An Overview of Oscillatory Integrals and Integral Operators in High Frequency Scattering

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and (in part) with **Peter Monk** (Delaware)

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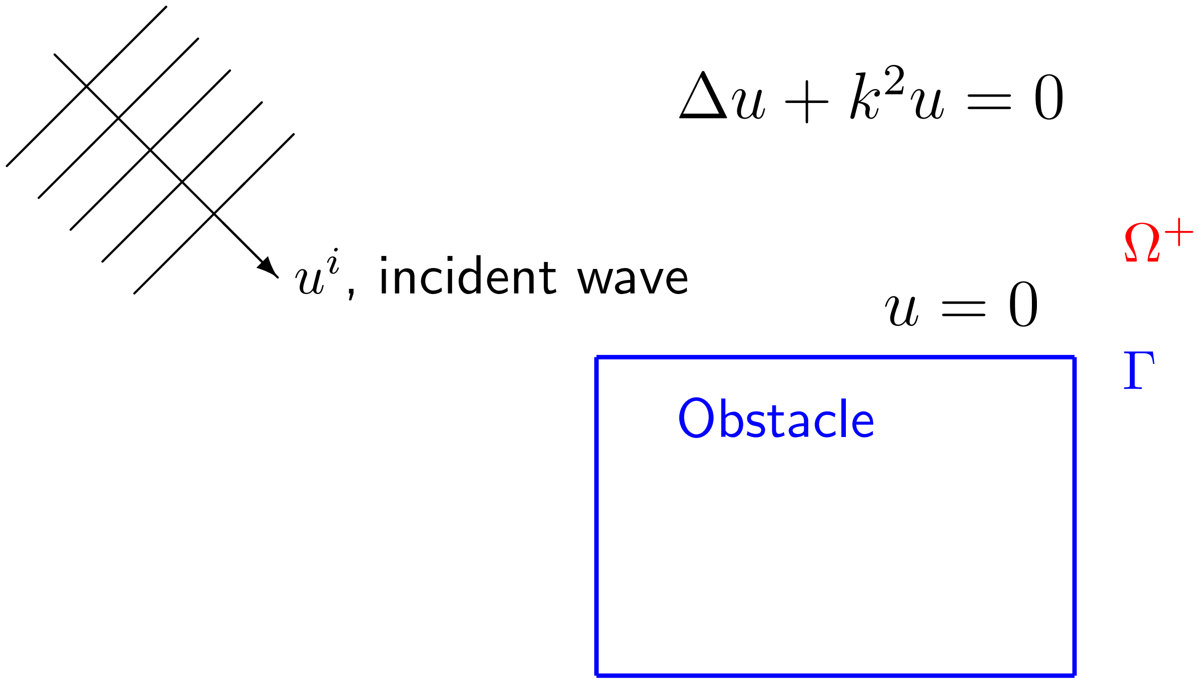
A Typical Scattering Problem

$$\Delta u + k^2 u = 0$$



Background

For simulation of wave scattering, solving the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

the number of degrees of freedom in a conventional boundary element method (BEM) or finite element method needs to increase as the **wave number** $k = \dfrac{2\pi}{\lambda}$ increases.

Part of the scattered field

Correction to the Kirchhoff approximation

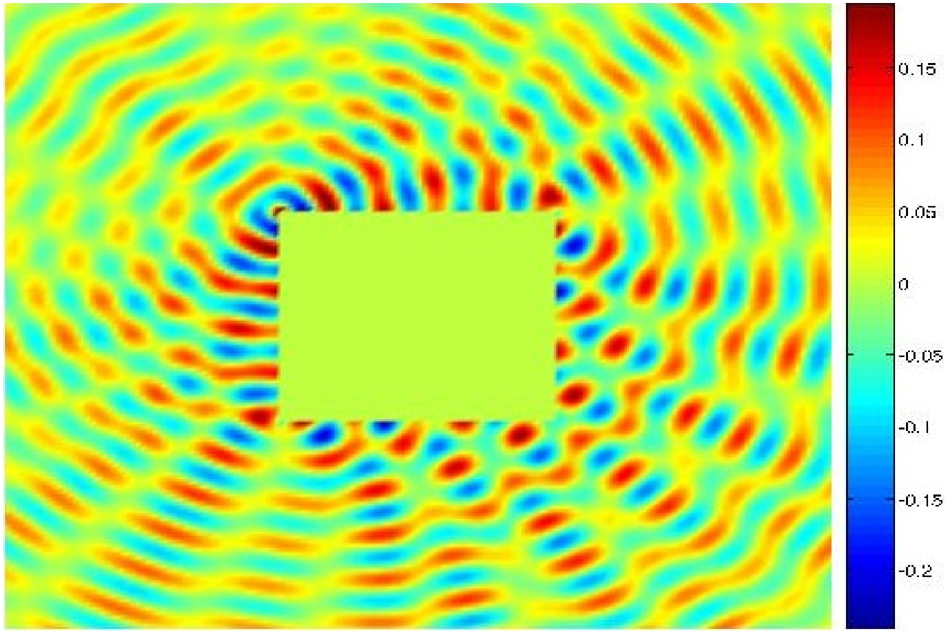


Figure 1: square, $k = 5$

Part of the scattered field

Correction to the Kirchhoff approximation

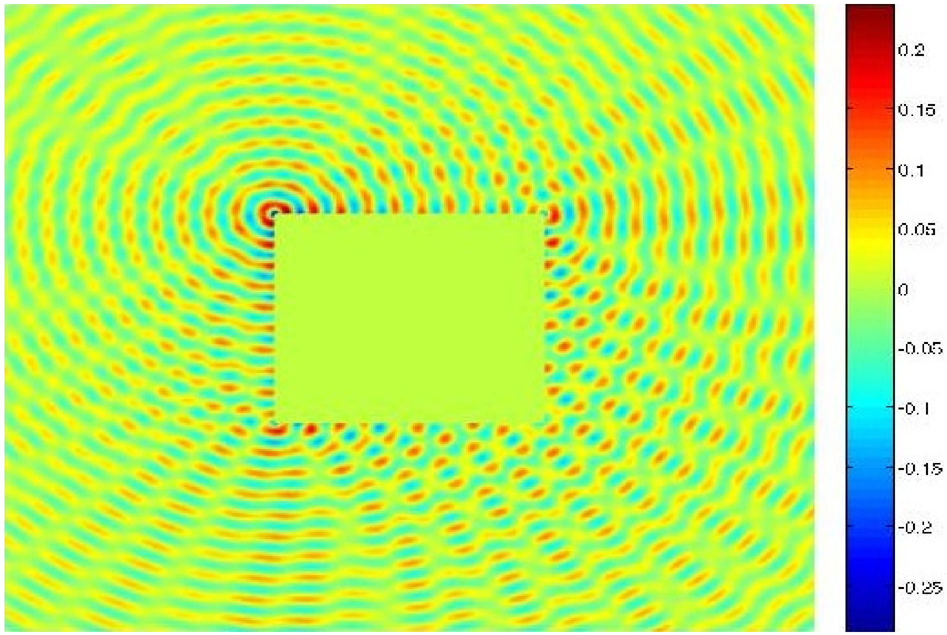


Figure 2: square, $k = 10$

Part of the scattered field

Correction to the Kirchhoff approximation

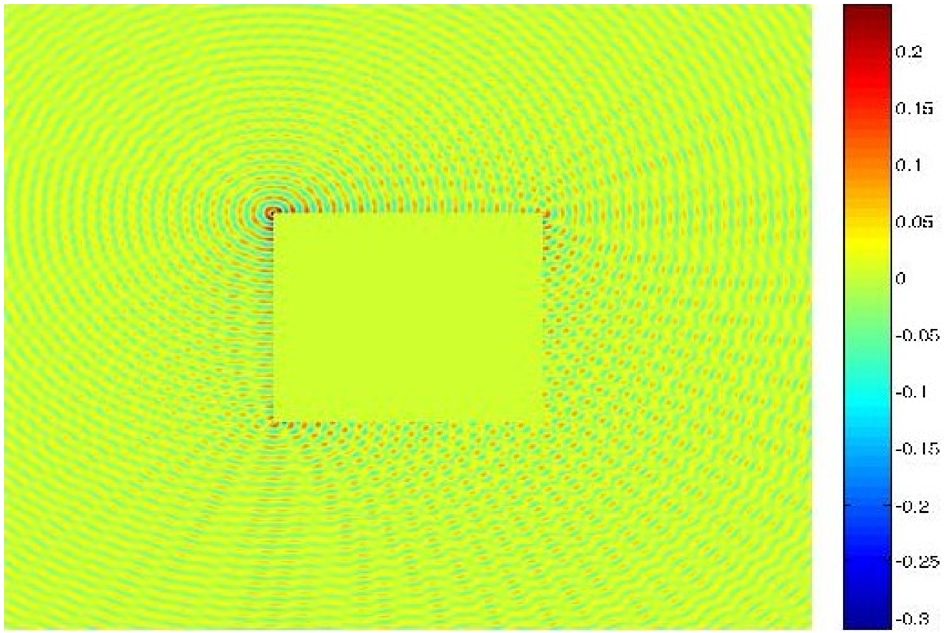


Figure 3: square, $k = 20$

This afternoon's talk

For simulation of wave scattering, solving the **Helmholtz equation**

$$\Delta u + k^2 u = 0,$$

the number of degrees of freedom in a conventional boundary element method (BEM) or finite element method needs to increase as the **wave number** $k = \dfrac{2\pi}{\lambda}$ increases.

