

**Monodromy-free Schrödinger operators  
and Painlevé-IV transcendents**

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## Plan

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Generalised Stieltjes relations for PIV transcendents

Application: new class of monodromy-free operators

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## Stieltjes' "electrostatic" interpretation of zeroes of Hermite polynomials

**T. Stieltjes (1885):** Consider  $n$  particles on the line interacting pairwise with repulsive logarithmic potential in the harmonic field. Then the equilibrium of this system is exactly the set of zeroes of Hermite polynomials  $H_n(z)$ :

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$

More precisely, the extremum condition for the function

$$U(z_1, \dots, z_n) = \sum_{j=1}^n z_j^2 - \sum_{j < k}^n \ln(z_j - z_k)^2,$$

which is the system of the **Stieltjes relations**

$$\sum_{j \neq k}^n (z_k - z_j)^{-1} - z_k = 0, \quad k = 1, \dots, n,$$

determines exactly the roots of the equation  $H_n(z) = 0$ .

**F. Calogero (1977):** another remarkable interpretation of zeroes of Hermite polynomials as equilibriums in the Calogero-Moser system with the potential

$$V_{CM}(z_1, \dots, z_n) = \sum_{j=1}^n z_j^2 + 2 \sum_{j < k}^n (z_j - z_k)^{-2}.$$

So these zeroes satisfy also the following system of **Calogero relations**

$$2 \sum_{j \neq k}^n (z_k - z_j)^{-3} - z_k = 0, k = 1, \dots, n.$$

However in contrast to the Stieltjes relation in the complex domain these relations have other solutions as well (although on the real these two systems are equivalent).

## Monodromy-free Schrödinger operators

**Definition.** A Schrödinger operator

$$L = -\frac{d^2}{dz^2} + u(z)$$

with meromorphic potential is **monodromy-free** if all the solutions of the equation

$$L\psi = -\psi'' + u\psi = \lambda\psi$$

are also meromorphic in the whole complex plane for **all**  $\lambda$ . In other words monodromy of the corresponding equation in the complex plane is trivial for all  $\lambda$ .

The simplest examples are

$$u(z) = \frac{m(m+1)}{z^2}$$

with integer  $m$ . More generally, all **finite-gap** (or algebro-geometric) potentials are monodromy-free.

Other examples:  $u(z) = P(z)$ , where  $P(z)$  is arbitrary polynomial.

**Problem.** Classify all monodromy-free Schrödinger operators

**Duistermaat and Grünbaum (1986)** : rational potentials decaying at infinity

**Gesztesy and Weikard (1996)**: elliptic potentials

**Oblomkov (1999)**: rational potentials with a quadratic growth at infinity

**Gibbons, Veselov (2003)**: partial results for rational potential with sextic growth at infinity

## Stieltjes and Calogero relations as trivial monodromy conditions

Consider Schrödinger equation

$$-\varphi'' + u(z)\varphi = \lambda\varphi$$

near a pole  $z = 0$  where the potential has a Laurent expansion

$$u = \sum_{i=-2}^{\infty} c_i z^i.$$

Classical Frobenius arguments show that it has a meromorphic solution

$$\varphi = z^{-m} \left( 1 + \sum_{i=1}^{\infty} \xi_i z^i \right)$$

for all  $\lambda$  only if  $c_{-2} = m(m+1)$  and all the first  $m+1$  odd coefficients at the Laurent expansion of the potential are vanishing:

$$c_{2k-1} = 0, \quad k = 0, 1, \dots, m.$$

Let  $f$  be a meromorphic function having the poles of the first order with integer residues and the potential  $u$  has the form

$$u = f' + f^2.$$

**Lemma.** *The corresponding Schrödinger operator  $L$  is monodromy-free if and only if at any pole  $z_0$  with  $\text{Res}_{z_0} f = m$  the following relations are satisfied:*

$$\text{Res}_{z_0} f^2 = \text{Res}_{z_0} f^4 = \dots = \text{Res}_{z_0} f^{2|m|} = 0.$$

We will call them **generalised Stieltjes relations** since for

$$f = -z + (\ln H_n(z))' = -z + \sum_{i=1}^n \frac{1}{z - z_i}$$

the condition  $\text{Res} f^2 = 0$  gives classical Stieltjes relations.

In this case the potential is

$$u = z^2 + \sum_{i=1}^n \frac{2}{(z - z_i)^2}$$

and the corresponding trivial monodromy conditions

$$c_1 = 0$$

are exactly the Calogero relations, which proves the following

**Theorem (Calogero (1977), Perelomov (1978)).**  
*Stieltjes relations imply Calogero relations.*

Converse is not true: we will show that the zeroes of the wronskians of Hermite polynomials  $W(H_{k_1}, H_{k_2}, \dots, H_{k_n})$  satisfy Calogero relations (but not the Stieltjes relations, of course).

