critical line.**
- **The repulsion between neighboring eigenvalues is very different from Poisson spacings.**



Universality 1: Local eigenvalue statistics

- This local behavior of eigenvalues is not special for GUE.
  - It holds for large class of unitary ensembles, these are ensembles that have the same symmetry property as GUE.
  - Physicists knew this for a long time, but mathematicians could only prove it some years ago Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)
- Local eigenvalue statistics is different for GOE and GSE which have different symmetry properties. Mathematical proof of universality for orthogonal and symplectic ensembles is a recent result

  Deift, Gioev (arxiv 2004)
- Universality fails at special points, such as end points of the spectrum, or points where limiting mean eigenvalue density vanishes.
  - This gives rise to new universality classes $\Longrightarrow$ Painlevé functions

Universality 2: Largest eigenvalue

- Fluctuations of the largest eigenvalues of random matrices also show a universal behavior (depending on the symmetry class).
- For $n \times n$ GUE matrix, the largest eigenvalue grows like $\sqrt{2n}$ and has a standard deviation of the order $n^{-1/6}$.
- Centered and rescaled largest eigenvalue

$$\sqrt{2}n^{1/6}\left(\lambda_{\max} - \sqrt{2n}\right)$$

converges in distribution as $n \to \infty$ to a random variable with the Tracy-Widom distribution   Tracy, Widom (1994)

  - Same limit holds generically for unitary random matrix ensembles.
  - It also appears in certain combinatorial and statistical models, e.g. the length of the longest increasing subsequence of a random permutation (Ulam's problem)   Baik, Deift, Johansson (1999)
  - Different TW-distributions for orthogonal and symplectic ensembles.

Tracy-Widom distribution

- There is **no simple formula** for the Tracy-Widom distribution

- First formula is as a **Fredholm determinant**: $\boxed{F(s) = \det(I - A_s)}$

  where $A_s : L^2(s,\infty) \to L^2(s,\infty)$ is the integral operator with kernel

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

Airy kernel

- Second formula

$$F(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)dx\right)$$

  where q is the Hastings-McLeod solution of the **Painlevé II equation**

$$q''(s) = sq(s) + 2q^3(s)$$

Unitary ensembles

- **Unitary invariant random $n \times n$ Hermitian matrices**

$$\frac{1}{Z_n} e^{-\operatorname{Tr} V(M)} dM, \qquad V(x) \text{ grows as } x \to \pm\infty$$

- **This is GUE if $V(x) = \frac{1}{2}x^2$.**
- **Eigenvalues follow determinantal point process, which means that there is a kernel $K_n(x, y)$ so that all eigenvalue correlation functions are expressed as determinants**

$$\mathcal{R}_m(x_1, x_2, \ldots, x_k) = \det \left[K_n(x_i, x_j)\right]_{i,j=1,\ldots,m}$$

  - $\int_a^b K_n(x, x)dx$ **is expected number of eigenvalues in $[a, b]$**
  - $\det\left(I - (K_n)_{[a,b]}\right)$ **is the probability to have no eigenvalues in interval $[a, b]$, etc.**
  - $\Longrightarrow$ **All information is in the kernel K_n.**

Universality 1: precise statement

- **Scale $V \mapsto nV$ in order to balance the repulsion between eigenvalues and the confinement due to V**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

- **Limiting mean eigenvalue density** $\boxed{\rho_V(x) = \lim_{n\to\infty} \frac{1}{n} K_n(x,x)}$ **exists but depends on V.**

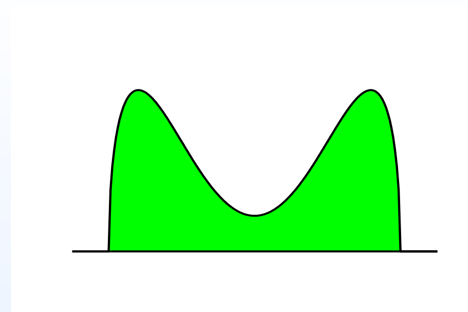
- **Take reference point x^* with $\rho_V(x^*) > 0$ and shift and rescale the eigenvalues** $\lambda \mapsto cn(\lambda - x^*)$ **with $c = \rho_V(x^*)$. Then rescaled kernel tends to the sine kernel,**

$$\lim_{n\to\infty} \frac{1}{cn} K_n\left(x^* + \frac{x}{cn}, x^* + \frac{y}{cn}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

- **This is due to Dyson for GUE. For more general V:** **Pastur, Shcherbina (1997)**
Bleher, Its (1999), Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

Edge universality: Airy kernel

- At **edge points** the mean eigenvalue density typically vanishes vanishes like a square root.



