Trends in Noncommutative Geometry
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POINCARÉ-BIRKHOFF-WITT DEFORMATIONS OF CALABI-YAU ALGEBRAS

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PART I - N-Koszul algebras

Introduced by R.B. (2001, J. Algebra). For $N = 2$, we recover Koszul algebras as defined by Priddy (1970).

Why higher N's?

1. There are some relevant examples coming from

- noncommutative projective geometry: cubic Artin-Schelter regular algebras (1987) of global dimension 3, such as

$$A = \frac{\mathbb{C}\langle x, y\rangle}{(ay^2x + byxy + axy^2 + cx^3, x \leftrightarrow y)}$$

- theoretical physics: Yang-Mills algebras (A. Connes and M. Dubois-Violette 2002)

$$A = \frac{\mathbb{C}\langle \nabla_0, \ldots, \nabla_s\rangle}{(\sum_{\lambda\mu} g^{\lambda\mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]])}$$

where $g^{\lambda\mu}$ are entries of an invertible symmetric real matrix.

- representation theory: antisymmetriser algebras (R.B. 2001, V. Ginzburg and R.B. 2006)

$$A = \frac{\mathbb{C}\langle x_1, \ldots, x_n\rangle}{(\sum_\sigma \mathrm{sgn}(\sigma)\ x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(N)}})}$$

for $2 \leq N \leq n$.

2. Poincaré duality in Hochschild (co)homology (N. Marconnet and R.B. 2003). Using Van den Bergh's duality theorem, we proved that if A is N-Koszul and AS-Gorenstein of global dimension d, then

$$HH^i(A, M) \cong HH_{d-i}(A, {}_{\varepsilon^{d+1}\phi}M).$$

Moreover (R. Taillefer and R.B.), A is Calabi-Yau (of dimension d) if and only if the automorphism $\varepsilon^{d+1}\phi$ of A is the identity automorphism.

Some applications of the latter result: the first two examples are Calabi-Yau. For $N \geq 3$, the third one is Calabi-Yau if and only if N divides $n-1$ and n is odd.

3. N-Koszul algebras and quantum Mac-Mahon Master Theorem (qMMT)

Let A be an N-Koszul superalgebra. Extending Manin's quadratic approach of quantum groups (Montréal 1988) to higher N, the superbialgebra $B = \mathrm{end}(A)$ "of matrices over A" is still defined (M. Dubois-Violette, M. Wambst and R.B. 2003).

Then, Phùng Hô Hai, B.Kriegk, and M. Lorenz recently proved the following "N-Koszul MMT":

$$\left(\sum_{\ell\geq 0}\chi^s(A_\ell)t^\ell\right)\left(\sum_{m\geq 0}(-1)^m\chi^s(A^{!*}_{\nu_N(m)})t^{\nu_N(m)}\right)=1,$$

where $\chi^s(M)\in B$ denotes the supercharacter of any finite-dimensional B-comodule M.

Specialising $N=2$ and $A=$ multiparameter quantum space, qMMT (by Garoufalidis-Lè-Zeilberger and Konvalinka-Pak) is recovered.

PART II - Basic observation: Bocklandt's algebras A are N-Koszul

A is a graded Calabi-Yau algebra of dimension 3. As proved by Bocklandt, A is defined by a homogeneous potential W_{N+1} of degree $N+1$:

$$A = A(Q, W_{N+1}) := \mathbb{C}Q/(\partial_a W_{N+1};\ a \in Q_1).$$

Assume $\#Q_0 = 1$ and $Q_1 = \{x_1, \ldots, x_n\}$. Then we have the following projective resolution of A as an A-bimodule:

$$0 \to A^e \xrightarrow{\delta_3} (A^e)^n \xrightarrow{\delta_2} (A^e)^n \xrightarrow{\delta_1} A^e \xrightarrow{\mu} A \to 0,$$

where

$$\delta_1((a_j \otimes b_j)_{1 \le j \le n}) = \sum_{1 \le j \le n} a_j x_j \otimes b_j - a_j \otimes x_j b_j$$

$$\delta_2((a_i \otimes b_i)_{1 \le i \le n}) = \left(\sum_{1 \le i \le n} a_i \ \frac{\partial^2 W_{N+1}}{\partial x_i \partial x_j} \ b_i \right)_{1 \le j \le n}$$

$$\delta_3(a \otimes b) = (a \otimes x_i b - a x_i \otimes b)_{1 \le i \le n}.$$

PART III - PBW deformations of a graded algebra $A = \mathbb{C}Q/(R)$, where R is any $\mathbb{C}Q_0$-subbimodule of $\mathbb{C}Q_N$.

Let P be a $\mathbb{C}Q_0$-subbimodule of $\mathbb{C}Q_{\leq N}$ such that $\pi(P) = R$, where $\pi : \mathbb{C}Q_{\leq N} \to \mathbb{C}Q_N$ is the natural projection.

Set $U = \mathbb{C}Q/(P)$. It is a filtered algebra, and there is a natural graded algebra morphism $p : A \to \mathrm{gr}(U)$.

Definition. U is called a PBW deformation of A if p is an isomorphism, that is, if any monomial basis of A is also a basis of U.

Theorem (V. Ginzburg and R.B. 2006) Assume that A is N-Koszul. Then U is a PBW deformation of A if and only if we have

$$P \cap \mathbb{C}Q_{\leq N-1} = 0, \tag{1}$$

$$(P \otimes_{\mathbb{C}Q_0} V + V \otimes_{\mathbb{C}Q_0} P) \cap \mathbb{C}Q_{\leq N} \subseteq P\,. \tag{2}$$

Condition (1) is a non-triviality condition.