- In the BEM context, can we avoid this by using clever basis functions, e.g. solutions of the Helmholtz equation or solutions of the Helmholtz equation multiplied by standard basis functions?
- Does it help if we know enough about high frequency ($k \to \infty$) asymptotics of the solution? What is this high frequency behaviour?
- By doing this, is a solver achievable with $O(1)$ cost in the limit as $k \to \infty$?

In fact, can we achieve

'prescribed error tolerances within fixed computational times for scattering problems of arbitrarily high frequency'

to quote from the title of Bruno, Geuzaine, Monro, and Reitich, Phil Trans R Soc Lond A (2004) [7]

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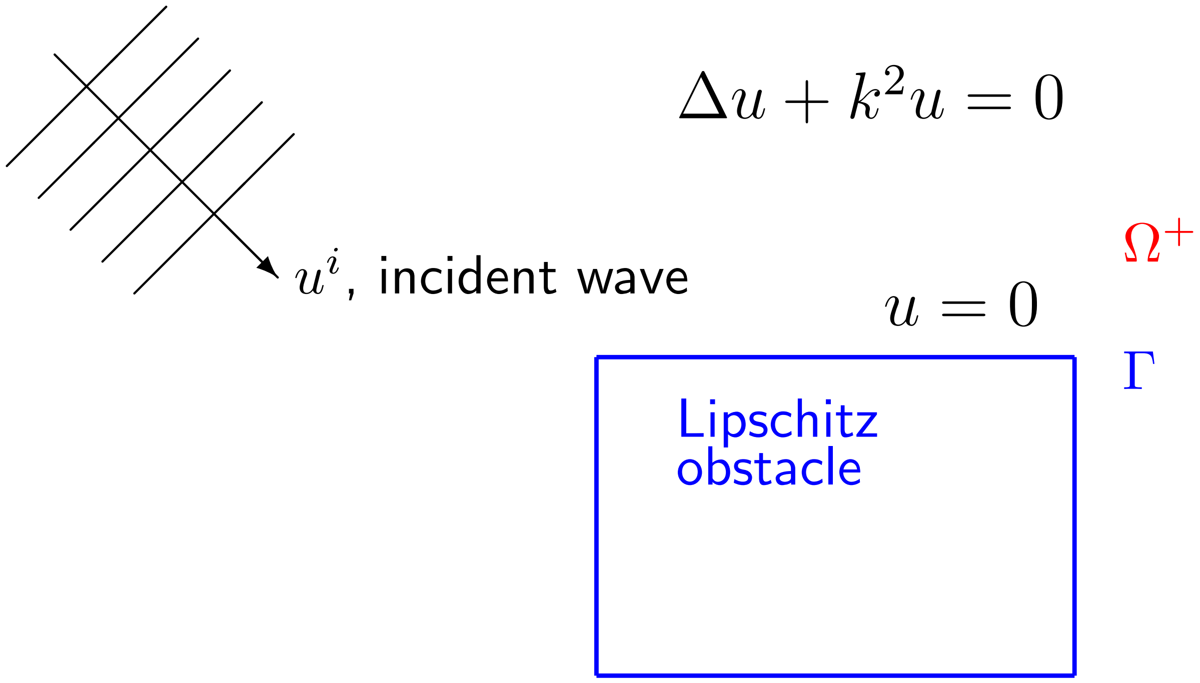
The answer will be:

- for some 2D problems, definitely yes, or at least something very close to this
- for significant classes of 3D problems probably yes; certainly some significant improvement on conventional methods is possible, and this is a promising research area

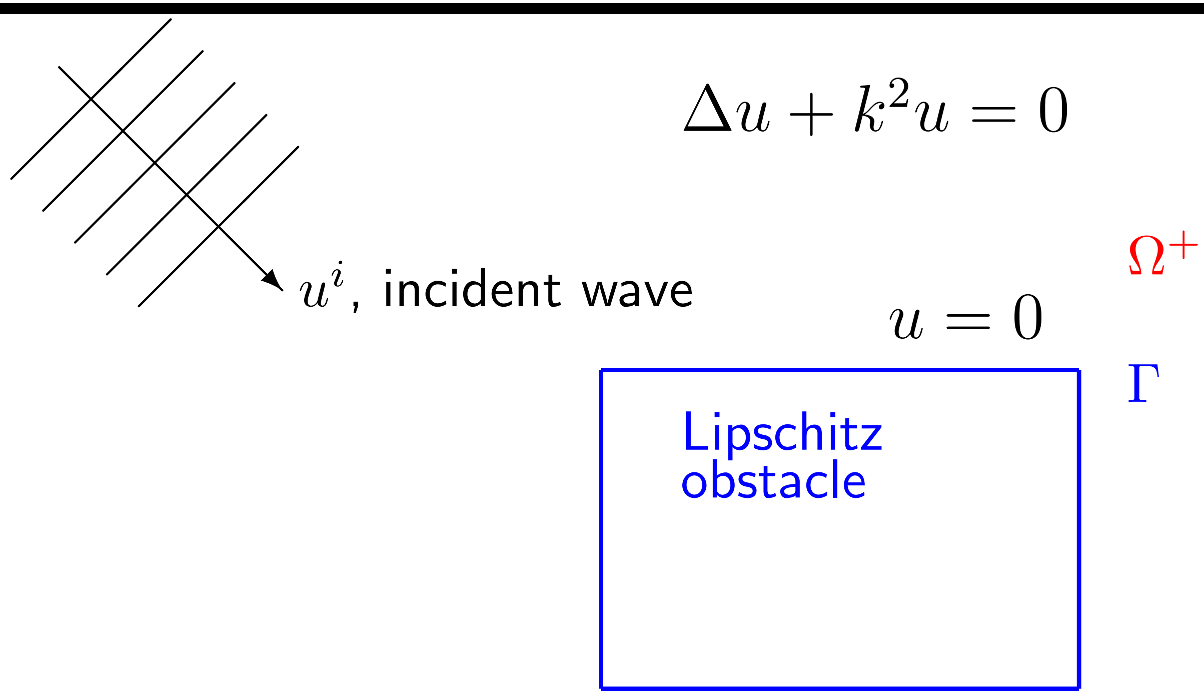
This is a Currently Popular Area, e.g.

- BIRS, Advances in Computational Scattering, Banff, February 2006
- Zurich Summer School on Numerical Methods for High Frequency Wave Propagation, August 2006, ETH Zurich
- 23rd Annual GAMM-Seminar, Integral Equation Methods for High Frequency Scattering Problems, January 2007, Leipzig
- Oberwolfach, Computational Electromagnetics and Acoustics, February 2007
- Our own programme, many of its participants working in this or very closely related areas (Bruno, Buffa, C-W, Engquist, Ganesh, Graham, Hiptmair, Huybrechs, Langdon, Monk, Sloan, Smyshlyaev, Vandewalle)

The Scattering Problem



$$\Delta u + k^2 u = 0$$

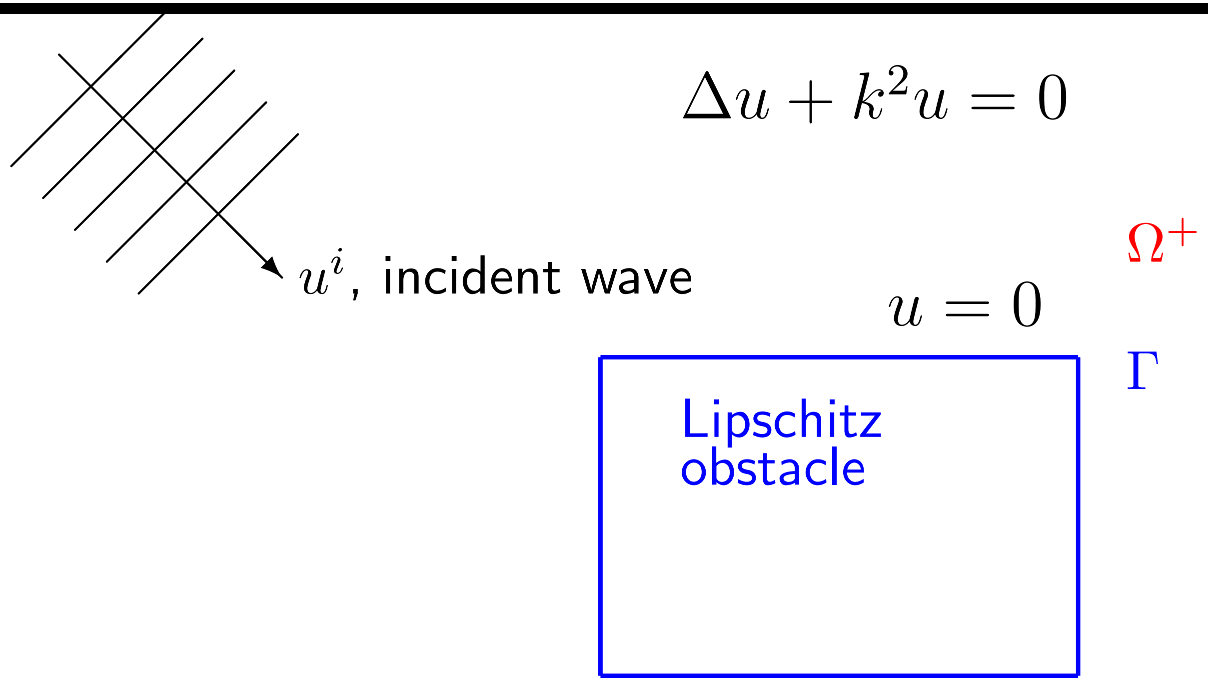


Green's representation theorem:

$$u(x) = u^i(x) - \int_\Gamma G(x,y) \frac{\partial u}{\partial n}(y) ds(y), \quad x \in \Omega^+,$$

where

$$G(x,y) := \tfrac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ (2D)}, \qquad := \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$