- Different scaling

$$\lambda \mapsto (cn)^{2/3}(\lambda - x^*)$$

$\rho_V(x)$ **vanishes like** $c|x - x^*|^{1/2}$

near edge point x^*

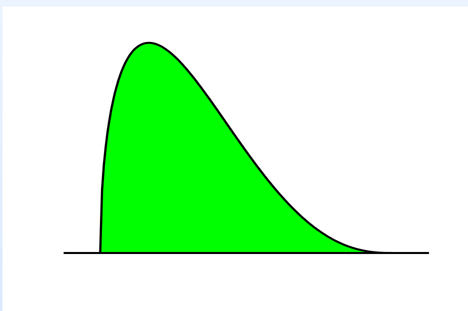
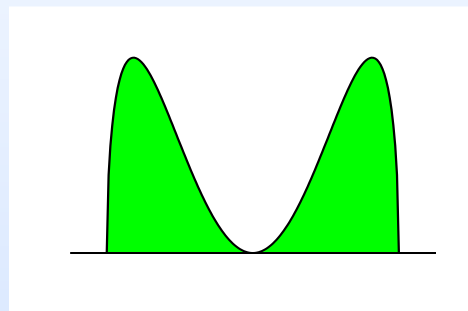
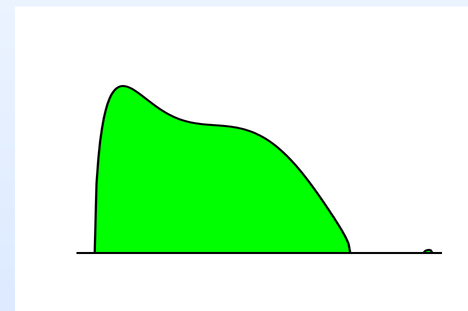
- Rescaled kernels tend (as $n \to \infty$) to the **Airy kernel**

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}$$

  - $\Longrightarrow$ Distribution of largest eigenvalue is a Fredholm determinant with Airy kernel
  - $\Longrightarrow$ Tracy-Widom distribution

Singular cases

- **Singular case I: ρ_V vanishes to higher order at endpoint.**
- **Singular case II: ρ_V vanishes at interior point.**
- **Singular case III: Variational conditions that characterize ρ_V are not strict at some point outside the spectrum.**

- **Singular cases correspond to possible change in number of intervals if parameters in the external field V change.**

Singular cases

- While regular cases are described by classical special functions, singular cases are described by **non-classical special functions**
  - Singular case I $\Longrightarrow$ Painlevé I + hierarchy
  - Singular case II $\Longrightarrow$ Painlevé II + hierarchy
  - Singular case III $\Longrightarrow$??
- Singular cases are considered in double scaling limit

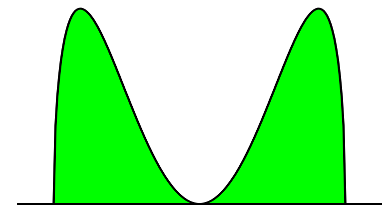
$$\frac{1}{Z_{n,N}} e^{-N \operatorname{Tr} V(M)} dM$$

where $n, N \to \infty, \quad n/N = t \to 1$ such that $n^{\beta}(t-1)$ is constant.

Singular case II: double scaling limit

- **Singular case II**

$$\frac{1}{Z_{n,N}} e^{-N \operatorname{Tr} V(M)} dM$$



and ρ_V vanishes quadratically at x^*.

- **Rescaled kernels**

$$\frac{1}{(c_1 n)^{\gamma}} K_{n,N}\left(x^* + \frac{x}{(c_1 n)^{\gamma}}, x^* + \frac{y}{(c_1 n)^{\gamma}}\right)$$

then have limits as $n, N \to \infty$ with $n^{\beta}(n/N - 1) = c_2 s$.

