Monodromy-free Schrödinger operators and Painlevé-IV transcendents

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Stieltjes' "electrostatic" interpretation of zeroes of Hermite polynomials

T. Stieltjes (1885): Consider n particles on the line interacting pairwise with repulsive logarithmic potential in the harmonic field. Then the equilibrium of this system is exactly the set of zeroes of Hermite polynomials $H_n(z)$:

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$

More precisely, the extremum condition for the function

$$U(z_1, ..., z_n) = \sum_{j=1}^{n} z_j^2 - \sum_{j<k}^{n} \ln(z_j - z_k)^2,$$

which is the system of the **Stieltjes relations**

$$\sum_{j \neq k}^{n} (z_k - z_j)^{-1} - z_k = 0, \quad k = 1, ..., n,$$

determines exactly the roots of the equation $H_n(z) = 0$.

F. Calogero (1977): another remarkable interpretation of zeroes of Hermite polynomials as equilibriums in the Calogero-Moser system with the potential

$$V_{CM}(z_1, ..., z_n) = \sum_{j=1}^{n} z_j^2 + 2 \sum_{j<k}^{n} (z_j - z_k)^{-2}.$$

So these zeroes satisfy also the following system of **Calogero relations**

$$2 \sum_{j \neq k}^{n} (z_k - z_j)^{-3} - z_k = 0, k = 1, ..., n.$$

However in contrast to the Stieltjes relation in the complex domain these relations have other solutions as well (although on the real these two systems are equivalent).

Monodromy-free Schrödinger operators

Definition. A Schrödinger operator

$$L = -\frac{d^2}{dz^2} + u(z)$$

with meromorphic potential is **monodromy-free** if all the solutions of the equation

$$L\psi = -\psi'' + u\psi = \lambda\psi$$

are also meromorphic in the whole complex plane for **all** λ. In other words monodromy of the corresponding equation in the complex plane is trivial for all λ.

The simplest examples are

$$u(z) = \frac{m(m+1)}{z^2}$$

with integer m. More generally, all **finite-gap** (or algebro-geometric) potentials are monodromy-free.

Other examples: $u(z) = P(z)$, where $P(z)$ is arbitrary polynomial.

Problem. Classify all monodromy-free Schrödinger operators

Duistermaat and Grünbaum (1986) : rational potentials decaying at infinity

Gesztesy and Weikard (1996): elliptic potentials

Oblomkov (1999): rational potentials with a quadratic growth at infinity

Gibbons, Veselov (2003): partial results for rational potential with sextic growth at infinity

Stieltjes and Calogero relations as trivial monodromy conditions

Consider Schrödinger equation

$$-\varphi'' + u(z)\varphi = \lambda\varphi$$

near a pole $z = 0$ where the potential has a Laurent expansion

$$u = \sum_{i=-2}^{\infty} c_i z^i.$$

Classical Frobenius arguments show that it has a meromorphic solution

$$\varphi = z^{-m}(1 + \sum_{i=1}^{\infty} \xi_i z^i)$$

for all λ only if $c_{-2} = m(m+1)$ and all the first $m+1$ odd coefficients at the Laurent expansion of the potential are vanishing:

$$c_{2k-1} = 0, \quad k = 0, 1, ..., m.$$

Let f be a meromorphic function having the poles of the first order with integer residues and the potential u has the form

$$u = f' + f^2.$$

Lemma. *The corresponding Schrödinger operator L is monodromy-free if and only if at any pole z_0 with $\mathrm{Res}_{z_0} f = m$ the following relations are satisfied:*

$$\mathrm{Res}_{z_0} f^2 = \mathrm{Res}_{z_0} f^4 = ... = \mathrm{Res}_{z_0} f^{2|m|} = 0.$$

We will call them **generalised Stieltjes relations** since for

$$f = -z + (\ln H_n(z))' = -z + \sum_{i=1}^{n} \frac{1}{z - z_i}$$

the condition $\mathrm{Res}\, f^2 = 0$ gives classical Stieltjes relations.

In this case the potential is

$$u = z^2 + \sum_{i=1}^{n} \frac{2}{(z - z_i)^2}$$

and the corresponding trivial monodromy conditions

$$c_1 = 0$$

are exactly the Calogero relations, which proves the following

Theorem (**Calogero** (1977), **Perelomov** (1978)). *Stieltjes relations imply Calogero relations.*

Converse is not true: we will show that the zeroes of the wronskians of Hermite polynomials $W(H_{k_1}, H_{k_2}, ..., H_{k_n})$ satisfy Calogero relations (but not the Stieltjes relations, of course).