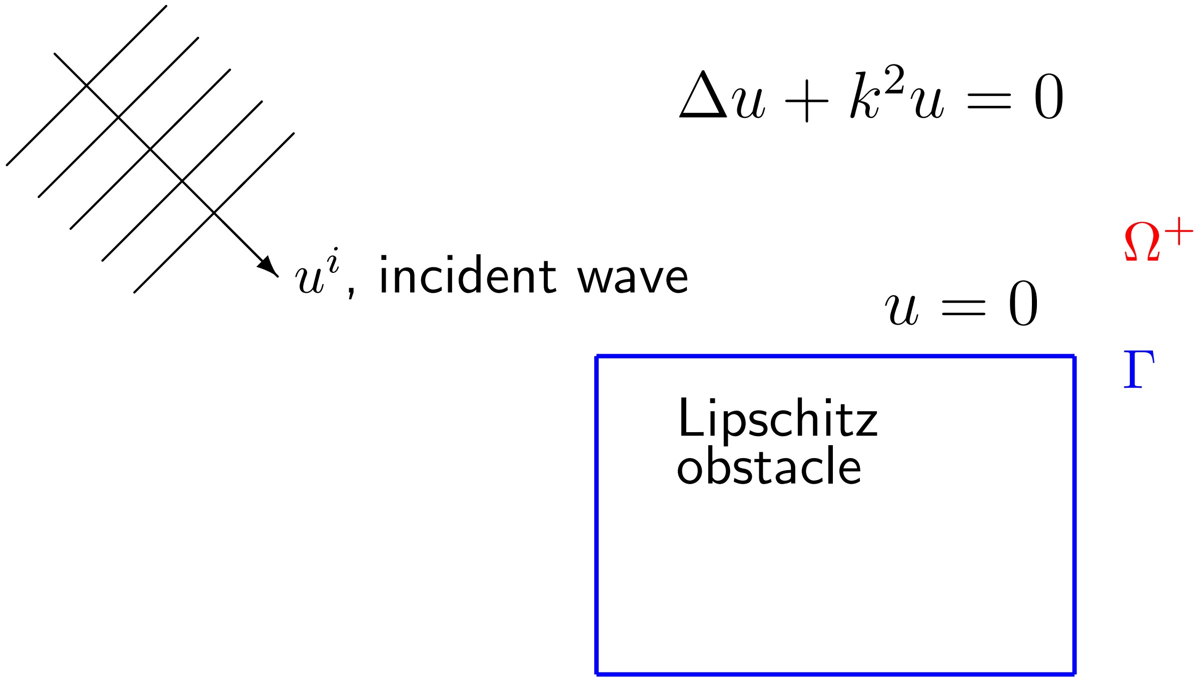


Taking a linear combination of Dirichlet and Neumann traces of the previous equation, we get the **boundary integral equation**

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial G(x,y)}{\partial n(x)} + \mathrm{i}\eta G(x,y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma,$$

where

$$f(x) := \frac{\partial u^i}{\partial n}(x) + \mathrm{i}\eta u^i(x).$$



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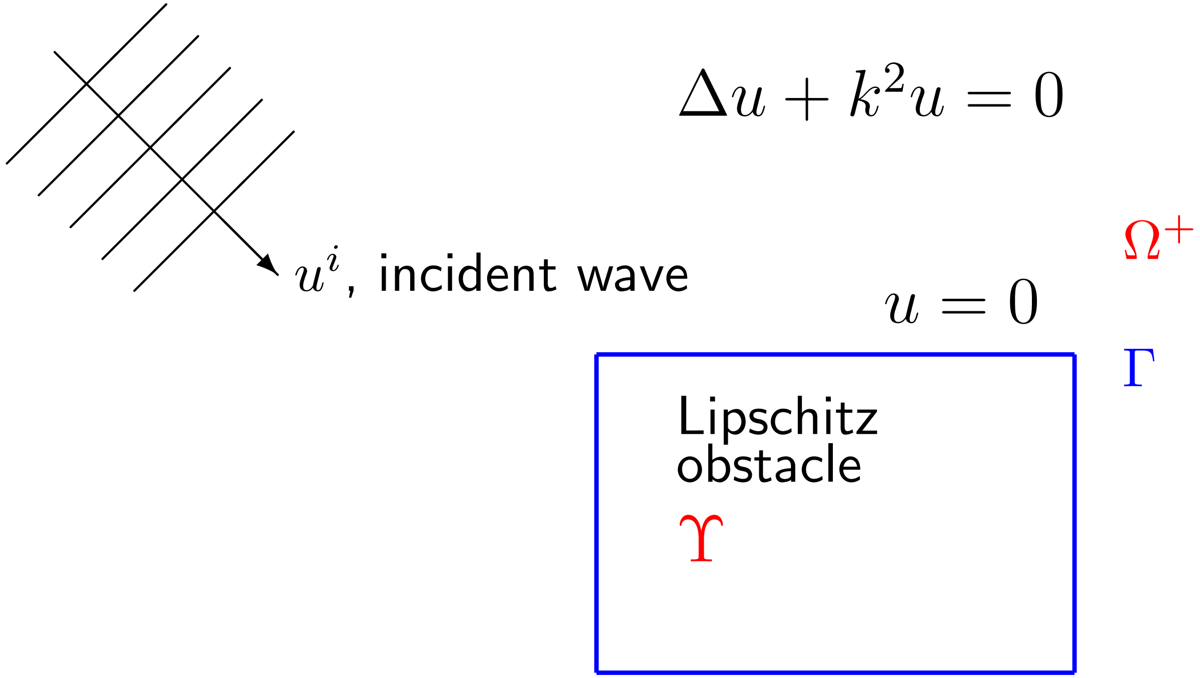
Theorem (Mitrea [28]) If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$.

$$\Delta u + k^2 u = 0$$



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Theorem If $\eta \in \mathbb{R}$, $\eta \neq 0$, then this integral equation is uniquely solvable in $L^2(\Gamma)$. In fact (C-W & Monk [9]), if Υ is **starlike** and $\eta = k$ then the inverse operator is bounded independently of k, e.g. $\|\frac{\partial u}{\partial n}\|_2 \leq 11\,\|f\|_2$ for a square.

$$\frac{1}{2}\frac{\partial u}{\partial n}(x) + \int_\Gamma \left(\frac{\partial G(x,y)}{\partial n(x)} + \mathrm{i}\eta G(x,y) \right) \frac{\partial u}{\partial n}(y) ds(y) = f(x), \quad x \in \Gamma.$$

Conventional BEM: Approximate $\partial u / \partial n$ by a piecewise polynomial, i.e.

$$\frac{\partial u}{\partial n}(x) \approx \sum_{j=1}^{N} a_j \mathbf{b}_j(x),$$

where $\mathbf{b}_1(x), \ldots, \mathbf{b}_N(x)$ are the piecewise polynomial basis functions (more precisely, if the boundary is curved, these functions are the images of conventional FEM basis functions under a mapping from a reference element in $\mathbb{R}^{d-1}$ to Γ).

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Applying a **Galerkin method** or a **collocation method** (which means: stick the approximation into the integral equation and force the integral equation to hold at N carefully chosen points – the **collocation points**) we get a linear system to solve with N degrees of freedom, namely the unknown values of $a_1, \ldots, a_N$.

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Conventional BEM: Apply a Galerkin method, approximating $\partial u/\partial n$ by a piecewise polynomial of degree P, leading to a linear system to solve with N degrees of freedom. **Problem:** N of order of $(kL)^{d-1}$, where L is a linear dimension, and cost is $O(N^2)$ to compute full matrix and apply iterative solver.

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Conventional BEM: Apply a Galerkin method, approximating $\partial u/\partial n$ by a piecewise polynomial of some degree p, leading to a linear system to solve with N degrees of freedom. **Problem:** N of order of kL^{d-1}, where L is linear dimension, and cost is $O(N^2)$ to compute full matrix and apply iterative solver. ... or close to $O(N)$ if a fast multipole method is used.

This is **fantastic** but still infeasible as $kL \to \infty$.

