Matrix integrals as isomonodromic tau functions

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Introduction.

There are many examples of solutions to isomonodromic and KP-type equations appearing as matrix integrals:

- Partition functions
- Orthogonal polynomials
- Gap probabilities / Fredholm determinants
- Expectation values of spectral invariants
- Spectral correlation functions

Question: Is there an explanation of all these

Sub specie aeternitatis?

\[
\begin{bmatrix}
\text{Matrix model} \\
\text{integrals}
\end{bmatrix} \leftrightarrow \begin{bmatrix}
\text{Isomonodromic} \\
\tau - \text{functions}
\end{bmatrix} \leftrightarrow \begin{bmatrix}
\text{M} - \text{KP} \\
\tau - \text{functions}
\end{bmatrix}
\]
Summary and background.

The inclusion $\tau_{\text{isomon}} \hookrightarrow \text{multi-KP}$ is not yet understood in general. But in many known cases, the isomonodromic systems can be deduced as *multi-scaling* reductions.

The identification $1 - \text{Matrix integrals} \sim \tau_{KP}$ is well understood since long (Krachev, Marshakov, Mironov, Orlov, Zabrodin (1991)). The case of 2-Matrix integrals $\sim \tau_{2-\text{Toda}}$ was explained more recently (Adler, Van Moerbeke (1999), Harnad, Orlov (2002)).

The identification of 1-Matrix integrals $\sim \tau_{\text{isomon}}$ was understood long ago in special cases (Moore (1990), Fokas, Its, Kitaev (1992), Harnad, Tracy, Widom (1993)). By now it is quite well understood, for finite $N$, in general for all “semi-classical models” (Bertola, Eynard, Harnad (2003, 2006)). For the case of 2-Matrix integrals (for finite $N$), the isomonodromic system is understood for polynomial potentials (BEH1(2002), BEH2(2003)). Integral representations of the fundamental system and a Riemann-Hilbert characterization are known (BEH2, McLaughlin & Ercolani, Kuijlaars & McLaughlin), but the identification of 2-Matrix integrals $\sim \tau_{\text{isomon}}$ remains to be shown. (This requires an extension of the definition of $\tau_{\text{isomon}}$ for highly irregular singularities (Bertola, Mo (2006)).)
**KP τ-functions** (Sato-Segal-Wilson):

Segal-Wilson Grassmannian

\[ \mathcal{H} := L^2(S^1) = \mathcal{H}_+ + \mathcal{H}_- \]

\[ \mathcal{H}_- = \text{span}\{z^i\}_{i \geq 0}, \quad \mathcal{H}_+ = \text{span}\{z^{-i}\}_{i > 0}, \quad z \in S^1 \]

\[ w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \quad t := (t_1, t_2, \ldots) \]

Homogeneous coordinates

\[ w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix} = \begin{pmatrix} w_+h \\ w_+h \end{pmatrix}, \quad \forall h \in \text{GL}(\mathcal{H}_+) \]

The infinite abelian (flow) group:

\[ \Gamma_+ := \{ \gamma(t) := e^{\sum_{i=1}^{\infty} t_iz^i} \} \]

acts linearly on \( \mathcal{H} \), \( \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \).

\[ \Gamma_+: \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \]

\[ \gamma(t) : w \mapsto \gamma(t)w := w(t) = \begin{pmatrix} w_+(t) \\ w_-(t) \end{pmatrix} \]

Dual determinantal line bundle

\[ \text{Det}^* \]

\[ \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \]

Holomorphic (square integrable) sections

\[ H^0(\text{Gr}_{\mathcal{H}_+}(\mathcal{H}), \text{Det}^*) \equiv \mathcal{F}^* \]

where \( \mathcal{F} \) is the *Fermionic Fock space*:

\[ \mathcal{F} := \Lambda \mathcal{H} \]
Vacuum vector:

\[ |0> := z^0 \wedge z^1 \wedge \ldots \]