Condition (2) is an N-generalisation of the Jacobi identity: for $N = 2$, assume $\#Q_0 = 1$ and $Q_1 = \{x_1, \ldots, x_n\}$, and choose an alternating bilinear map

$$[\ ,\ ] : \mathbb{C}Q_1 \times \mathbb{C}Q_1 \to \mathbb{C}Q_1.$$

Let P be spanned over $\mathbb{C}$ by the elements

$$x \otimes y - y \otimes x - [x, y],\ x \in Q_1,\ y \in Q_1.$$

Then (1) holds, and (2) is equivalent to the Jacobi identity of the bracket $[\ ,\ ]$. In this case, U is the enveloping algebra of the Lie algebra $(\mathbb{C}Q_1, [\ ,\ ])$.

PART IV - PBW deformations of a Bocklandt algebra $A = A(Q, W_{N+1})$. Then

$$R = \mathrm{span}_{\mathbb{C}Q_0}\{\partial_a W_{N+1};\ a \in Q_1\}.$$

We apply the previous theorem to get the following.

Theorem (R. Taillefer and R.B. 2006) Let W' be a potential of degree $< N+1$. Then

$$U = A(Q, W = W_{N+1} + W')$$

is a PBW deformation of A, and U is Calabi-Yau of dimension 3.

Let us discuss only the first point. We have

$$P = \mathrm{span}_{\mathbb{C}Q_0}\{\partial_a W;\ a \in Q_1\},$$

$$P \cap \mathbb{C}Q_{\leq N-1} = \{\textstyle\sum_a \lambda_a \partial_a W; \sum_a \lambda_a \partial_a W_{N+1} = 0\}.$$

Since A is Calabi-Yau of dimension 3, $\#Q_1 = \dim R$, thus the $\partial_a W_{N+1}$'s are linearly independent, and condition (1) holds.

PART V - A noncommutative Euler identity

Euler identity: for a (commutative) polynomial W in variables $x_1, \ldots, x_n$,

$$\sum_{1 \leq i \leq n} x_i \frac{\partial W}{\partial x_i} = \mathfrak{c}(W),$$

where $\mathfrak{c}$ is the cyclic sum of each monomial.

Analogue in circular calculus (one vertex case): for a potential W in $x_1, \ldots, x_n$,

$$\sum_{1 \leq i \leq n} x_i \, \partial_{x_i} W = \mathfrak{c}(W) = \sum_{1 \leq i \leq n} \partial_{x_i} W \, x_i.$$

For example, if $W = xyz - yxz$, then

$\partial_x W = yz - zy, \ \partial_y W = zx - xz, \ \partial_z W = xy - yx$

and $(x\partial_x + y\partial_y + z\partial_z)(W) = \mathfrak{c}(W)$ is the totally antisymmetric tensor in x, y, z.

General case: for each vertex e of a quiver Q, we have

$$\sum_{a \in eQ_1} a \, \partial_a W = e.\mathfrak{c}(W) = \sum_{b \in Q_1 e} \partial_b W \, b.$$

PART VI - The space $\mathcal{W}$ of volume forms of $A = A(Q, W_{N+1})$

It is $\mathcal{W} = (R \otimes_{\mathbb{C}Q_0} \mathbb{C}Q_1) \cap (\mathbb{C}Q_1 \otimes_{\mathbb{C}Q_0} R)$. For each vertex e, $e.\mathfrak{c}(W_{N+1})$ belongs to $\mathcal{W}$.

Since A is Calabi-Yau of dimension 3, $\#Q_0 = \dim \mathcal{W}$, thus $(e.\mathfrak{c}(W_{N+1}))_{e \in Q_0}$ is a basis of $\mathcal{W}$.

Recall that $U = A(Q, W_{N+1} + W')$.

Define $\varphi : R \to \mathbb{C}Q_{\leq N-1}$ by $\varphi(\partial_a W_{N+1}) = -\partial_a W'$, so that relations of U are

$$\partial_a W_{N+1} = \varphi(\partial_a W_{N+1}), \ a \in Q_1.$$

Clearly $\varphi \otimes 1 := \varphi \otimes_{\mathbb{C}Q_0} \mathrm{id}_{\mathbb{C}Q_1}$ and $1 \otimes \varphi := \mathrm{id}_{\mathbb{C}Q_1} \otimes_{\mathbb{C}Q_0} \varphi$ are defined on $\mathcal{W}$.

We know (V. Ginzburg and R.B.) that condition (2) is equivalent to

$$(\varphi \otimes 1 - 1 \otimes \varphi)(\mathcal{W}) \subseteq P.$$

Using circular Euler identity, we calculate for each vertex e,

$$(\varphi \otimes 1 - 1 \otimes \varphi)(e.\mathfrak{c}(W_{N+1}))$$

$$= \sum_{a \in eQ_1} a\, \partial_a W' - \sum_{b \in Q_1 e} \partial_b W'\, b,$$

which is zero, again from circular Euler identity.

Thus we get a condition *stronger* than (2):

$$(P \otimes_{\mathbb{C}Q_0} V + V \otimes_{\mathbb{C}Q_0} P) \cap \mathbb{C}Q_{\leq N} = 0. \qquad (3)$$

The converse statement is true:

Theorem (R. Taillefer and R.B. 2006) If $U = \mathbb{C}Q/(P)$ is a PBW deformation of a Bocklandt algebra $A = A(Q, W_{N+1})$ such that condition (3) holds, then there exists a unique potential W' without constant term such that

$$U = A(Q, W_{N+1} + W').$$