Alternative: Reduce N by using new basis functions, namely oscillatory basis functions which can represent the solution well. Specifically, let's try an approximation of the form

$$\frac{\partial u}{\partial n}(x) \approx \sum_{i=1}^{M} \sum_{j=1}^{N_i} a_{ij} \mathrm{e}^{\mathrm{i}kg_i(x)} \mathbf{b}_{ij}(x),$$

with $a_{ij} \in \mathbb{C}$ the unknown coefficients,

$g_1(x), \ldots, g_M(x)$ known **phase functions**,

$\mathbf{b}_{ij}(x)$ **conventional BEM basis functions**.

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Moreover, let's have a total #dof $N = \sum_{i=1}^{M} N_i$ much less than in the conventional BEM, maybe even $N = O(1)$ as $k \to \infty$, the **'high frequency $O(1)$ algorithm'** holy grail.

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All the implementations I will describe have $g_i(x) = x \cdot \hat{d}_i$, for some unit vector $\hat{d}_i$, so

$$\mathrm{e}^{\mathrm{i}kg_i(x)} = \exp(\mathrm{i}kx \cdot \hat{d}_i)$$

is a **plane wave** travelling in direction $\hat{d}_i$.

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The Plan: let's have a total #dof $N = \sum_{i=1}^{M} N_i$ which is $N = O(1)$ as $k \to \infty$, and then we will have achieved the **'high frequency $O(1)$ CPU time algorithm'** holy grail.

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No! Unfortunately, $N = O(1) \not\Rightarrow$ CPU time $= O(1)$.

The Snag: our N^2 matrix entries are highly oscillatory integrals

E.g. if the integral equation is

$$\int_\Gamma G(x,y)\frac{\partial u}{\partial n}(y)ds(y) = f(x), \quad x \in \Gamma,$$

and we use a collocation method, collocating at points x_ℓ, $\ell = 1, \ldots, N$, then the matrix entries have the form

$$\int_{\Gamma_{ij}} G(x_\ell, y)\exp(\mathrm{i}kg_i(y))\mathbf{b}_{ij}(y)\, ds(y)$$

where Γ_{ij} is the support of b_{ij}.

If $N = O(1)$ then, where $h = \max_{ij}\mathrm{diam}(\Gamma_{ij})$, necessarily $kh = 2\pi h/\lambda \to \infty$ as $k \to \infty$.

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$$\int_{\Gamma_{ij}} \frac{1}{4\pi|x_\ell - y|}\exp[\mathrm{i}k(|x_\ell - y| + g_i(y))]\mathbf{b}_{ij}(y)\,ds(y)$$

where Γ_{ij} is the support of b_{ij}.

The integrand is increasingly oscillatory as $k \to \infty$ but at least we **know what this oscillation is**.

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and we use a **Galerkin method**, then the matrix entries have the form (in 3D)

$$\int_{\Gamma_{ij}}\int_{\Gamma_{mn}} \frac{1}{4\pi|x-y|}\exp[\mathrm{i}k(|x-y|+g_i(y)-g_m(x))]\mathbf{b}_{ij}(y)\mathbf{b}_{mn}(x)\,ds(y)ds(x).$$

Each entry is a 4-dimensional, increasingly oscillatory integral as $k \to \infty$.

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Recent research on evaluation of oscillatory integrals is developing tools to attack these problems – numerical stationary phase and steepest descent methods. See Iserles et al. [20, 21, 22], Levin [26], Bruno et al. [7, 6], Huybrechs et al. [18], Ganesh, Langdon, Sloan [17], Langdon & Melenk in preparation.

How are people choosing $\hat{d}_i$ and b_{ij}??

$$\frac{\partial u}{\partial n}(x) \approx \sum_{i=1}^{M} \sum_{j=1}^{N_i} a_{ij} \exp(\mathrm{i}kx \cdot \hat{d}_i) \mathbf{b}_{ij}(x),$$

with $a_{ij} \in \mathbb{C}$ the unknown coefficients,
$\hat{d}_1, \ldots, \hat{d}_N$ distinct unit vectors,
$\mathrm{b}_{ij}(x)$ **conventional BEM basis functions**.

Approach 1. M large.

Approach 2. $M = 1$.

Approach 3. M small, directions $\hat{d}_i$ carefully chosen to match high frequency solution behaviour.

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with $\hat{d}_1, \ldots, \hat{d}_N$ distinct unit vectors and $\mathbf{b}_{ij}(x)$ conventional BEM basis functions.

Approach 1. Fix $N_i = N^*$ so $N = MN^*$, use conventional, fixed degree boundary elements on a (usually uniform) mesh, and have M largish (e.g. 18 in 2D, 200 in 3D) and the directions $\hat{d}_i$ uniformly spread, e.g., in 2D ($d = 2$),

$\hat{d}_i = (\cos(2\pi i/N^*), \sin(2\pi i/N^*))$, $i = 1, \ldots, N^*$.

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This is very successful, reducing number of degrees of freedom per wavelength from e.g. 6-10 to close to 2. However N still increases proportional to kL. There are also severe conditioning problems (the basis is almost linearly dependent). See de La Bourdonnaye et al. [13, 14], Perrey-Debain et al. [31, 32, 30, 33].

Some similarities to conventional high order (p large) BEMs.

How are people choosing $\hat{d}_i$ and $\mathbf{b}_{ij}$??

$$\frac{\partial u}{\partial n}(x) \approx \sum_{i=1}^{M} \sum_{j=1}^{N_i} a_{ij} \exp(\mathrm{i}kx \cdot \hat{d}_i) \mathbf{b}_{ij}(x),$$

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Approach 2. $M = 1$.

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$$\frac{\partial u}{\partial n}(x) \approx \exp(\mathrm{i}kx \cdot \hat{d}) \sum_{j=1}^{N^*} a_j \mathbf{b}_j(x),$$

with $\mathbf{b}_j(x)$ conventional BEM basis functions.

Approach 2. $M = 1$, with $\hat{d}$ the direction of the incident plane wave.

How are people choosing $\hat{d}_i$ and $\mathbf{b}_{ij}$??

$$\frac{\partial u}{\partial n}(x) \approx \exp(\mathrm{i}kx \cdot \hat{d}) \sum_{j=1}^{N^*} a_j \mathbf{b}_j(x),$$

with $\mathbf{b}_j(x)$ conventional BEM basis functions.

Approach 2. $M = 1$, with $\hat{d}$ the direction of the incident plane wave. In other words, we remove some of the oscillation by **factoring out the oscillation of the incident wave**. A slight variant on this is to write

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times \mu(y)$$

and then approximate μ by a conventional BEM.

Approach 2. We remove some of the oscillation by **factoring out the oscillation of the incident wave**, e.g.

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times \mu(y) \quad (*)$$

and then approximate μ by a conventional BEM.

For **smooth convex obstacles** this should work well: equation $(*)$ holds with $F(y) \approx 2$ on the illuminated side and $F(y) \approx 0$ in the shadow zone (this is the high frequency **Kirchhoff** or **physical optics** approximation).

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This is an old idea (e.g. R J Uncles, Dundee NA Report 1974), but has seen sophisticated analysis, algorithmic ideas, and numerical analysis applied in recent years, see Zhou et al. [1], Darrigrand [12], Bruno et al. [7, 8, 6], Dominguez et al. [15], Ecevit [16, 3], Huybrechs and Vanderwalle [19].

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To understand how algorithms in this class work we have to look at the solution to scattering by smooth convex obstacles - in fact let us digress and look at high frequency asymptotics more generally.

The Geometrical Theory of Diffraction – see Keller et al. [24, 23]

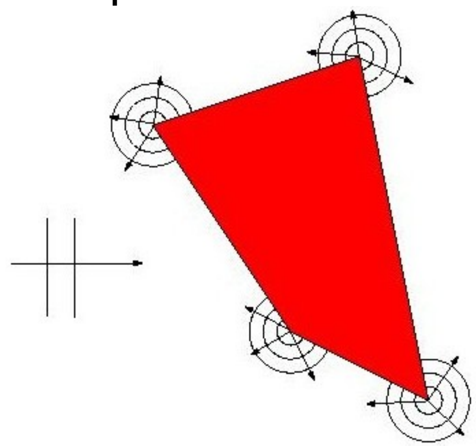
A partly heuristic, semi-rigorous theory, whose principles are:

- At high frequency a ray model is appropriate
- The paths of rays are determined by Fermat's principle, i.e. rays take the quickest route
- Phase of the field on a ray is determined by distance along the ray, i.e. $u(x) = |u(x)|\mathrm{e}^{\mathrm{i}ks}$, s distance along ray
- Localization: interaction with obstacles depends only on the geometry local to the point where the ray hits the obstacle, and so can be determined by solving **canonical scattering problems**

Two Examples. If obstacle has corners then rays are reflected from sides but also diffracted from corners. Each diffracted ray (in 2D) has the form:

$$u^{diff}(x) = u^i(x_c)D(\theta,\theta_0)\frac{e^{ikr}}{\sqrt{r}}$$

where x_c is the corner, (r,θ) are polar coordinates of x relative to the corner (i.e. of $x - x_c$), θ_0 is the angle of incidence and $D(\theta,\theta_0)$ is a diffraction coefficient which depends on the local geometry.