Free Fermi creation and annihilation operators:

\[ f_i := \iota z^i, \quad \bar{f}_i := z^i \wedge \]

Plücker embedding

\[ \mathfrak{P} : \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \hookrightarrow P(\mathcal{F}) \]

\[ \mathfrak{P} : \text{span}(v_1, v_2, v_3, \ldots) \mapsto [v_1 \wedge v_2 \wedge \ldots] \]

Plücker coordinates

For each partition

\[ \lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq 0, \ldots, 0, \ldots \]

there is a Plücker coordinate

\[ \pi_\lambda(w) := \det(w_\lambda) \]

(where \( w_\lambda \) is the semi-\( \infty \) block spanned by \( \{z^i - \lambda_i + 1\}_{i \in \mathbb{N}} \)

Then the KP \( \tau \)-function associated to \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \) is

\[ \tau_w(t) = \pi_0(\mathfrak{P}(w(t)) = \det(w_+(t)) \]

The other Plücker coordinates are:

\[ \pi_\lambda = s_\lambda \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots \right) \tau_w(t) \]

\( (s_\lambda = \text{Schur function}) \) and the Hirota bilinear relations are just the Plücker relations.
The $GL(H)$ action on $Gr_{H^+}(H)$

$$GL(H) : Gr_{H^+}(H) \rightarrow Gr_{H^+}(H)$$

$$g : w \mapsto gw$$

lifts to an action on $\mathcal{F}$

$$GL(H) : \mathcal{F} \rightarrow \mathcal{F}$$

$$g : v \mapsto \exp \sum_{i,j \in \mathbb{Z}} \xi_{ij} f_i \bar{f}_j$$

$$g = \exp \xi \in GL(H)$$

The Plücker map intertwines the $GL(H)$ action:

$$Gr_{H^+}(H) \xrightarrow{\mathfrak{P}} \mathcal{P} \mathcal{F}$$

$$Gl(H) \downarrow \quad Gl(H) \downarrow$$

$$Gr_{H^+}(H) \xrightarrow{\mathfrak{P}} \mathcal{P} \mathcal{F}$$

The image of $\mathcal{H}^+$ is the (projectivized) vacuum

$$\mathfrak{P}(\mathcal{H}^+) = \left[ |0> \right]$$

The $\tau$ function then becomes (up to projective equivalence)

$$\tau_w(t) = <0 | e^{\sum_{i=1}^{\infty} t_j H_j} g | 0>$$

where

$$H_j := \sum_{i \in \mathbb{Z}} f_i \bar{f}_{i+j}, \quad w = g(H_+)$$
More generally, define the charge-$n$ vacuum

\[ |n> := f_{n-1} f_{n-2} \cdots f_0 |0> \]

and change the splitting

\[ \mathcal{H} := \mathcal{H}_+ + \mathcal{H}_- \]

\[ \mathcal{H}_+ := \text{span}(z^{i+n})_{i \geq 0}, \quad \mathcal{H}_- := \text{span}(z^{i+n})_{i < 0} \]

In homogeneous coordinates, \( w \in \text{Gr}_{\mathcal{H}_+}(\mathcal{H}) \) is expressed

\[ w = \left[ \begin{pmatrix} w_{n+} \\ w_n \end{pmatrix} \right] \]

Then

\[ \mathfrak{P}(\mathcal{H}_+) = [|n>] \]

and, for \( w = g(\mathcal{H}_+) \), we can define the \( \tau \)-function as

\[ \tau_{n,w}(t) = \det(w_{n+}(t)) \]

\[ = < n|e^{\sum_{i=1}^\infty t_j H_j} g|n> \]
2. Matrix model integrals as KP $\tau$-functions

2.1 Unitary invariant Matrix model

Let $d\mu$ be a measure supported on a curve $C$ (e.g., the real axis, or a segment, or union of segments in the complex plane).

The integral

$$Z_n(t) := \int_C d\mu(x_1) \cdots \int_C d\mu(x_n) \Delta^2(x) e^{\sum_{j=1}^{\infty} \sum_{a=1}^{n} t_j x_a^j}$$

is the type of integral obtained from $U(n)$-invariant matrix integrals, after reduction to the space of eigenvalues.