  - **Scaling exponents $\gamma = 1/3$ and $\beta = 2/3$ are such that expected distance between scaled eigenvalues around x^* becomes $O(1)$**
  - **Constants c_1, c_2 are V-dependent.**
  - **Limiting kernels $K^{crit}(x, y; s)$ are universal: they only depend on s.**

Limiting kernels

THEOREM (critical quartic potential: Bleher, Its (2003); real analytic V: Claeys, AK (2006)):

- Suppose V is real analytic such that ρ_V vanishes quadratically at x^*. Suppose x^* is the only singular point. Then rescaled kernels have limit

$$K^{crit}(x,y;s) = -\frac{\Phi_1(x;s)\Phi_2(y;s) - \Phi_2(x;s)\Phi_1(y;s)}{2\pi i(x-y)}$$

- $\Phi_1(\zeta;s)$ and $\Phi_2(\zeta;s)$ are solutions of a linear ODE

$$\frac{d}{d\zeta}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = \begin{pmatrix}-4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir\\ 4\zeta q - 2ir & 4i\zeta^2 + i(s+2q^2)\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$$

$$\begin{pmatrix}\Phi_1(\zeta)\\ \Phi_2(\zeta)\end{pmatrix} e^{i(\frac{4}{3}\zeta^3 + s\zeta)} = \begin{pmatrix}1\\ 0\end{pmatrix} + O(\zeta^{-1}) \quad \text{as } \zeta \to \infty, \arg\zeta = \pi/2.$$

and $q = q(s)$ satisfies $\boxed{q'' = sq + 2q^3}$ $r = q'(s)$, and $q(s) \sim \mathrm{Ai}(s)$ as $s \to +\infty$.

Painlevé II equation

- **Painlevé II equation** $\boxed{q'' = sq + 2q^3}$ **is the compatibility condition for linear ODEs** **Flaschka, Newell (1980)**

$$\frac{\partial}{\partial \zeta}\Psi = \begin{pmatrix} -4i\zeta^2 - i(s+2q^2) & 4\zeta q + 2ir \\ 4\zeta q - 2ir & 4i\zeta^2 + i(s+2q^2) \end{pmatrix} \Psi$$

$$\frac{\partial}{\partial s}\Psi = \begin{pmatrix} -i\zeta & q \\ q & i\zeta \end{pmatrix} \Psi$$

Extension: Spectral singularity

- **Additional factor $|\det M|^{2\alpha}$ in random matrix ensemble**

$$\frac{1}{Z_{n,N}} |\det M|^{2\alpha} e^{-N \operatorname{Tr} V(M)} dM, \qquad \alpha > -1/2,$$

introduces a singularity at 0.

  - **It does not change the global eigenvalue regime.**
  - **If $\rho_V(0) > 0$, then scaling limit at 0 is a Bessel kernel with Bessel functions with parameters $\alpha \pm \frac{1}{2}$**

$$\pi \sqrt{x}\sqrt{y} \frac{J_{\alpha+\frac{1}{2}}(\pi x) J_{\alpha-\frac{1}{2}}(\pi y) - J_{\alpha-\frac{1}{2}}(\pi x) J_{\alpha+\frac{1}{2}}(\pi y)}{2(x-y)}$$

Akemann, Damgaard, Magnea, Nishigaki (1997)

Kanzieper, Freilikher (1998)

AK, Vanlessen (2003)

Multicritical case

$$\frac{1}{Z_{n,N}} |\det M|^{2\alpha} e^{-N \operatorname{Tr} V(M)} dM, \qquad \alpha > -1/2$$

- If ρ_V vanishes quadratically at 0, then the double scaling limit is associated with a special solution of the general form of the Painlevé II equation **Claeys, AK, Vanlessen (arxiv 2005)**

$$\boxed{q''(s) = sq + 2q^3 - \alpha}$$

  - The special solution is the analogue of the Hastings-McLeod solution. We prove that it has no poles on the real line.

- If 0 is a regular edge point, then connection with a special solution of the Painlevé II equation with different α **Its, AK, Östensson (in preparation)**

$$\boxed{q''(s) = sq + 2q^3 - 2\alpha - 1/2}$$

  - The special solution is not the Hastings-McLeod solution.