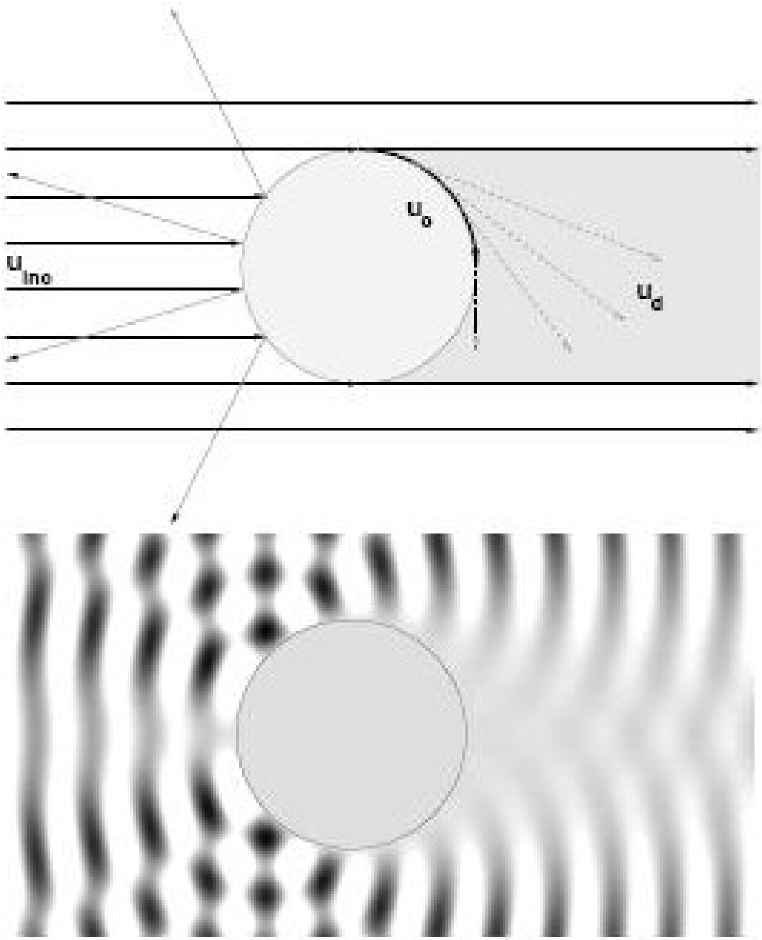


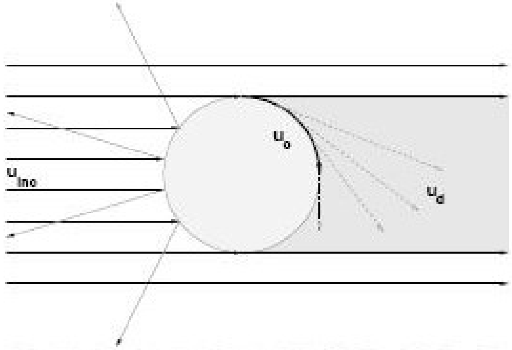
Figure 4: If obstacle is smooth then reflected and creeping rays are generated (graphic from [29]).

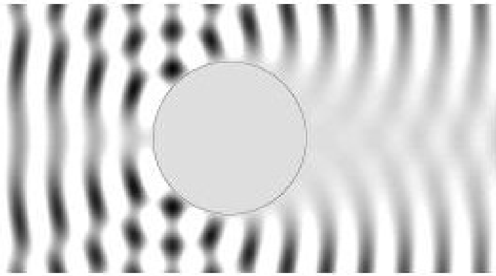
Exact and/or rigorous High Frequency Asymptotics??

There exist very powerful formal methods for generating high frequency asymptotics, e.g. the method of matched asymptotic expansions [23].

Exact solutions are known for simple geometries, mainly 2D, which are a strong guide to general behaviour, e.g. work of Rawlins on diffraction by knife-edges, wedges, etc.

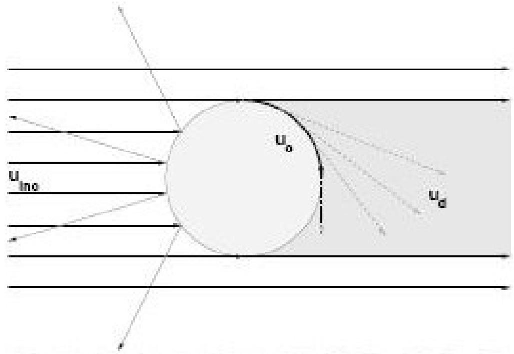
A little exact, rigorous asymptotics is known for general scatterers. E.g. scattering by a smooth, convex, positive curvature obstacle in 2D/3D (Melrose and Taylor [27]).

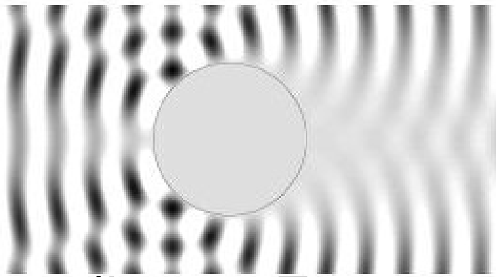




Rigorous asymptotics [27] predicts on Γ:

- Kirchhoff approximation works on illuminated side, i.e. $\dfrac{\partial u}{\partial n} \approx 2\dfrac{\partial u^i}{\partial n}$ (for $u = 0$)



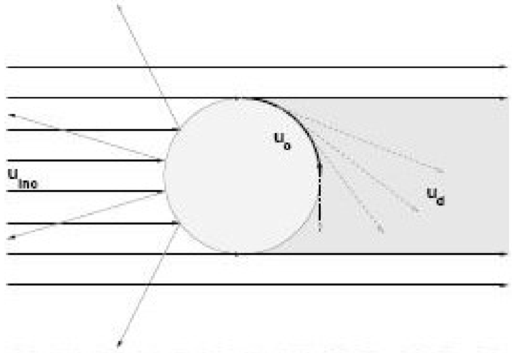


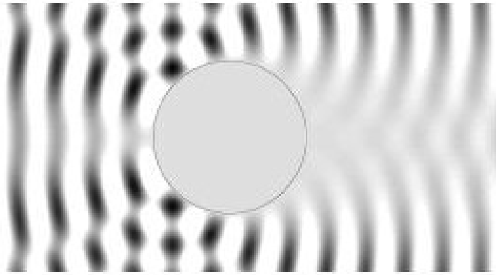
Rigorous asymptotics [27] predicts on Γ:

- on the shadow side there are two creeping rays, the normal derivative of each creeping ray field having the form

$$\frac{\partial u^{creep}}{\partial n}(x) = A \exp(\mathrm{i}(ks - C_0 F(s) k^{1/3} s)) \exp(-C_1 F(s) k^{1/3} s),$$

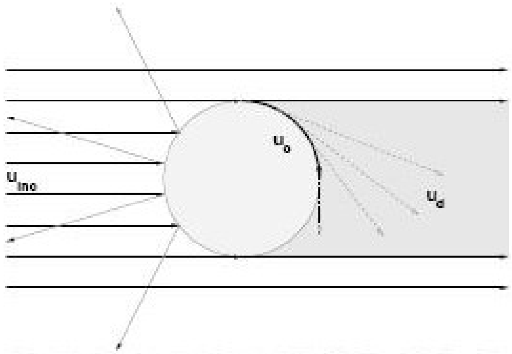
  where C_0 and C_1 are known positive constants, s is arc-length, and $c_1 s \leq F(s) \leq c_2 s$

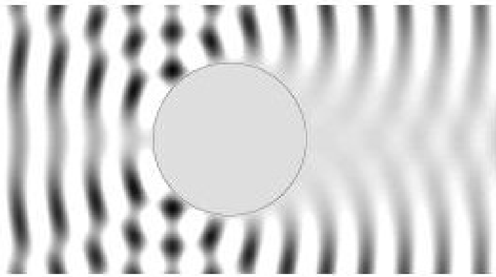




Rigorous asymptotics [27] predicts on Γ:

- something complicated happens in the so-called **transition zones**, or **Fock-Leontovich** zones, around the tangency points (the North and South poles), in intervals of length $\approx R^{2/3}k^{-1/3}$ around the tangency points, where R is the radius of curvature at the tangency point. (Complicated, but smooth on the length scale $R^{2/3}k^{-1/3}$.)





Rigorous asymptotics [27] predicts on Γ:

- For further details see Melrose and Taylor [27] (which is pretty incomprehensible to me), or see Dominguez, Graham, Smyshlyaev [15] (where [27] is interpreted for numerical analysts, though I don't follow completely how they get their Theorem 5.1 from [27]).

Approach 2. We remove some of the oscillation by **factoring out the oscillation of the incident wave**, e.g.

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times \mu(y) \quad (*)$$

and then approximate μ by a conventional BEM.

For **smooth convex obstacles** this should work well: equation $(*)$ holds with $F(y) \approx 2$ on the illuminated side and $F(y) \approx 0$ in the shadow zone (this is the high frequency **Kirchhoff** or **physical optics** approximation).

To understand how algorithms in this class work we have to look at the solution to scattering by smooth convex obstacles - in fact let us digress and look at high frequency asymptotics more generally.

Approach 2. We remove some of the oscillation by **factoring out the oscillation of the incident wave**, e.g.

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times \mu(y) \quad (*)$$

and then approximate μ by a conventional BEM.