Let $\{p_j(x)\}_{j=0,1,...}$ be the associated sequence of monic orthogonal polynomials

$$\int_C p_j(x)p_k(x)d\mu(x) = h_j \delta_{jk}$$

Let

$$w_{n,\mu} := \text{span}\{(p_{j+n}(x))_{j \in \mathbb{N}} \to \text{Gr}_{\mathcal{H}+}(\mathcal{H})$$

Then:

$$\frac{Z_n(t)}{Z_n(0)} = \tau_{n, w_{n,\mu}}(t)$$
2.2 Two-matrix model integrals as 2-Toda $\tau$-functions

A similar construction gives 2-Toda $\tau$-functions in terms of 2-component fermions:

$$f^{(\alpha)}_j := f_{2j + \alpha - 1}, \quad \bar{f}^{(\alpha)}_j := \bar{f}_{2j + \alpha - 1}, \quad \alpha = 1, 2$$

Define new charged vacua (for $n, m \geq 0$)

$$|n, -m> := f^{(2)}_{-m} \cdots f^{(2)}_{-1} f^{(2)}_n \cdots f^{(2)}_0 |0>$$

Let

$$d\mu_{\alpha\beta}(x, y), \quad \alpha, \beta = 1, 2$$

be a $2 \times 2$ matrix of bimeasures supported on a sum of products of curve segments $\Sigma$ in $\mathbb{C} \times \mathbb{C}$.

Define

$$g := e^A, \quad A : \sum_{\alpha, \beta} \int \int_{\Sigma} d\mu_{\alpha\beta}(x, y) : f^{(\alpha)}(x) \bar{f}^{(\beta)}(y) :$$

where the free Fermi fields (generators of Clifford algebra) are:

$$f^{(\alpha)}(x) := \sum_{i \in \mathbb{Z}} f_i^{(\alpha)} x^i, \quad \bar{f}^{(\alpha)}(x) := \sum_{i \in \mathbb{Z}} \bar{f}_i^{(\alpha)} x^{-i - 1},$$

and

$$: f^{(\alpha)} \bar{f}^{(\beta)} : = f^{(\alpha)} \bar{f}^{(\beta)} - <0, 0|f^{(\alpha)} \bar{f}^{(\beta)}|0, 0>$$

Define

$$H^{(\alpha)}_k := \sum_{n = -\infty}^{+\infty} f^{(\alpha)}_n \bar{f}^{(\alpha)}_{n+k}, \quad k \neq 0, \quad \alpha = 1, 2.$$
These give two infinite linear commutative subspaces quadratic operators

\[
H(t) = \sum_{k=1}^{\infty} H_k^{(1)} t_k^{(1)} - \sum_{k=1}^{\infty} H_k^{(2)} t_k^{(2)}
\]

\[
\bar{H}(\bar{t}) = \sum_{k=1}^{\infty} H_{-k}^{(2)} \bar{t}_k^{(2)} - \sum_{k=1}^{\infty} H_{-k}^{(1)} \bar{t}_k^{(1)}
\]

Then (Date et al (1983), Takasaki, Ueno (1984))

\[
\tau_N(t, n, m, \bar{t}) = \langle N + n, -N - m | e^{H(t)} g e^{\bar{H}(\bar{t})} | n, -m \rangle
\]

for any \( N, n, m \in \mathbb{Z} \) is a \( \tau \) function of the two-component Toda Lattice (TL) hierarchy (or, equivalently, the coupled two-component KP hierarchy).

