Matrix integrals as isomonodromic tau functions

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Introduction.

There are many examples of solutions to isomonodromic and KP-type equations appearing as matrix integrals:

- Partition functions
- Orthogonal polynomials
- Gap probabilities / Fredholm determinants
- Expectation values of spectral invariants
- Spectral correlation functions

Question: Is there an explanation of all these

Sub specie aeternitatis?

$$\begin{bmatrix}\text{Matrix model}\\ \text{integrals}\end{bmatrix} \hookrightarrow \begin{bmatrix}\text{Isomonodromic}\\ \tau - \text{functions}\end{bmatrix} \hookrightarrow \begin{bmatrix}\text{M} - \text{KP}\\ \tau - \text{functions}\end{bmatrix}$$

Summary and background.

The inclusion $\tau_{\mathrm{isomon}} \hookrightarrow$ multi–KP is not yet understood in general. But in many known cases, the isomonodromic systems can be deduced as *multi-scaling* reductions.

The identification $1-$ Matrix integrals $\sim \tau_{KP}$ is well understood since long (Krachev, Marshakov, Mironov, Orlov, Zabrodin (1991)). The case of 2-Matrix integrals $\sim \tau_{2-\mathrm{Toda}}$ was explained more recently (Adler, Van Moerbeke (1999), Harnad, Orlov (2002)).

The identification of 1-Matrix integrals $\sim \tau_{\mathrm{isomon}}$ was understood long ago in special cases (Moore (1990), Fokas, Its, Kitaev (1992), Harnad, Tracy, Widom (1993)). By now it is quite well understood, for finite N, in general for all "semi-classical models" (Bertola, Eynard, Harnad (2003, 2006)). For the case of 2-Matrix integrals (for finite N), the isomonodromic system is understood for polynomial potentials (BEH1(2002), BEH2(2003)). integral representations of the fundamental system and a Riemann-Hilbert chcraterization are known (BEH2, Mclaughlin & Ercolani, Kuijlaars & McLaughlin), but the identification of $2-$Matrix integrals $\sim \tau_{\mathrm{isomon}}$ remains to be shown. (This requires an extension of the definition of τ_{isomon} for highly irregular singularities (Bertola, Mo (2006).)

KP τ-functions (Sato-Segal-Wilson):

Segal-Wilson Grassmannian

$$\mathcal{H} := L^2(S^1) = \mathcal{H}_+ + \mathcal{H}_-$$

$$\mathcal{H}_- = \text{span}\{z^i\}_{i\geq 0}, \quad \mathcal{H}_: = \text{span}\{z^{-i}\}_{i>0}, \quad z \in S^1$$

$$w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \quad \mathbf{t} := (t_1, t_2, \ldots)$$

Homogeneous coordinates

$$w = \left[\begin{pmatrix} w_+ \\ w_- \end{pmatrix}\right] = \left[\begin{pmatrix} w_+ h \\ w_- h \end{pmatrix}\right], \quad \forall h \in GL(\mathcal{H}_+)$$

The infinite abelian (flow) group:

$$\Gamma_+ := \{\gamma(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i z^i}\}$$

acts linearly on $\mathcal{H}$, $\text{Gr}_{\mathcal{H}_+}(\mathcal{H})$.

$$\Gamma_+ : \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \rightarrow \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

$$\gamma(\mathbf{t}) : w \mapsto \gamma(\mathbf{t})w := w(\mathbf{t}) = \left[\begin{pmatrix} w_+(\mathbf{t}) \\ w_-(\mathbf{t}) \end{pmatrix}\right]$$

Dual determinantal line bundle

$$\begin{array}{c} Det^* \\ \downarrow \\ \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \end{array}$$