The research splits into two groups:

Group 1. Use a quasi-uniform mesh BEM to approximate μ, see Zhou et al. [1], where it is shown that the error is

$$N^{-p} + (k^{1/3}/N)^{p+1}$$

in 2D, using polynomial degree p BEMs, and see Darrigrand [12] for impressive 3D implementations (including for an aircraft wing).

Approach 2. We remove some of the oscillation by **factoring out the oscillation of the incident wave**, e.g.

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times \mu(y) \quad (*)$$

and then approximate μ by a conventional BEM.

Group 2 (2D only). Ignore the deep shadow zone (where field is zero), use a standard spectral approximation on the illuminated side, and then a refined mesh or spectral approximation in the transition zones of width $k^{-1/3}$.

See Bruno et al. [7, 6, 8], Huybrechs and Vanderwalle [18] for numerical results which suggest $N = O(1)$ works, and Dominguez, Graham Smyshlyaev [15] ditto, plus rigorous numerical analysis which shows $N = O(k^{1/9+\epsilon})$ works.

How are people choosing $\hat{d}_i$ and $\mathbf{b}_{ij}$??

$$\frac{\partial u}{\partial n}(x) \approx \sum_{i=1}^{M} \sum_{j=1}^{N_i} a_{ij} \exp(\mathrm{i}kx \cdot \hat{d}_i)\mathbf{b}_{ij}(x),$$

with $a_{ij} \in \mathbb{C}$ the unknown coefficients,
$\hat{d}_1, \ldots, \hat{d}_N$ distinct unit vectors,
$\mathbf{b}_{ij}(x)$ **conventional BEM basis functions**.

Approach 3 (2D so far). M small, directions $\hat{d}_i$ carefully chosen on the basis of the geometrical theory of diffraction to match high frequency solution behaviour.

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Approach 3 (2D so far). M small, directions $\hat{d}_i$ carefully chosen on the basis of the geometrical theory of diffraction to match high frequency solution behaviour.

E.g. Bruno et al. [7] suggest how this might work for a (not too) non-convex obstacle, and see Bruno et al. [8] and Ecevit et al. [16, 3] for how this can work in a multiple scattering context.

How are people choosing $\hat{d}_i$ and b_{ij}??

$$\frac{\partial u}{\partial n}(x) \approx \sum_{i=1}^{M} \sum_{j=1}^{N_i} a_{ij} \exp(\mathrm{i}kx \cdot \hat{d}_i)\mathbf{b}_{ij}(x),$$

with $a_{ij} \in \mathbb{C}$ the unknown coefficients,
$\hat{d}_1(x), \ldots, \hat{d}_N(x)$ distinct unit vectors,
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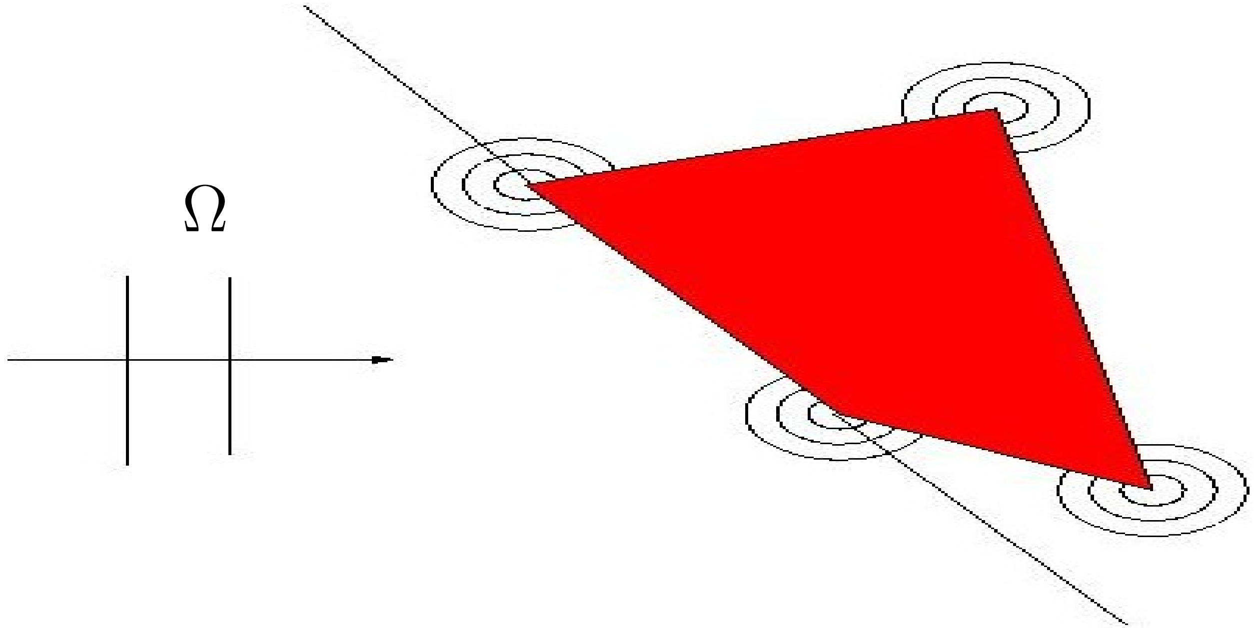
Approach 3 (2D). M small, directions $\hat{d}_i$ carefully chosen on the basis of the geometrical theory of diffraction to match high frequency solution behaviour.

C-W & Langdon have implemented and analysed methods in this vein for scattering by two specific scattering problems [11, 25, 5, 10], the second scattering by convex polygons.

A Simple Technique for Understanding Solution Behaviour for the Convex Polygon

Rigorous, high frequency asymptotics.

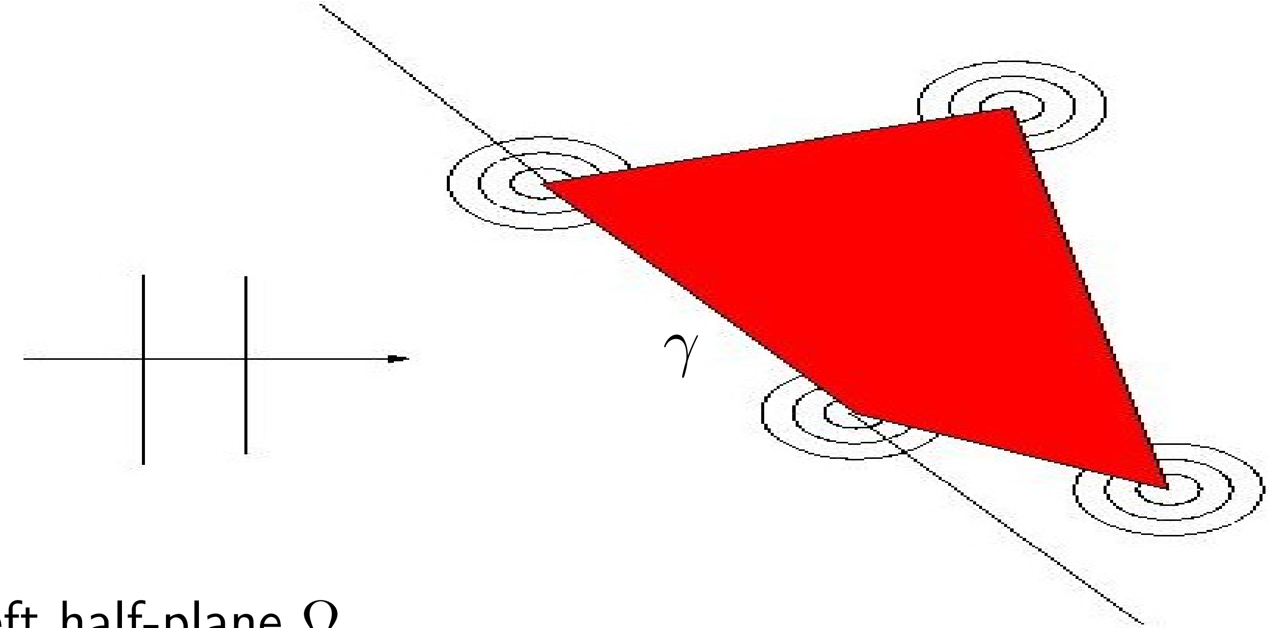


Let

$$G_D(x,y) := G(x,y) - G(x,y')$$

be the Dirichlet Green function for the left half-plane Ω. By Green's representation theorem,