**Theorem [HO2]**

Choose: \( d\mu_{11} = d\mu_{11} = d\mu_{11} = 0, d\mu_{12} := d\mu \), then

\[
\tau_N(t, n, m, \bar{t}) = \frac{1}{N!} (-1)^{\frac{1}{2}N(N+1)+mN} c(t, \bar{t}) Z_N(t, n, m, \bar{t})
\]

where \( c(t, \bar{t}) := e^{-\sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t^{(\alpha)}_k \bar{t}^{(\alpha)}_k} \),

\[
Z_N(t, n, m, \bar{t}) := \prod_{k=1}^{N} \int d\mu(x_k, y_k|t, n, m, \bar{t})) \Delta_N(x) \Delta_N(y)
\]

\[
d\mu(x, y|t, n, m, \bar{t}) := x^n y^m e^{V(x, t^{(1)}) + V(y, t^{(2)})} \times e^{V(x^{-1}, \bar{t}^{(1)}) + V(y^{-1}, \bar{t}^{(2)})} d\mu(x, y)
\]

\[
V(x, t^{(\alpha)}) = \sum_{k=1}^{\infty} t^{(\alpha)}_k x^k, \quad V(x^{-1}, \bar{t}^{(\alpha)}) = \sum_{k=1}^{\infty} \bar{t}^{(\alpha)}_k x^{-k}
\]
3. Semiclassical orthogonal polynomials, matrix models, isomonodromic tau functions (BEH3, BEH4)

Define a class of \textbf{semiclassical} measures that includes all reduced matrix integrals (for unitary invariant ensembles) *(Partition functions, gap probabilities, orthogonal polynomials, rational spectral correlators):*

\[
\mu(x) = e^{-V(x)}, \quad V(x) := \sum_{r=0}^{K} T_r(x)
\]

\[
T_0(x) := t_{0,0} + \sum_{J=1}^{d_0} t_{0,J} x^J
\]

\[
T_r(x) := \sum_{J=1}^{d_r} \frac{t_{r,J}}{J(x - c_r)^J} - t_{r,0} \ln(x - c_r)
\]

Define sectors for each pole and permissible contours:

\[
S_k^{(0)} := \{x \in \mathbb{C}; \frac{2k\pi - \arg(t_{0,d_0}) - \frac{\pi}{2}}{d_0} < \arg(x) < \frac{2k\pi - \arg(t_{0,d_0}) + \frac{\pi}{2}}{d_0} \}
\]

\[
k = 0 \ldots d_0 - 1 ;
\]

\[
S_k^{(r)} := \{x \in \mathbb{C}; \frac{2k\pi + \arg(t_{r,d_r}) - \frac{\pi}{2}}{d_r} < \arg(x - c_r) < \frac{2k\pi + \arg(t_{r,d_r}) + \frac{\pi}{2}}{d_r} \}
\]

\[
k = 0, \ldots, d_r - 1, \quad r = 1, \ldots, K
\]
These sectors are defined in such a way that approaching any of the essential singularities of $\mu(x)$ (i.e. a $c_r$ such that $d_r > 0$) within them, the function $\mu(x)$ tends to zero faster than any power.

Define the orthogonality contour:

$$\int_{\kappa} := \sum_{j=1}^{L} \kappa_j \int_{m_j} + \sum_{j=1}^{S} \kappa_{L+j} \int_{\sigma_j}.$$ 

and corresponding monic generalized orthogonal polynomials $\{p_n(x)\}$, satisfy:

$$\int_{\kappa} p_n(x)p_m(x)\mu(x)dx = h_n\delta_{nm}$$
Define also, normalized orthogonal polynomials:

\[ \pi_n(x) := \frac{1}{\sqrt{h_n}} p_n(x) \]

and the Cauchy transforms

\[ \phi_n(x) := e^{V(x)} \int_\kappa \frac{e^{-V(z)} \pi_n(z)}{x - z} \, dz. \]

These satisfy the usual 3-term recursion relations:

\[ x \pi_j(x) = \gamma_{j+1} \pi_{j+1}(x) + \beta_j \pi_j(x) + \gamma_j \pi_{j-1}(x). \]

Define the semi-infinite recursion matrix \( Q \) with components\n
\[ Q_{ij} = \gamma_j \delta_{i,j-1} + \beta_i \delta_{ij} + \gamma_i \delta_{i,j+1}, \quad i, j \in \mathbb{N}, \]