Sketch of the proof

- **Starting point is the Riemann-Hilbert problem for orthogonal polynomials**

  Fokas, Its, Kitaev (1992)

  - $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic**
  - $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}$ **for** $x \in \mathbb{R}$
  - $Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$

- **Solution Y contains orthogonal polynomials in the first column, and their Cauchy transforms in the second column.**

- **Correlation kernel is expressed directly in terms of Y:**

$$K_{n,N}(x,y) = \frac{\sqrt{e^{-NV(x)}}\sqrt{e^{-NV(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_\pm^{-1}(y) Y_\pm(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- **Asymptotics of $K_{n,N}$ is obtained via the Deift/Zhou steepest descent analysis of the RH problem**

  Deift, Zhou (1993)

  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

Deift/Zhou steepest descent

- **Sequence of explicit transformations $Y \mapsto T \mapsto S \mapsto R$ leading to RH problem for R on contour Σ_R such that**
  - $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ **is analytic,**
  - $R_+(z) = R_-(z)(I + O(n^{-1/3}))$ **uniformly for $z \in \Sigma_R$,**
  - $R(z) = (I + O(1/z))$ **as $z \to \infty$**
- **Then (in the presence of enough analyticity, which we assume)**

$$R(z) = I + O(n^{-1/3}) \quad \text{uniformly as } n \to \infty$$

- **We follow the effect of the transformations on $K_{n,N}$.**
  - **THEOREM follows from $R(z) \approx I$.**

Transformations $Y \mapsto T \mapsto S$

- First two transformations depend on properties of ρ_V Assume (for simplicity) that
  - ρ_V is supported on one interval $[a, b]$. Then

$$\rho_V(x) = h(x)\sqrt{(b-x)(x-a)}$$

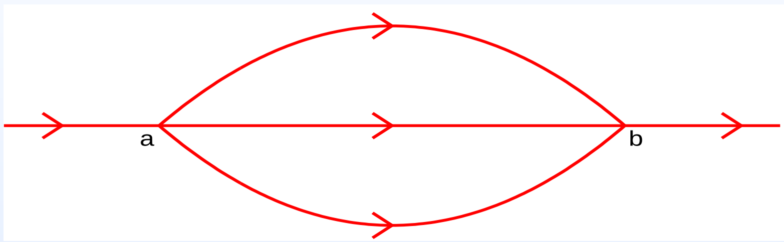
    with h analytic.
  - Regular case. Then $h(x) > 0$ for $x \in [a, b]$.
  - Singular case II. Then $h(x^*) = 0$ and otherwise $h(x) > 0$.
- Jumps for S depend on functions

$$\varphi(z) = \pi \int_b^z h(s)((s-b)(s-a))^{1/2} ds$$

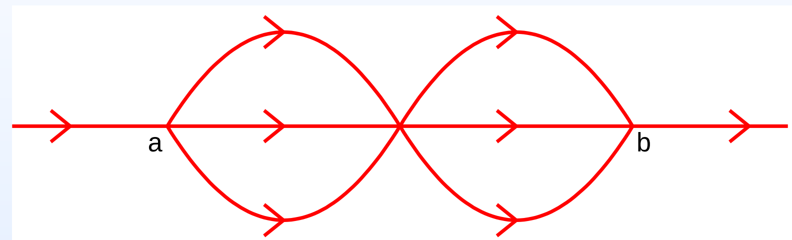
$$\tilde{\varphi}(z) = \pi \int_a^z h(s)((s-b)(s-a))^{1/2} ds$$

RH problem for S

- S satisfies a RH problem that is normalized at infinity and that has jumps on a contour that arises after opening a lens around the interval $[a, b]$



Regular case



Singular case II

- Jumps for S are

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ on } [a,b], \qquad \begin{pmatrix} 1 & 0 \\ e^{2n\varphi} & 1 \end{pmatrix} \text{ on lips of the lenses,}$$

$$\begin{pmatrix} 1 & e^{-2n\varphi} \\ 0 & 1 \end{pmatrix} \text{ on } [b,\infty), \qquad \text{and} \qquad \begin{pmatrix} 1 & e^{-2n\tilde{\varphi}} \\ 0 & 1 \end{pmatrix} \text{ on } (-\infty, a]$$

- All jumps tend to identity matrix as $n \to \infty$, except for jump on $[a, b]$.