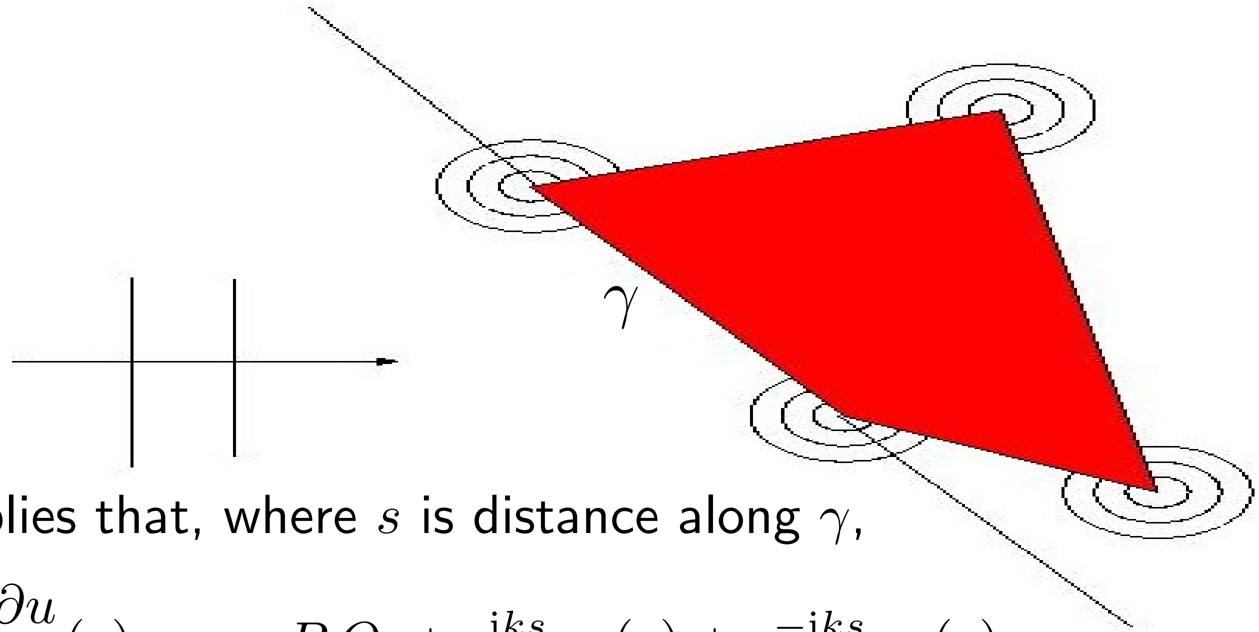
$$u(x) = u^i(x) + u^r(x) + \int_{\partial\Omega\setminus\Gamma} \frac{\partial G_D(x,y)}{\partial n(y)} u(y)ds(y), \quad x \in \Omega,$$



In the left half-plane Ω,

$$u(x) \quad = \quad u^i(x) + u^r(x) + \int_{\partial\Omega\setminus\Gamma} \frac{\partial G_D(x,y)}{\partial n(y)} u(y) ds(y)$$

$$\Rightarrow \frac{\partial u}{\partial n}(x) = 2\frac{\partial u^i}{\partial n}(x) + 2\int_{\partial\Omega\setminus\Gamma} \frac{\partial^2 G(x,y)}{\partial n(x)\partial n(y)} u(y) ds(y), \quad x \in \gamma = \partial\Omega \cap \Gamma.$$



This implies that, where s is distance along γ,

$$\frac{\partial u}{\partial n}(s) \quad = \quad P.O. + \mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s)$$

where

$$k^{-n} |v_+^{(n)}(s)| = O\left((ks)^{-1/2-n}\right) \text{ as } ks \to \infty$$

and (by separation of variables local to the corner),

$$k^{-n} |v_+^{(n)}(s)| = O\left((ks)^{-\alpha-n}\right) \text{ as } ks \to 0,$$

where $\alpha < 1/2$ depends on the corner angle.

A Numerical Scheme for the Convex Polygon Which Uses this Precise Understanding of Solution Behaviour

$$\frac{\partial u}{\partial n}(s) \quad = \quad P.O. + \mathrm{e}^{\mathrm{i}ks} v_+(s) + \mathrm{e}^{-\mathrm{i}ks} v_-(s)$$

where

$$k^{-n}|v_+^{(n)}(s)| = \begin{cases} O\left((ks)^{-1/2-n}\right) & \text{as } ks \to \infty \\ O\left((ks)^{-\alpha-n}\right) & \text{as } ks \to 0, \end{cases}$$

where $\alpha < 1/2$ depends on the corner angle.

Thus approximate

$$\frac{\partial u}{\partial n}(s) \quad \approx \quad P.O. + \mathrm{e}^{\mathrm{i}ks} V_+(s) + \mathrm{e}^{-\mathrm{i}ks} V_-(s),$$

where V_+ and V_- are piecewise polynomials on graded meshes, i.e. linear combinations of standard boundary element basis functions.

$$k^{-n}|v_+^{(n)}(s)| = \begin{cases} O\left((ks)^{-1/2-n}\right) & \text{as } ks \to \infty \\ O\left((ks)^{-\alpha-n}\right) & \text{as } ks \to 0. \end{cases}$$

Thus approximate

$$\frac{\partial u}{\partial n}(s) \quad \approx \quad P.O. + \mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s),$$

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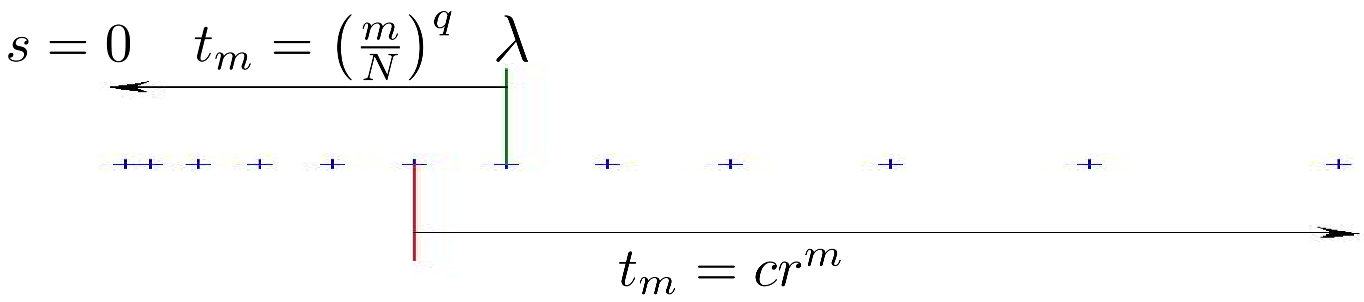

Thus approximate

$$\frac{\partial u}{\partial n}(s) \quad \approx \quad P.O. + \mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s),$$

where V_+ and V_- are piecewise polynomials on graded meshes.

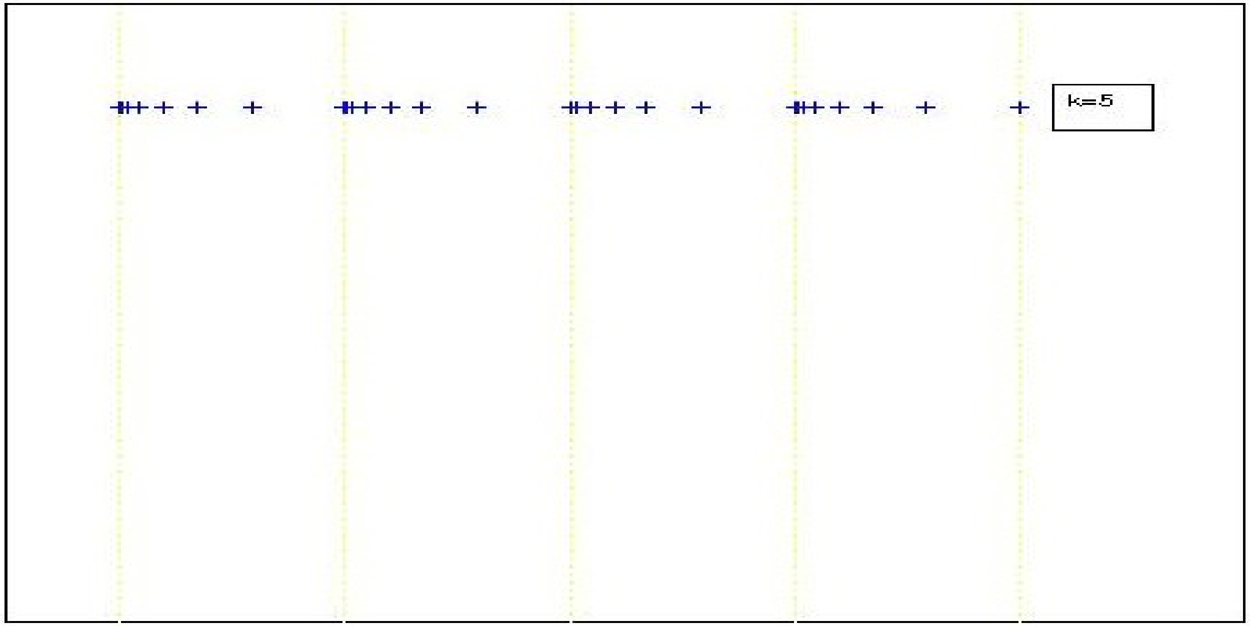

Figure 5: Scattering by a square

Thus approximate

$$\frac{\partial u}{\partial n}(s) \quad \approx \quad P.O. + \mathrm{e}^{\mathrm{i}ks}V_+(s) + \mathrm{e}^{-\mathrm{i}ks}V_-(s),$$

where V_+ and V_- are piecewise polynomials on graded meshes.