And the semi-infinite “wave vector”

\[ \Pi(x) := [\pi_0(x), \pi_1(x), \ldots, \pi_n(x), \ldots]^t \]

This may then be expressed as

\[ x \Pi(x) = Q \Pi(x) \]

This also satisfies a system of differential-deformation equations:

\[ \partial_x \Pi(x) = P \Pi(x) \]
\[ \partial_{a_i} \Pi(x) = A_i \Pi(x), \quad i = 1, \ldots, L \]
\[ \partial_{c_r} \Pi(x) = C_r \Pi(x), \quad r = 1, \ldots, K \]
\[ \partial_{t_{r,J}} \Pi(x) = T_{r,J} \Pi(x), \quad r = 0, \ldots, K, \quad J = 0, \ldots, d_r \]
where the matrices $P$, $A_i$, $C_r$ and $T_{r,J}$ are all lower semi-triangular (with $P$ strictly lower triangular), and given by

$$P = V'(Q)_- - \sum_{i=1}^{L} A_i = V'(Q)_- - \sum_{i=1}^{L} (A_i)_-$$

$$A_i = \kappa_i (\Pi(a_i)\Pi^t(a_i))_{-0}$$

$$C_r = - \sum_{J=0}^{d_r} t_{r,J}(Q - c_r)^{-J-1}_-, \quad r = 1, \ldots, K$$

$$T_{0,0} = \frac{1}{2} I, \quad T_{0,J} = \frac{1}{J} Q^J_-, \quad J = 1, \ldots, d_0$$

$$T_{r,J} = \frac{1}{J} (Q - c_r)^{-J}_-, \quad r = 1, \ldots, K, \quad J = 1, \ldots, d_r$$

$$T_{r,0} = - \ln(Q - c_r)_-, \quad r = 1, \ldots, K$$

where $(Q - c_r)^{-J}$ and $\ln(Q - c_r)$ are defined by the formulæ

$$(Q - c_r)^{-J}_{nm} := \int_{\kappa} \frac{\pi_n(z)\pi_m(z)}{(z - c_r)^J} \mu(z)dz$$

$$\ln(Q - c_r)_{nm} := \int_{\kappa} \ln(z - c_r)\pi_n(x)\pi_m(z)\mu(z)dz$$

The diagonal matrix elements of the above are given by

$$X_{jj} = - \frac{1}{2} \partial(\ln h_j) ,$$

where $\partial = \partial_x, \partial_{a_i}, \partial_{c_r}$ and $\partial_{t_{r,J}}$ for $X = P, A_i, C_r$ and $T_{r,J}$. and

$$V'(Q)_{jj} = \sum_{i=1}^{L} \kappa_i \psi_j(a_i)^2$$
Now define the $2 \times 2$ matrix

$$
\Gamma_n(x) := \begin{pmatrix} 
\pi_{n-1}(x) & \phi_{n-1}(x) \\
\pi_n(x) & \phi_n(x) 
\end{pmatrix}
$$

Using the recursion relations, we can “fold” all the above relations into a sequence of $2 \times 2$ “windows”, giving:

**Proposition [BEH2]** The folded forms of the recursion and differential relations for $\Gamma_n$ are

$$
\Gamma_{n+1}(x) = R_n(x)\Gamma_n(x), \quad n \geq 1,
$$

$$
\partial_x \Gamma_n(x) = D_n(x)\Gamma_n(x)
$$

$$
R_n := \begin{pmatrix} 
0 & 1 \\
-\gamma_n\gamma_{n+1} & x-\beta_n \gamma_{n+1}
\end{pmatrix}
$$

$$
D_n(x) = D_n^{(0)}(x)
$$

$$
+ \sum_{i=1}^{L} \frac{\kappa_i \gamma_n}{x-a_i} \begin{pmatrix} 
\psi_{n-1}(a_i)\psi_n(a_i) & -\psi^2_{n-1}(a_i) \\
\psi^2_n(a_i) & -\psi_{n-1}(a_i)\psi_n(a_i)
\end{pmatrix}
$$