Holomorphic (square integrable) sections

$$H^0(\text{Gr}_{\mathcal{H}_+}(\mathcal{H}), Det^*) \equiv \mathcal{F}^*$$

where $\mathcal{F}$ is the *Fermionic Fock space*:

$$\mathcal{F} := \Lambda \mathcal{H}$$

Vacuum vector:

$$|0 >:= z^0 \wedge z^1 \wedge \dots$$

Free Fermi creation and annihilation operators:

$$f_i := \iota_{z^i}, \quad \bar{f}_i := z^i \wedge$$

Plücker embedding

$$\mathfrak{P} : \mathrm{Gr}_{\mathcal{H}_+(\mathcal{H})} \hookrightarrow P(\mathcal{F})$$

$$\mathfrak{P} : \mathrm{span}(v_1, v_2, v_3, \dots) \mapsto [v_1 \wedge v_2 \wedge \dots]$$

Plücker coordinates

For each partition

$$\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq 0, \dots 0, \dots$$

there is a Plücker coordinate

$$\pi_\lambda(w) := \det(w_\lambda)$$

(where w_λ is the semi-∞ block spanned by $\{z^{i-\lambda_i+1}\}_{i\in\mathbb{N}}$

Then the KP τ-function associated to $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is

$$\tau_w(\mathbf{t}) = \pi_0(\mathfrak{P}(w(\mathbf{t}) = \det(w_+(\mathbf{t})$$

The other Plücker coordinates are:

$$\pi_\lambda = s_\lambda\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots\right)\tau_w(\mathbf{t})$$

($s_\lambda =$ Schur function) and the Hirota bilinear relations are just the Plücker relations.

The $GL(\mathcal{H})$ action on $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$

$$GL(\mathcal{H}) : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \rightarrow \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

$$g : w \mapsto gw$$

lifts to an action on $\mathcal{F}$

$$GL(\mathcal{H}) : \mathcal{F} \rightarrow \mathcal{F}$$

$$g : v \mapsto \exp \sum_{i,j \in \mathbb{Z}} \xi_{ij} f_i \bar{f}_j$$

$$g = \exp \xi \in GL(\mathcal{H})$$

The Plücker map intertwines the $GL(\mathcal{H})$ action:

$$\begin{array}{ccc} \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) & \overset{\mathfrak{P}}{\mapsto} & P\mathcal{F} \\ {\scriptstyle Gl(\mathcal{H})} \downarrow & & {\scriptstyle Gl(\mathcal{H})} \downarrow \\ \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) & \overset{\mathfrak{P}}{\mapsto} & P\mathcal{F} \end{array}$$

The image of $\mathcal{H}_+$ is the (projectivized) vacuum

$$\mathfrak{P}(\mathcal{H}_+) = [|0>]$$

The τ function then becomes (up to projective equivalence)

$$\tau_w(\mathbf{t}) = < 0|e^{\sum_{i=1}^{\infty} t_j H_j} g|0 >$$

where

$$H_j := \sum_{i \in \mathbb{Z}} f_i \bar{f}_{i+j}, \quad w = g(\mathcal{H}_+)$$

More generally, define the *charge-n vacuum*

$$|n>:= f_{n-1}f_{n-2}\cdots f_0|0>$$

and change the splitting

$$\mathcal{H} := \mathcal{H}_{n+} + \mathcal{H}_-^n$$
$$\mathcal{H}_{n+} := \mathrm{span}(z^{i+n})_{i\geq 0}, \quad \mathcal{H}_-^n := \mathrm{span}(z^{i+n})_{i<0}$$

In homogeneous coordinates, $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ is expressed

$$w = \left[\begin{pmatrix} w_{n+} \\ w_-^n \end{pmatrix}\right]$$

Then

$$\mathfrak{P}(\mathcal{H}_{n+}) = [|n>]$$

and, for $w = g(\mathcal{H}_+)$, we can define the τ-function as

$$\begin{aligned}\tau_{n,w}(\mathbf{t}) &= \det(w_{n+}(\mathbf{t})) \\ &= < n|e^{\sum_{i=1}^{\infty} t_j H_j} g|n>\end{aligned}$$

2. Matrix model integrals as KP τ-functions

2.1 Unitary invariant Matrix model

Let $d\mu$ be a measure supported on a curve $\mathcal{C}$ (e.g., the real axis, or a segment, or union of segments in the complex plane).