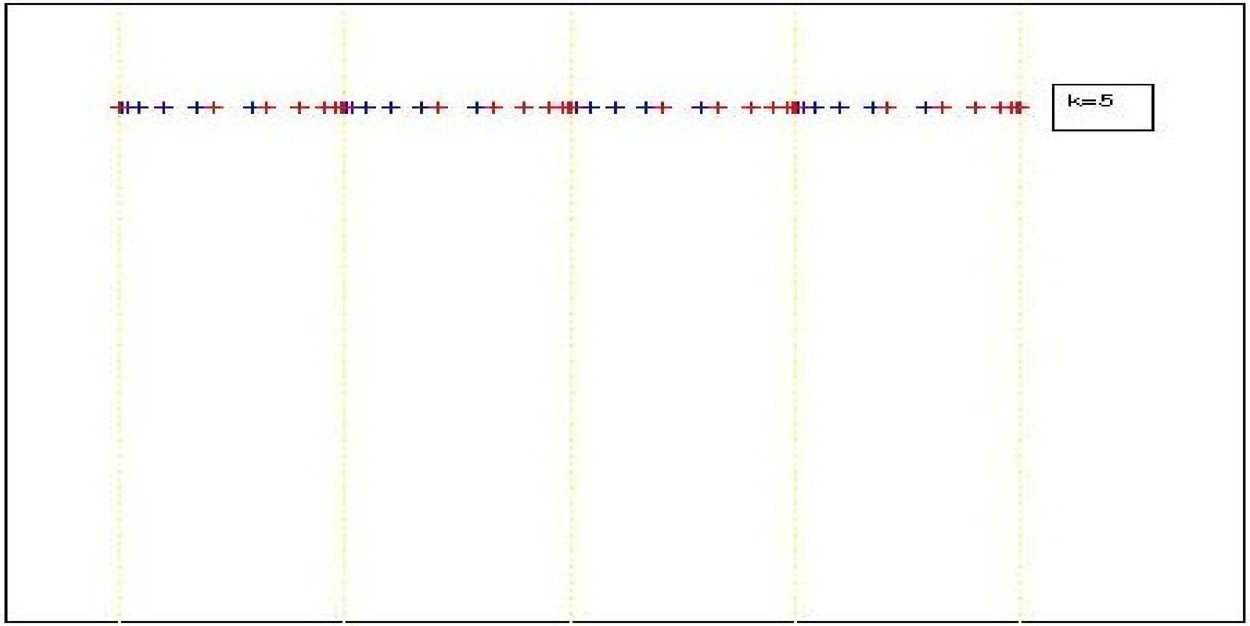

Figure 6: Scattering by a square

Thus approximate

$$\frac{\partial u}{\partial n}(s) \quad \approx \quad P.O. + \mathrm{e}^{\mathrm{i}ks} V_+(s) + \mathrm{e}^{-\mathrm{i}ks} V_-(s),$$

where V_+ and V_- are piecewise polynomials on graded meshes.

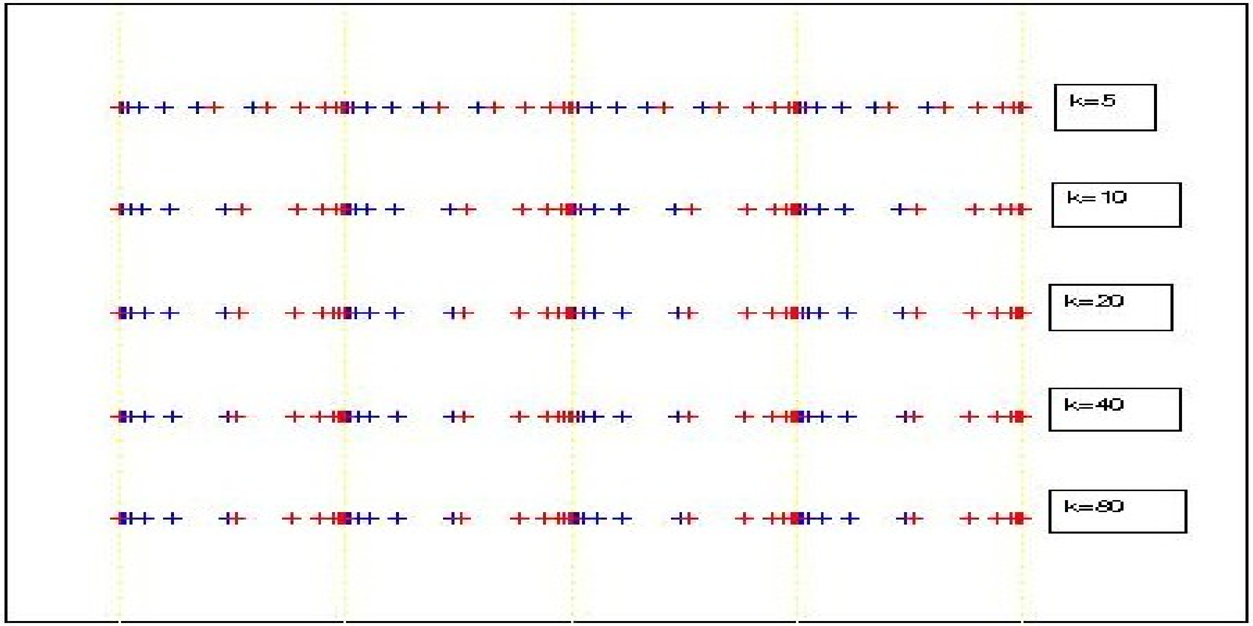

Figure 7: Scattering by a square

Thus approximate

$$\frac{\partial u}{\partial n}(s) \quad \approx \quad P.O. + e^{\mathrm{i}ks} V_+(s) + e^{-\mathrm{i}ks} V_-(s),$$

where V_+ and V_- are piecewise polynomials on graded meshes.

Theorem Where $\phi = \frac{\partial u}{\partial n}$, ϕ_N is the best L_2 approximation to ϕ from the approximation space, n is the number of sides, N the degrees of freedom, p the polynomial degree, and L the total arc-length,

$$k^{-1/2} ||\phi - \phi_N||_2 \leq C \sup_{x \in D} |u(x)| \frac{[n(1 + \log(kL/n))]^{p+3/2}}{N^{p+1}},$$

where C depends (only) on the corner angles and p.

Table 1: Relative errors, $k = 10$

k	N (#dof)	$\|\phi - \phi_N\|_2 / \|\phi\|_2$	EOC
10	24	$1.1187 \times 10^{+0}$	1.5
	48	4.0499×10^{-1}	0.7
	88	2.5348×10^{-1}	0.9
	176	1.3979×10^{-1}	1.3
	360	5.5216×10^{-2}	0.9
	712	3.0358×10^{-2}	

Table 2: Relative errors, $k = 160$

k	N (#dof)	$\|\phi - \phi_N\|_2/\|\phi\|_2$	EOC
160	32	$1.0350\times10^{+0}$	1.3
	56	4.2389×10^{-1}	0.5
	120	3.0406×10^{-1}	0.6
	240	2.0471×10^{-1}	1.5
	472	7.3763×10^{-2}	1.0
	944	3.6983×10^{-2}	

Summary/Conclusions

We've reviewed recent work on high frequency scattering that:

- Reduces the # D.O.F. by using oscillatory basis functions, e.g. plane waves $\times$ polynomials
- In many cases uses high frequency asymptotics, at least to deduce the **phases/oscillation** of important components of the field
- Requires ideas from asymptotics (stationary phase, steepest descents) to evaluate the integrals that arise
- Knowledge of rigorous high frequency asymptotics of solution and e.g. norms of integral operators and their inverses to prove complete numerical analysis results

Some Open Problems

- Dependence of norm of integral operators on k: the L^2 norm known to be $O(k^{1/3})$ [15] as $k \to \infty$ for circle/sphere, but only a crude $O(k)$ estimate known for general geometries.
- For the Dirichlet problem (and $\eta = k$) the norm of inverse operator is $O(1)$ as $k \to \infty$ if object is starlike [9], but what happens for trapping obstacles? (It is known, and hardly surprising, that conditioning is worse in domain formulations [9].)
- Design of new integral equation formulations/preconditioners which give $O(1)$ conditioning as $k \to \infty$. Some steps in this direction are [4, 2].
- For the Dirichlet case (and $\eta = k$), uniform coercivity of the operator is known [15] as $k \to \infty$ for a circle/sphere, but what about more general obstacles?

- Construction of hybrid asymptotic-numeric approximation spaces for general classes of scattering problems. Constructing spaces that are (provably) robust (i.e. $O(1)$ in DOF) with respect to k and the geometry.

- Efficient computation of oscillatory matrix entries; can we develop methods that are provably $O(1)$ in cost as $k \to \infty$, uniformly with respect to the geometry?

- Stability: can we say something about stability, not only for DOF$\to \infty$ (though even this is unknown for Galerkin method for second kind integral equations on Lipschitz domains) but also as $k \to \infty$ with DOF fixed (or growing very mildly).

Possible Interactions with Other Participants

I mentioned already that a number of participants are working in this or very closely related areas already: Bruno, Buffa, C-W, Engquist, Ganesh, Graham, Hiptmair, Huybrechs, Langdon, Monk, Sloan, Smyshlyaev, Vandewalle.

Some wider possibilities for interaction include:

- norms of oscillatory integral operators: Stein, ...
- evaluation of oscillatory integrals: Iserles, Levin, Nørsett, ...
- high frequency asymptotics, billiard flows, semi-classical physics: Nerry, Howls, Olde Daalhuis, ...
- ... ?

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