## Chains of Darboux transformations and PIV hierarchy

**A.B. Shabat, A.V. (1993), V.E. Adler (1994)**

Consider

$$L = -\frac{d^2}{dz^2} + u(z)$$

and factorise it as

$$L = -\left(\frac{d}{dz} + f(z)\right)\left(\frac{d}{dz} - f(z)\right)$$

which equivalent to solving the Riccati equation

$$f' + f^2 = u.$$

The corresponding  $f(z)$  can be found as logarithmic derivative

$$f = \frac{d \log \psi}{dz}$$

of any solution of

$$-\psi'' + u\psi = 0.$$

The transformation

$$L \rightarrow \tilde{L} = - - \left( \frac{d}{dz} - f(z) \right) \left( \frac{d}{dz} + f(z) \right)$$

is called **Darboux transformation**. The new potential  $\tilde{u}$  is

$$\tilde{u} = f^2 - f' = u - 2f' = u - 2(\log \psi)''.$$

The solutions of the corresponding equations  $L\psi = \lambda\psi$  and  $\tilde{L}\tilde{\psi} = \lambda\tilde{\psi}$  are related as

$$\tilde{\psi} = \left( \frac{d}{dz} - f(z) \right) \psi, \quad \psi = \left( \frac{d}{dz} + f(z) \right) \tilde{\psi}.$$

In particular, if  $L$  is monodromy-free then  $\tilde{L}$  is monodromy-free as well.

One can repeat this procedure for the operator  $\tilde{L} + \alpha$  to come to the following **dressing chain**

$$(f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i, \quad i = 1, 2, \dots$$

Following **Shabat and A.V. (1993)** consider **periodic** version of dressing chain with **odd** period:  $i = 1, 2, \dots, N = 2g + 1$ .

For  $N = 3$  and  $\alpha = \sum \alpha_i = -2$  it is equivalent to Painlevé-IV:  $w = -(z + f_1)$  satisfies

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - a)w^2 + 2b$$

with

$$a = \frac{1}{2}(\alpha_3 - \alpha_1), b = -\frac{1}{2}\alpha_2^2.$$

Thus periodic dressing chain can be considered as **higher analogue of PIV-equation**. It is equivalent to  $A_{2g}^{(1)}$  **system** considered by **Noumi and Yamada (1998)** written in the variables

$$g_i = f_i + f_{i+1}.$$

**Theorem 1.** *For any meromorphic solution of the dressing chain the function  $f = f_1$  has the poles of the first order with integer residues. At any pole  $z_0$  with  $\text{Res}_{z=z_0} f = m$  the generalised Stieltjes relations are satisfied:*

$$\text{Res}_{z_0} f^2 = \text{Res}_{z_0} f^4 = \dots = \text{Res}_{z_0} f^{2|m|} = 0.$$

*In particular, for any solution  $w$  of PIV*

$$\text{Res}(z + w)^2 = 0.$$

For special solutions  $w = -(\log H_n(z))'$  we have classical Stieltjes relations.

**Corollary.** *For any solution  $w$  of PIV Schrödinger operator  $L$  with potential  $u = w' + (w + z)^2$  is monodromy-free. The same is true for the operators with the potentials  $u = f' + f^2$  where  $f = f_1$  for any meromorphic solution of some dressing chain.*

**Conjecture.** *All solutions of the dressing chain (or PIV-hierarchy) are meromorphic in the whole complex plane.*

**Question.** *Do the generalised Stieltjes relations characterise the class of higher PIV transcendents ?*

The answer is positive in the following class of rational functions

$$f = \sum_{i=1}^n \frac{m_i}{z - z_i} + \nu - \mu z,$$

where  $m_i$  are some integers.

**Theorem 2.** *If a rational function  $f$  from this class satisfies the generalised Stieltjes relations then  $f$  is a rational solution of some higher Painlevé-IV equation.*

The proof is based on the results of **Duistermaat-Grünbaum (1986)** and **Oblomkov (1999)**.

## Some open problems

Can one make precise the interpretation of the poles of PIV transcendents as the **infinite** "electrostatic" equilibrium configurations ?

What are the analogues of Stieltjes relations for other Painlevé hierarchies ?

What are the properties of the higher Painlevé transcendents (meromorhicity, Nevanlinna order, asymptotic behaviour of poles) ?

Other classes of monodromy-free potentials ?

## Reference.

A.P. Veselov "On Stieltjes relations, Painlevé-IV hierarchy and complex monodromy." J. Phys. A: Math. Gen. **34** (2001), 3511-3519.