Model problem

- **Solve model RH problem for M**
  - M **is analytic on** $\mathbb{C} \setminus [a,b]$
  - $M_+ = M_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **on** $[a,b]$
  - $M(z) = I + O(1/z)$ **as** $z \to \infty$
- **This can be explicitly solved.**
- **We then expect that $S \approx M$ if n gets large.**
  - **Problem: this does not hold near the endpoints a and b, and near interior point x^*.**

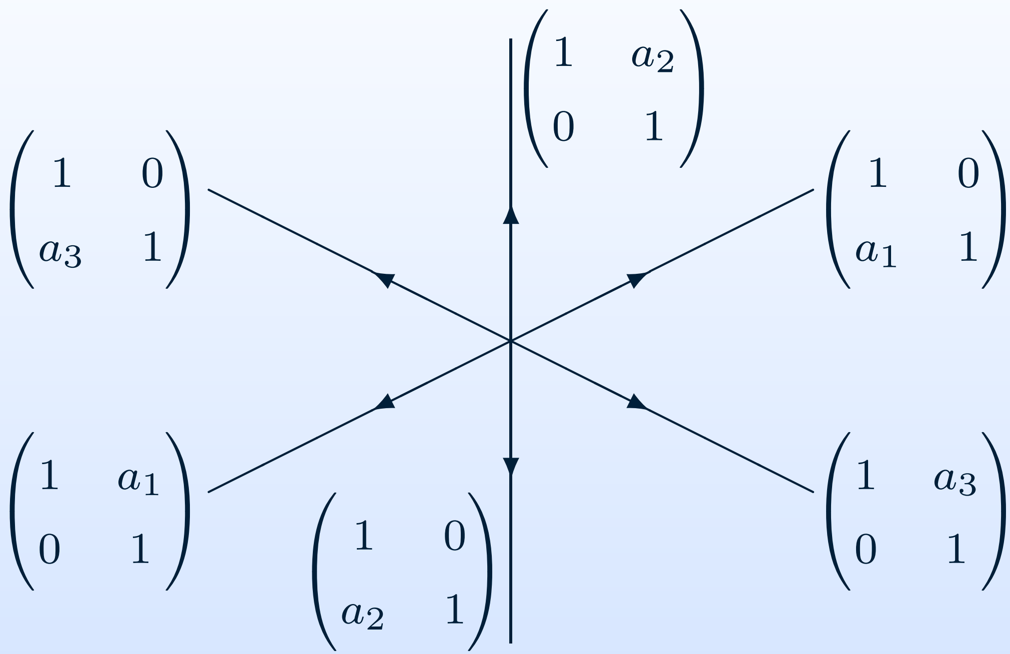
Local parametrices

- Find P in small disks around a, b, and x^* such that
  - P has the same jumps as S has inside the disks
  - $P(z) = M(z)(I + O(1/n^\kappa))$ uniformly for z on the boundary of the disks
- Local parametrix at a and b can be constructed with Airy functions
  - This leads to the Airy kernel
- The main step is the construction of a local parametrix at x^* with the help of Ψ functions for the Hastings-McLeod solution of Painlevé II.

Baik, Deift, Johansson (1999)

Riemann Hilbert problem for Painlevé II (general case)

- Ψ **functions satisfy a RH problem with jump on rays** $\arg\zeta = -\frac{1}{6}\pi + \frac{k}{3}\pi$**,**

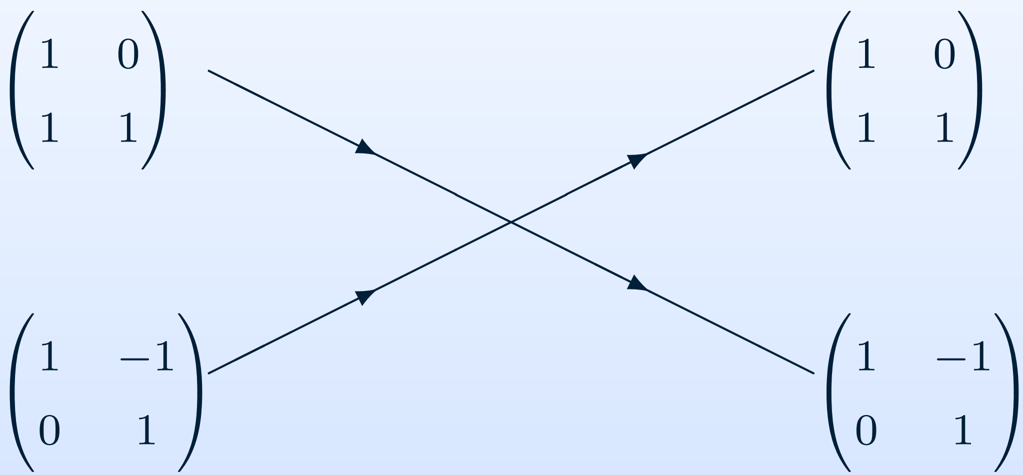


- **Stokes multipliers satisfy** $a_1 + a_2 + a_3 + a_1 a_2 a_3 = 0$**.**
- **Asymptotic behavior as** $\zeta \to \infty$ **in any sector**

$$\Psi(\zeta; s) = \left(I + O\left(\frac{1}{\zeta}\right)\right) \begin{pmatrix} e^{-i(\frac{4}{3}\zeta^3 + s\zeta)} & 0 \\ 0 & e^{i(\frac{4}{3}\zeta^3 + s\zeta)} \end{pmatrix}$$