The integral

$$\mathbf{Z}_n(\mathbf{t}) := \int_{\mathcal{C}} d\mu(x_1) \cdots \int_{\mathcal{C}} d\mu(x_n) \Delta^2(\mathbf{x}) e^{\sum_j^\infty \sum_{a=1}^n t_j x_a^j}$$

is the type of integral obtained from $U(n)$-invariant matrix integrals, after reduction to the space of eigenvalues.

Let $\{p_j(x)\}_{j=0,1,\ldots}$ be the associated sequence of monic orthogonal polynomials

$$\int_{\mathcal{C}} p_j(x) p_k(x) d\mu(x) = h_j \delta_{jk}$$

Let

$$w_{n,\mu} := \text{span}\{(p_{j+n}(x)\}_{j\in\mathbf{N}} \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

Then:

$$\frac{\mathbf{Z}_n(\mathbf{t})}{\mathbf{Z}_n(\mathbf{0})} = \tau_{n,w_{n,\mu}}(\mathbf{t})$$

2.2 Two-matrix model integrals as 2-Toda τ-functions

A similar construction gives 2-Toda τ-functions in terms of 2-component fermions:

$$f_j^{\alpha)} := f_{2j+\alpha-1}, \quad \bar{f}_j^{\alpha)} := \bar{f}_{2j+\alpha-1}, \quad \alpha = 1, 2$$

Define new *charged vacua* (for $n, m \geq 0$)

$$|n, -m >:= \bar{f}^{(2)}_{-m} \cdots \bar{f}^{(2)}_{-1} f^{(2)}_{n-1} \cdots f^{(2)}_0 |0>$$

Let

$$d\mu_{\alpha\beta}(x, y), \quad \alpha, \beta = 1, 2$$

be a 2×2 matrix of bimeasures supported on a sum of products of curve segments Σ in $\mathbf{C} \times \mathbf{C}$.

Define

$$g := e^{\mathcal{A}}, \quad \mathcal{A} : \sum_{\alpha,\beta} \int \int_{\Sigma} d\mu_{\alpha\beta}(x, y) : f^{\alpha}(x) \bar{f}^{(\beta)}(y) :$$

where the free Fermi fields (generators of Clifford algebra) are:

$$f^{(\alpha)}(x) := \sum_{i \in \mathbb{Z}} f_i^{(\alpha} x^i, \quad \bar{f}^{(\alpha)}(x) := \sum_{i \in \mathbb{Z}} \bar{f}_i^{(\alpha} x^{-i-1},$$

and

$$: f^{(\alpha)} \bar{f}^{(\beta)} := f^{(\alpha)} \bar{f}^{(\beta)} - < 0, 0| f(^{\alpha)} \bar{f}^{(\beta)} |0, 0>$$

Define

$$H_k^{(\alpha)} := \sum_{n=-\infty}^{+\infty} f_n^{(\alpha)} \bar{f}^{(\alpha)}_{n+k}\ , \quad k \neq 0, \quad \alpha = 1, 2.$$

These give two infinite linear commutative subspaces quadratic operators

$$H(\mathbf{t}) = \sum_{k=1}^{\infty} H_k^{(1)} t_k^{(1)} - \sum_{k=1}^{\infty} H_k^{(2)} t_k^{(2)}$$
$$\bar{H}(\bar{\mathbf{t}}) = \sum_{k=1}^{\infty} H_{-k}^{(2)} \bar{t}_k^{(2)} - \sum_{k=1}^{\infty} H_{-k}^{(1)} \bar{t}_k^{(1)}$$

Then (Date et al (1983), Takasaki, Ueno (1984))

$$\tau_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \langle N+n, -N-m | e^{H(\mathbf{t})} g e^{\bar{H}(\bar{\mathbf{t}})} | n, -m \rangle$$

for any $N, n, m \in \mathbf{Z}$ is a τ function of the *two-component Toda Lattice (TL)* hierarchy (or, equivalently, the *coupled two-component KP hierarchy*).