Chains of Darboux transformations and PIV hierarchy

A.B. Shabat, A.V. (1993), **V.E. Adler** (1994)

Consider

$$L = -\frac{d^2}{dz^2} + u(z)$$

and factorise it as

$$L = -(\frac{d}{dz} + f(z))(\frac{d}{dz} - f(z))$$

which equivalent to solving the Riccati equation

$$f' + f^2 = u.$$

The corresponding $f(z)$ can be found as logarithmic derivative

$$f = \frac{d \log \psi}{dz}$$

of any solution of

$$-\psi'' + u\psi = 0.$$

The transformation

$$L \to \tilde{L} = - - (\frac{d}{dz} - f(z))(\frac{d}{dz} + f(z))$$

is called **Darboux transformation.** The new potential $\tilde{u}$ is

$$\tilde{u} = f^2 - f' = u - 2f' = u - 2(\log \psi)''.$$

The solutions of the corresponding equations $L\psi = \lambda\psi$ and $\tilde{L}\tilde{\psi} = \lambda\tilde{\psi}$ are related as

$$\tilde{\psi} = (\frac{d}{dz} - f(z))\psi, \quad \psi = (\frac{d}{dz} + f(z))\tilde{\psi}.$$

In particular, if L is monodromy-free then $\tilde{L}$ is monodromy-free as well.

One can repeat this procedure for the operator $\tilde{L}+\alpha$ to come to the following **dressing chain**

$$(f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i, \quad i = 1, 2, ...$$

Following **Shabat and A.V. (1993)** consider **periodic** version of dressing chain with **odd** period: $i = 1, 2, ..., N = 2g + 1$.

For $N = 3$ and $\alpha = \sum \alpha_i = -2$ it is equivalent to Painlevé-IV: $w = -(z + f_1)$ satisfies

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - a)w^2 + 2b$$

with

$$a = \frac{1}{2}(\alpha_3 - \alpha_1), b = -\frac{1}{2}\alpha_2^2.$$

Thus periodic dressing chain can be considered as **higher analogue of PIV-equation**. It is equivalent to $A_{2g}^{(1)}$ **system** considered by **Noumi and Yamada (1998)** written in the variables

$$g_i = f_i + f_{i+1}.$$

Theorem 1. *For any meromorphic solution of the dressing chain the function $f = f_1$ has the poles of the first order with integer residues. At any pole z_0 with $\mathrm{Res}_{z=z_0} f = m$ the generalised Stieltjes relations are satisfied:*

$$\mathrm{Res}_{z_0} f^2 = \mathrm{Res}_{z_0} f^4 = ... = \mathrm{Res}_{z_0} f^{2|m|} = 0.$$

In particular, for any solution w of PIV

$$\mathrm{Res}\,(z + w)^2 = 0.$$

For special solutions $w = -(\log H_n(z))'$ we have classical Stieltjes relations.

Corollary. *For any solution w of PIV Schrödinger operator L with potential $u = w' + (w + z)^2$ is monodromy-free. The same is true for the operators with the potentials $u = f' + f^2$ where $f = f_1$ for any meromorphic solution of some dressing chain.*

Conjecture. *All solutions of the dressing chain (or PIV-hierarchy) are meromorphic in the whole complex plane.*

Question. *Do the generalised Stieltjes relations characterise the class of higher PIV transcendents ?*

The answer is positive in the following class of rational functions

$$f = \sum_{i=1}^{n} \frac{m_i}{z - z_i} + \nu - \mu z,$$

where m_i are some integers.

Theorem 2. *If a rational function f from this class satisfies the generalised Stieltjes relations then f is a rational solution of some higher Painlevé-IV equation.*

The proof is based on the results of **Duistermaat-Grünbaum (1986)** and **Oblomkov (1999)**.

Some open problems

Can one make precise the interpretation of the poles of PIV transcendents as the **infinite** "electrostatic" equilibrium configurations ?

What are the analogues of Stieltjes relations for other Painlevé hierarchies ?

What are the properties of the higher Painlevé transcendents (meromorhicity, Nevanlinna order, asymptotic behaviour of poles) ?

Other classes of monodromy-free potentials ?

Reference.

A.P. Veselov *"On Stieltjes relations, Painlevé-IV hierarchy and complex monodromy."* J. Phys. A: Math. Gen. **34** (2001), 3511-3519.