RH problem for Painlevé II (special case)

- **There is one-to-one correspondence between solutions of Painlevé II and choices of Stokes multipliers.**
- **Hastings-McLeod solution corresponds to** $\boxed{a_1 = 1,\ a_2 = 0,\ a_3 = -1}$

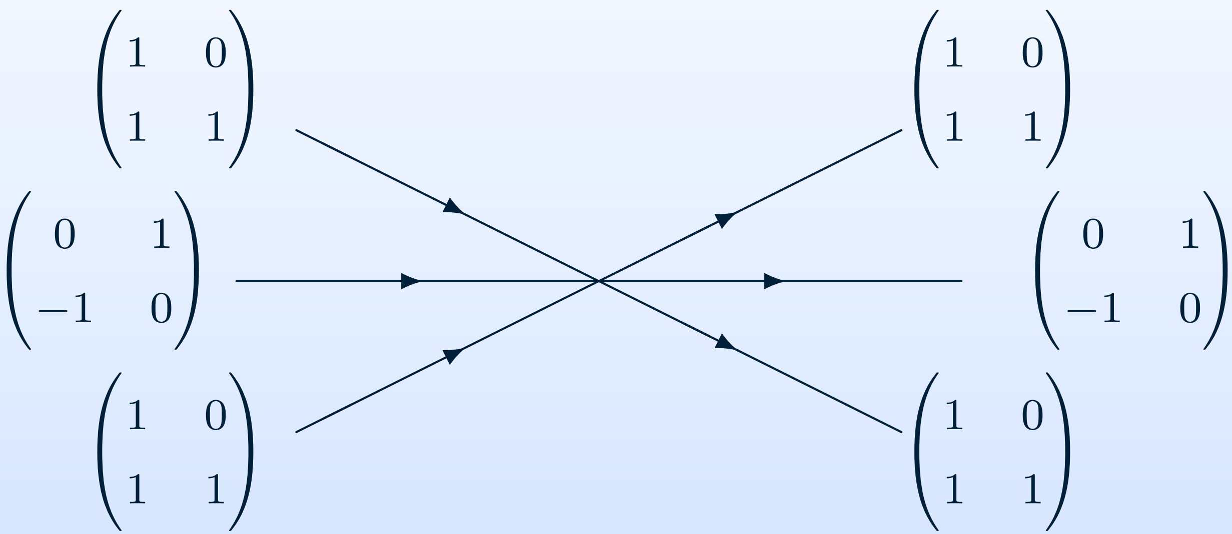


the orientation of the rays $\arg\zeta = \pm 5\pi/6$ is reversed

RH problem for Painlevé II (special case)

- Change Ψ in lower half plane to $\Psi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- This leads to a jump on $\mathbb{R}$ and changes the jumps in the lower half-plane.



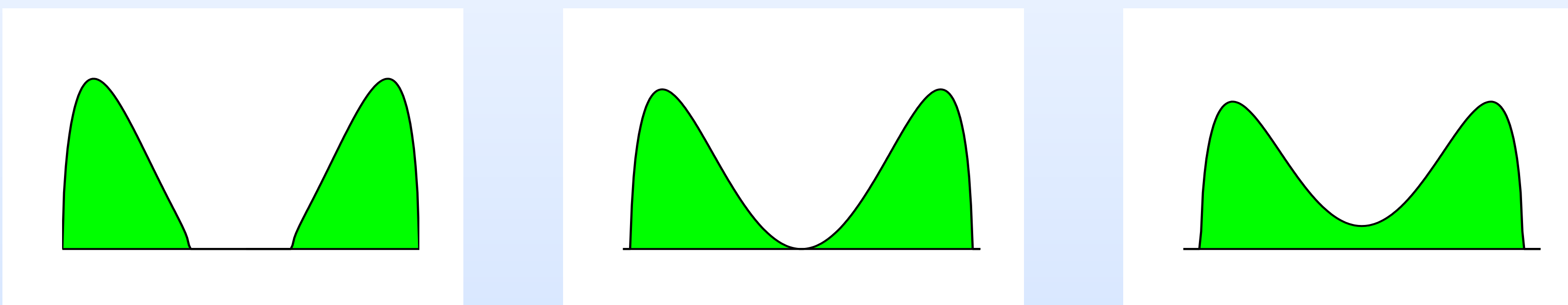
- We use a conformal map $\zeta = f(z)$ to map a neighborhood of x^* to the ζ-plane. The contours for the RH problem for S are mapped to the above half-rays. Main ingredient in the construction of the local parametrix is