Theorem [HO2]

Choose: $d\mu_{11} = d\mu_{11} = d\mu_{11} = 0, d\mu_{12} := d\mu$, then

$$\tau_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \frac{1}{N!}(-1)^{\frac{1}{2}N(N+1)+mN} c(\mathbf{t}, \bar{\mathbf{t}}) Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$$

where $c(\mathbf{t}, \bar{\mathbf{t}}) := e^{-\sum_{\alpha=1}^{2}\sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)}}$,

$$Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) := \prod_{k=1}^{N} \int_{\Sigma} d\mu(x_k, y_k | \mathbf{t}, n, m, \bar{\mathbf{t}})) \Delta_N(x) \Delta_N(y)$$

$$d\mu(x, y | \mathbf{t}, n, m, \bar{\mathbf{t}}) := x^n y^m e^{V(x, \mathbf{t}^{(1)}) + V(y, \mathbf{t}^{(2)})} \times e^{V(x^{-1}, \bar{\mathbf{t}}^{(1)}) + V(y^{-1}, \bar{\mathbf{t}}^{(2)})} d\mu(x, y)$$

$$V(x, \mathbf{t}^{(\alpha)}) = \sum_{k=1}^{\infty} t_k^{(\alpha)} x^k, \quad V(x^{-1}, \bar{\mathbf{t}}^{(\alpha)}) = \sum_{k=1}^{\infty} \bar{t}_k^{(\alpha)} x^{-k}$$

3. Semiclassical orthogonal polynomials, matrix models, isomonodromic tau functions (BEH3, BEH4)

Define a class of **semiclassical** measures that includes all reduced matrix integrals (for unitary invariant ensembles) (*Partition functions, gap probabilities, orthogonal polynomials, rational spectral correlators*):

$$\mu(x) = e^{-V(x))}, \quad V(x) := \sum_{r=0}^{K} T_r(x)$$

$$T_0(x) := t_{0,0} + \sum_{J=1}^{d_0} \frac{t_{0,J}}{J} x^J$$

$$T_r(x) := \sum_{J=1}^{d_r} \frac{t_{r,J}}{J(x-c_r)^J} - t_{r,0}\ln(x-c_r)$$

Define sectors for each pole and permissible contours:

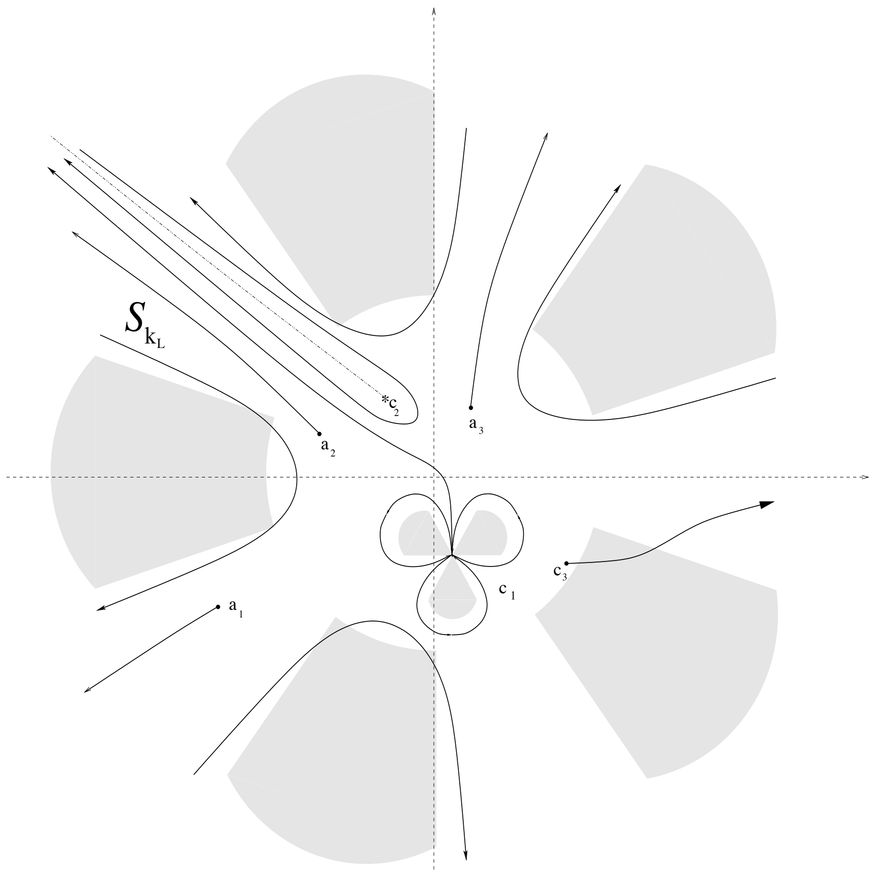
$$S_k^{(0)} := \{x \in \mathbb{C};\ \frac{2k\pi - \arg(t_{0,d_0}) - \frac{\pi}{2}}{d_0} < \arg(x) < \frac{2k\pi - \arg(t_{0,d_0}) + \frac{\pi}{2}}{d_0}\}$$

$$k = 0 \ldots d_0 - 1\ ;$$

$$S_k^{(r)} := \{x \in \mathbb{C};\ \frac{2k\pi + \arg(t_{r,d_r}) - \frac{\pi}{2}}{d_r} < \arg(x-c_r) < \frac{2k\pi + \arg(t_{r,d_r}) + \frac{\pi}{2}}{d_r}\},$$

$$k = 0, \ldots, d_r - 1, \quad r = 1, \ldots, K$$

These sectors are defined in such a way that approaching any of the essential singularities of $\mu(x)$ (i.e. a c_r such that $d_r > 0$) within them, the function $\mu(x)$ tends to zero faster than any power.



Define the orthogonality contour:

$$\int_{\kappa} := \sum_{j=1}^{L} \kappa_j \int_{m_j} + \sum_{j=1}^{S} \kappa_{L+j} \int_{\sigma_j} \quad .$$

and corresponding **monic generalized orthogonal polynomials** $\{p_n(x)\}$, satisfy:

$$\int_{\kappa} p_n(x) p_m(x) \mu(x) dx = h_n \delta_{nm}$$

Define also, **normalized orthogonal polynomials**:

$$\pi_n(x) := \frac{1}{\sqrt{h_n}} p_n(x)$$

and the Cauchy transforms

$$\phi_n(x) := \mathrm{e}^{V(x)} \int_\kappa \frac{\mathrm{e}^{-V(z)}\pi_n(z)}{x-z} dz.$$

These satisfy the usual 3-term recursion relations:

$$x\pi_j(x) = \gamma_{j+1}\pi_{j+1}(x) + \beta_j \pi_j(x) + \gamma_j \pi_{j-1}(x) \ .$$

Define the semi-infinite recursion matrix Q with components

$$Q_{ij} = \gamma_j \delta_{i,j-1} + \beta_i \delta_{ij} + \gamma_i \delta_{i,j+1}, \quad i, j \in \mathbb{N} \ ,$$