$$
D_n^{(0)}(x) = \begin{pmatrix} 
V'(x) & 0 \\
0 & 0
\end{pmatrix}
$$

$$
+ \gamma_n \begin{pmatrix} 
(\nabla_Q V'(x))_{n-1,n} - (\nabla_Q V'(x))_{n-1,n-1} \\
(\nabla_Q V'(x))_{n,n} - (\nabla_Q V'(x))_{n,n-1}
\end{pmatrix}
$$

where

$$
(\nabla_Q v(x))_{nm} := \int_\kappa \frac{v(x) - v(z)}{x-z} \pi_n(z)\pi_m(z)e^{-V(z)} dz
$$
Proposition [BEH2] The deformation equations are equivalent to the infinite sequence of $2 \times 2$ equations

$$\delta_v \Gamma_n(x) = \mathcal{V}_n(x) \Gamma_n(x)$$

where the folded matrix of the deformation is defined by

$$\mathcal{V}_n(x) = \left( \begin{array}{cc} v(x) - \frac{1}{2} v(Q)_{n-1,n-1} & 0 \\ 0 & \frac{1}{2} v(Q)_{nn} \end{array} \right) + \gamma_n \left( \begin{array}{cc} \nabla_Q v(x)_{n-1,n} & -\nabla_Q v(x)_{n-1,n-1} \\ \nabla_Q v(x)_{nn} & -\nabla_Q v(x)_{n,n-1} \end{array} \right)$$

For the deformations this gives the following equations corresponding to changes in the potential.

$$\partial_{c_r} \Gamma_n(x) = C_{r;n} (x) \Gamma_n(x)$$

$$\partial_{t_{r,J}} \Gamma_n(x) = T_{r,J;n} (x) \Gamma_n(x)$$

where the $2 \times 2$ matrices $C_{r;n}$ and $T_{r,J;n}(x)$ are rational in $x$, with poles at the points $\{ c_r \}$, obtained by making the following substitutions

$$C_r : v(x) \to - \sum_{J=0}^{d_r} t_{r,J}(x - c_r)^{-J-1}$$

$$T_{r,J} : v(x) \to \frac{1}{J}(x - c_r)^{-J}$$

$$T_{0,J} : v(x) \to \frac{1}{J}x^J$$

$$T_{r,0} : v(x) \to - \ln(x - c_r)$$
Endpoint deformations

$$\partial_{a_i} \Gamma_n(x) = A_{i,n}(x) \Gamma_n(x)$$

where

$$A_{i,n} := \frac{\kappa_i \gamma_n}{a_i - x} \left( \begin{array}{cc} \psi_{n-1}(a_i) \psi_n(a_i) & -\psi_{n-1}^2(a_i) \\ \psi_n^2(a_i) & -\psi_{n-1}(a_i) \psi_n(a_i) \end{array} \right)$$

$$+ \frac{\kappa_i}{2} \left( \begin{array}{cc} -\psi_{n-1}^2(a_i) & 0 \\ 0 & \psi_n^2(a_i) \end{array} \right)$$

Isomonodromic deformations

This gives a compatible system, showing that the (generalized) monodromy) data of the rational covariant derivative operator:

$$\partial_x - D_n(x)$$

are independent of all deformation parameters \(\{c_r, a_i, t_r, J, n\}\).