$$\Psi(n^{1/3} f(z); s)$$

Modifications

- **All this works only if $n \equiv N$ (single scaling limit) so that $t \equiv 1$ and $s = 0$.**
  - **In double scaling limit we have $t \neq 1$ (but $t = n/N \to 1$) and it is more natural to work with ρ_{V_t} where $V_t = \frac{1}{t}V$ instead of ρ_V.**

    Baik, Deift, Johansson
  - **Density ρ_{V_t} as a function of $t = n/N$:**

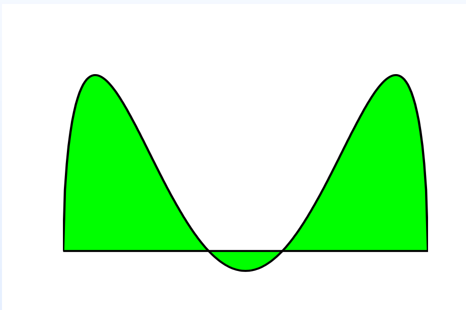


$t < 1$ $\qquad$ $t = 1$ $\qquad$ $t > 1$

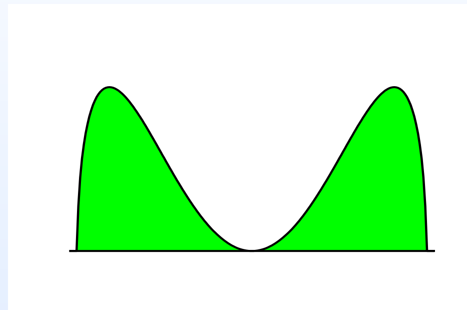
- **We introduce two modifications in the construction of Baik et al.**

Modification 1

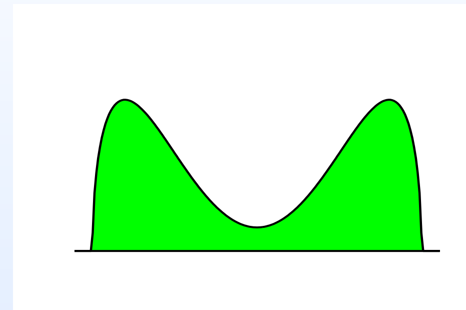
- **A modification of the density simplifies the arguments for $t < 1$. We do not impose non-negativity near x^***



$t < 1$



$t = 1$



$t > 1$

Modification 2

- Local parametrix P should have same jumps in a fixed neighborhood of x^* and match with outside parametrix M.
- If we use $\Psi(n^{1/3}f(z); s)$ for a certain conformal map f, we get correct jumps but not the matching.
  - Baik et al. change f, which leads to matching but has the correct jumps only approximately.
- We get both matching and correct jumps exactly by letting s depend on z

$$\Psi\left(n^{1/3}f(z); n^{2/3}s_t(z)\right)$$

- Explicit formulas exist for f and s_t.
  - If $n \to \infty$ and $z \to x^*$, then $n^{2/3}s_t(z) \to s$.

Third transformation

- **The final transformation**

$$R(z) = \begin{cases} S(z)P(z)^{-1} & \textbf{inside disks} \\ S(z)M(z)^{-1} & \textbf{outside disks} \end{cases}$$

leads to R which solves a RH problem on contour Σ_R such that

  - $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ **is analytic,**
  - $R_+(z) = R_-(z)(I + O(n^{-1/3}))$ **uniformly for $z \in \Sigma_R$,**
  - $R(z) = (I + O(1/z))$ **as $z \to \infty$**

- **Then $R(z) = I + O(n^{-1/3})$ as $n \to \infty$, uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.**