And the semi-infinite "wave vector"

$$\Pi(x) := [\pi_0(x), \pi_1(x), \ldots, \pi_n(x), \ldots]^t$$

This may then be expressed as

$$x\Pi(x) = Q\Pi(x)$$

This also satisfies a system of differential-deformation equations:

$$\begin{aligned} \partial_x \Pi(x) &= P\Pi(x) \\ \partial_{a_i} \Pi(x) &= A_i \Pi(x) \ , \qquad i = 1, \ldots, L \\ \partial_{c_r} \Pi(x) &= C_r \Pi(x) \ , \qquad r = 1, \ldots, K \\ \partial_{t_{r,J}} \Pi(x) &= T_{r,J} \Pi(x) \ , \qquad r = 0, \ldots, K, \ J = 0, \ldots, d_r \end{aligned}$$

where the matrices P, A_i, C_r and $T_{r,J}$ are all lower semi-triangular (with P strictly lower triangular) , and given by

$$
\begin{aligned}
P &= V'(Q)_{-0} - \sum_{i=1}^{L} A_i = V'(Q)_{-} - \sum_{i=1}^{L} (A_i)_{-} \\
A_i &= \kappa_i (\Pi(a_i)\Pi^t(a_i))_{-0} \\
C_r &= -\sum_{J=0}^{d_r} t_{r,J} (Q - c_r)_{-0}^{-J-1} \ , \quad r = 1, \ldots, K \\
T_{0,0} &= \frac{1}{2}\mathbf{I}, \quad T_{0,J} = \frac{1}{J} Q_{-0}^{J} \ , \quad J = 1, \ldots, d_0 \\
T_{r,J} &= \frac{1}{J}(Q - c_r)_{-0}^{-J}, \quad r = 1, \ldots, K, \quad J = 1, \ldots, d_r \\
T_{r,0} &= -\ln(Q - c_r)_{-0}, \quad r = 1, \ldots, K
\end{aligned}
$$

where $(Q - c_r)^{-J}$ and $\ln(Q - c_r)$ are defined by the formulæ

$$
\begin{aligned}
(Q - c_r)_{nm}^{-J} &:= \int_{\kappa} \frac{\pi_n(z)\pi_m(z)}{(z - c_r)^J} \mu(z) dz \\
\ln(Q - c_r)_{nm} &:= \int_{\kappa} \ln(z - c_r) \pi_n(x) \pi_m(z) \mu(z) dz
\end{aligned}
$$

The diagonal matrix elements of the above are given by

$$
X_{jj} = -\frac{1}{2}\partial(\ln h_j) \ ,
$$

where $\partial = \partial_x$, ∂_{a_i}, ∂_{c_r} and $\partial_{t_{r,J}}$ for $X = P$, A_i, C_r and $T_{r,J}$. and

$$
V'(Q)_{jj} = \sum_{i=1}^{L} \kappa_i \psi_j(a_i)^2
$$

Now define the 2×2 matrix

$$\Gamma_n(x) := \begin{pmatrix} \pi_{n-1}(x) & \phi_{n-1}(x) \\ \pi_n(x) & \phi_n(x) \end{pmatrix}$$

Using the recursion relations, we can "fold" all the above relations into a sequence of 2×2 "windows", giving:

Proposition [BEH2] The folded forms of the recursion and differential relations for Γ_n are

$$\begin{aligned}
\Gamma_{n+1}(x) &= R_n(x)\Gamma_n(x) \ , \quad n \geq 1, \\
\partial_x \Gamma_n(x) &= \mathcal{D}_n(x)\Gamma_n(x) \\
R_n &:= \begin{pmatrix} 0 & 1 \\ -\frac{\gamma_n}{\gamma_{n+1}} & \frac{x-\beta_n}{\gamma_{n+1}} \end{pmatrix} \\
\mathcal{D}_n(x) &= \mathcal{D}_n^{(0)}(x) \\
&\quad + \sum_{i=1}^{L} \frac{\kappa_i \gamma_n}{x - a_i} \begin{pmatrix} \psi_{n-1}(a_i)\psi_n(a_i) & -\psi_{n-1}^2(a_i) \\ \psi_n^2(a_i) & -\psi_{n-1}(a_i)\psi_n(a_i) \end{pmatrix} \\
\mathcal{D}_n^{(0)}(x) &= \begin{pmatrix} V'(x) & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \gamma_n \begin{pmatrix} \left(\nabla_Q V'(x)\right)_{n-1,n} & -\left(\nabla_Q V'(x)\right)_{n-1,n-1} \\ \left(\nabla_Q V'(x)\right)_{nn} & -\left(\nabla_Q V'(x)\right)_{n,n-1} \end{pmatrix}
\end{aligned}$$

where

$$(\nabla_Q v(x))_{nm} := \int_\kappa \frac{v(x) - v(z)}{x - z} \pi_n(z)\pi_m(z) e^{-V(z)} dz$$