Traceless gauge

Let

$$\Psi_n(x) := e^{-\frac{1}{2}V(x)} \Gamma_n(x) = \left( \begin{array}{c} \psi_{n-1}(x) \\ \tilde{\psi}_{n-1}(x) \end{array} \right)$$

$$\Psi'_n(x) = A_n(x) \Psi_n(x)$$

$$A_n(x) = D_n(x) - \frac{1}{2} V'(x) I$$

where

$$\tilde{\psi}_n := e^{-\frac{1}{2}V(x)} \psi_n$$
Local asymptotics near singular points \( \{c_r\} \)

\[ \Psi(x) \sim C_r Y_r(x) \exp \left( \frac{1}{2} T_r(x) + \delta_{r0} \left( n + \frac{1}{2} \sum_{r \geq 1} t_{r,0} \right) \ln(x) \right) \sigma_3 \]

where

\[
Y_0(x) := \mathbf{I} + \sum_{k=1}^{\infty} \frac{Y_{0;k}}{x^k},
\]

\[
Y_r(x) := \mathbf{I} + \sum_{k=1}^{\infty} Y_{r;k}(x - c_r)^k
\]

\[
C_0 = \begin{pmatrix} 0 & \sqrt{h_{n-1}} \\ \frac{1}{\sqrt{h_n}} & 0 \end{pmatrix}
\]

\[
C_r = \begin{pmatrix} \pi_{n-1}(c_r)e^{-\frac{V_r(c_r)}{2}} & (c_r - Q)_{n-0,1}\sqrt{h_0}e^{-\frac{V_r(c_r)}{2}} \\ \pi_n(c_r)e^{-\frac{V_r(c_r)}{2}} & (c_r - Q)_{n,0}\sqrt{h_0}e^{-\frac{V_r(c_r)}{2}} \end{pmatrix}
\]

where

\[
\tilde{V}_r(x) := V(x) - T_r(x)
\]

Local asymptotics near endpoints \( \{a_j\} \)

\[ \Psi(x) \sim A_j \cdot Y_j(x) \cdot \exp \left[ -\kappa_j \ln(x - a_j) \sigma_+ \right], \quad \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

\[
A_j = \begin{pmatrix} \pi_{n-1}(a_j)e^{-\frac{V(a_j)}{2}} & e^{-\frac{1}{2} V(z)}(\pi_{n-1}(z) - \pi_{n-1}(a_j))a_j - z \int_\kappa dz e^{-\frac{1}{2} V(z)}(\pi_n(z) - \pi_n(a_j))a_j - z \\ \pi_n(a_j)e^{-\frac{V(a_j)}{2}} & e^{-\frac{1}{2} V(z)}(\pi_n(z) - \pi_n(a_j))a_j - z \int_\kappa dz e^{-\frac{1}{2} V(z)}(\pi_n(z) - \pi_n(a_j))a_j - z \end{pmatrix}
\]
Isomonodromic $\tau$-function

$$d\ln \tau_{n}^{IM} = -\frac{1}{2} \sum_{r=0} \text{res}_{x=c_r} dT_{r}(x) \text{tr} \left( Y_{r}^{-1} Y'_{r} \sigma_{3} \right)$$

$$+ \sum_{j} \text{res}_{x=a_j} \frac{\kappa_j da_j}{x - a_j} \text{tr} \left( Y_{j}^{-1} Y'_{j} \sigma_{+} \right)$$

**Theorem [BEH2]** Up to multiplicative terms that are independent of the isomonodromic deformation parameters $\{t_{r,J}, c_r, a_j, n\}$, the partition function

$$Z_{n}(\{t_{r,J}, c_r, a_j\}|[\kappa]) := C_n \int_{\text{spec}(M) \in \kappa} dMe^{-\text{tr}V(M)}$$

$$= \int_{\kappa} dx_1 \cdots \int_{\kappa} dx_n \Delta(x)^2 e^{-\sum_{j=1}^{n} V(x_j)}$$

$$= n! \prod_{j=0}^{n-1} h_j$$

of the generalized random matrix model and the isomonodromic tau function $\tau_{n}^{IM}$ for the associated ODE are related by

$$Z_{n} = \tau_{n}^{IM} \mathcal{F}_{n},$$

where

$$\ln \left( \frac{\mathcal{F}_{n}}{n!} \right) = \frac{1}{2h^2} \sum_{0 \leq q < r \leq K} \text{res}_{x=c_r} T'_{r}(x) T_{q}(x)$$
References