Proposition [BEH2] The deformation equations are equivalent to the infinite sequence of 2×2 equations

$$\delta_v \Gamma_n(x) = \mathcal{V}_n(x)\Gamma_n(x)$$

where the folded matrix of the deformation is defined by

$$\mathcal{V}_n(x) = \begin{pmatrix} v(x) - \frac{1}{2}v(Q)_{n-1,n-1} & 0 \\ 0 & \frac{1}{2}v(Q)_{nn} \end{pmatrix} + \gamma_n \begin{pmatrix} \nabla_Q v(x)_{n-1,n} & -\nabla_Q v(x)_{n-1,n-1} \\ \nabla_Q v(x)_{nn} & -\nabla_Q v(x)_{n,n-1} \end{pmatrix}$$

For the deformations this gives the following equations corresponding to changes in the potential.

$$\partial_{c_r}\Gamma_n(x) = \mathcal{C}_{r;n}(x)\Gamma_n(x)$$

$$\partial_{t_{r,J}}\Gamma_n(x) = \mathcal{T}_{r,J;n}(x)\Gamma_n(x)$$

where the 2×2 matrices $\mathcal{C}_{r;n}$ and $\mathcal{T}_{r,J;n}(x)$ are rational in x, with poles at the points $\{c_r\}$, obtained by making the following substitutions

$$C_r : \ v(x) \to -\sum_{J=0}^{d_r} t_{r,J}(x - c_r)^{-J-1}$$

$$T_{r,J} : \ v(x) \to \frac{1}{J}(x - c_r)^{-J}$$

$$T_{0,J} : \ v(x) \to \frac{1}{J}x^J$$

$$T_{r,0} : \ v(x) \to -\ln(x - c_r)$$

Endpoint deformations

$$\partial_{a_i}\Gamma_n(x) = \mathcal{A}_{i,n}(x)\Gamma_n(x)$$

where

$$\mathcal{A}_{i,n} := \frac{\kappa_i\gamma_n}{a_i - x}\begin{pmatrix} \psi_{n-1}(a_i)\psi_n(a_i) & -\psi_{n-1}^2(a_i) \\ \psi_n^2(a_i) & -\psi_{n-1}(a_i)\psi_n(a_i) \end{pmatrix} + \frac{\kappa_i}{2}\begin{pmatrix} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{pmatrix}$$

Isomonodromic deformations

This gives a compatible system, showing that the (generalized) monodromy) data of the rational covariant derivative operator:

$$\partial_x - \mathcal{D}_n(x)$$

are independent of all deformation parameters $\{c_r, a_i, t_{r,J}, n\}$.

Traceless gauge

Let

$$\Psi_n(x) := \mathrm{e}^{-\frac{1}{2}V(x)}\Gamma_n(x) = \begin{pmatrix} \psi_{n-1}(x) & \widetilde{\psi}_{n-1}(x) \\ \psi_n(x) & \widetilde{\psi}_n(x) \end{pmatrix}$$

$$\Psi_n'(x) = \mathcal{A}_n(x)\Psi_n(x)$$

$$\mathcal{A}_n(x) = \mathcal{D}_n(x) - \frac{1}{2}V'(x)\mathbf{I}$$

where

$$\widetilde{\psi}_n := e^{-\frac{1}{2}V(x)}\psi_n$$

Local asymptotics near singular points $\{c_r\}$

$$\Psi(x) \sim C_r Y_r(x)\ \exp\left(\frac{1}{2}T_r(x) + \delta_{r0}\left(n + \frac{1}{2}\sum_{r\geq 1} t_{r,0}\right)\ln(x)\right)\sigma_3$$

where

$$Y_0(x) := \mathbf{I} + \sum_{k=1}^{\infty}\frac{Y_{0;k}}{x^k},$$

$$Y_r(x) := \mathbf{I} + \sum_{k=1}^{\infty} Y_{r;k}(x-c_r)^k$$

$$C_0 = \begin{pmatrix} 0 & \sqrt{h_{n-1}} \\ \frac{1}{\sqrt{h_n}} & 0 \end{pmatrix}$$

$$C_r = \begin{pmatrix} \pi_{n-1}(c_r)\mathrm{e}^{-\frac{\check{V}_r(c_r)}{2}} & (c_r - Q)^{-1}_{n-1,0}\sqrt{h_0}e^{\frac{\check{V}_r(c_r)}{2}}, \\ \pi_n(c_r)\mathrm{e}^{-\frac{\check{V}_r(c_r)}{2}} & (c_r - Q)^{-1}_{n,0}\sqrt{h_0}e^{\frac{\check{V}_r(c_r)}{2}} \end{pmatrix}$$

where

$$\check{V}_r(x) := V(x) - T_r(x)$$

Local asymptotics near endpoints $\{a_j\}$

$$\Psi(x) \sim A_j \cdot Y_j(x) \cdot \exp\left[-\kappa_j \ln(x-a_j)\sigma_+\right], \quad \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A_j \begin{pmatrix} \pi_{n-1}(a_j)\mathrm{e}^{-\frac{V(a_j)}{2}} & e^{\frac{V(a_j)}{2}}\int_\kappa dz \frac{e^{-\frac{1}{V(z)}}(\pi_{n-1}(z)-\pi_{n-1}(a_j))}{a_j - z} \\ \pi_n(a_j)\mathrm{e}^{-\frac{V(a_j)}{2}} & \mathrm{e}^{\frac{V(a_j)}{2}}\int_\kappa dz \frac{e^{\frac{-1}{V(z)}}(\pi_n(z)-\pi_n(a_j))}{a_j - z} \end{pmatrix}$$

Isomonodromic τ-function

$$
\begin{aligned}
d\ln\tau_n^{IM} = & -\frac{1}{2}\sum_{r=0} \operatorname*{res}_{x=c_r} dT_r(x)\operatorname{tr}\left(Y_r^{-1}Y_r'\sigma_3\right) \\
& + \sum_j \operatorname*{res}_{x=a_j} \frac{\kappa_j da_j}{x-a_j}\operatorname{tr}\left(Y_j^{-1}Y_j'\sigma_+\right)
\end{aligned}
$$

Theorem [BEH2] Up to multiplicative terms that are independent of the isomonodoromic deformation parameters $\{t_{r,J}, c_r, a_j, n$ the partition function

$$
\begin{aligned}
\mathbf{Z}_n(\{t_{r,J}, c_r, a_j\}|[\kappa]) &:= C_n \int_{spec(M)\in\kappa} dM e^{-\operatorname{tr}V(M)} \\
&= \int_\kappa dx_1 \cdots \int_\kappa dx_n \, \Delta(\underline{x})^2 \mathrm{e}^{-\sum_{j=1}^n V(x_j)} \\
&= n! \prod_{j=0}^{n-1} h_j
\end{aligned}
$$

of the generalized random matrix model and the isomonodromic tau function τ_n^{IM} for the associated ODE are related by

$$
\mathbf{Z}_n = \tau_n^{IM}\mathcal{F}_n,
$$

where

$$
\ln\left(\frac{\mathcal{F}_n}{n!}\right) = \frac{1}{2\hbar^2}\sum_{0\le q<r\le K} \operatorname*{res}_{x=c_r} T_r'(x)T_q(x)
$$

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