Analysis on Graphs and its Applications
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Liouville theorems for equations on coverings of graphs and manifolds

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- Kuchment, P., Pinchover, Y.: Integral representations and Liouville theorems for solutions of periodic elliptic equations, *J. Funct. Anal.* **181** (2001), 402–446.

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1. Classical Liouville Theorems

Theorem 1. *If u is a harmonic function $\Delta u = 0$ in $\mathbb{R}^n$ growing polynomially $|u(x)| \leq C(1 + |x|)^N$, then it is a polynomial of degree at most N. In particular, the dimension of the space $V_N(\Delta)$ of such solutions is equal to* $\begin{pmatrix} n+N \\ N \end{pmatrix} - \begin{pmatrix} n+N-2 \\ N-2 \end{pmatrix}$.

Theorem 2. *If u is an entire function $\overline{\partial} u = 0$ in $\mathbb{C}^n$ growing polynomially $|u(z)| \leq C(1 + |z|)^N$, then it is a holomorphic polynomial of degree at most N. In particular, the dimension of the space $V_N(\overline{\partial})$ of such solutions is equal to* $\begin{pmatrix} n+N \\ N \end{pmatrix}$.

2. **Generalizations sought for**

- *Laplace-Beltrami operators on manifolds of non-negative Ricci curvature*

  Conjecture 3. *(S. T. Yau '74)*

  For open manifold M^n with $R \geq 0$,
  $dim\, V_N(\Delta_M) < \infty$ for any N.

  Is it also true that

  $$dim\, V_N(\Delta_{M^n}) < dim\, V_N(\Delta_{\mathbb{R}^n})??$$

  The conjecture proven by Colding and Minicozzi '97 (partial results by P. Li and co-authors and by Donnely and Fefferman)

Corollary 4. *(of C&M result)*

If M *covers a compact base with nilpotent deck group, then for any* N, $dim\, V_N(\Delta_M) < \infty$.

- *More general elliptic operators*

  Avellaneda&F. H. Lin '89, Moser&Struwe '92:

$$L = \sum_{i,j} \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial}{\partial x_j} \text{ in } \mathbb{R}^n$$

  with periodic coefficients. Then

$$dim\, V_N(L) = dim\, V_N(\Delta_{\mathbb{R}^n}).$$

  Technique of homogenization theory used.

  Generalization by P. Li and J. Wang to

$$\sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

Problems:

i) why divergence type? when does the Liouville theorem hold for periodic equations?

ii) are V_N always finite-dimensional **simultaneously**?

iii) systems (e.g. overdetermined, $\overline{\partial}$)?

iv) periodic equations on abelian (virtually nilpotent) coverings?

- *Holomorphic functions*

  E.g. (V. Lin): on any nilpotent covering of a compact complex manifold $dim\, V_0(\overline{\partial}) = 1$.

  $dim\, V_N(\overline{\partial}) < \infty$ was not known even for $\mathbb{Z}$-coverings, except the Kähler case
  (A. Brudnyi).

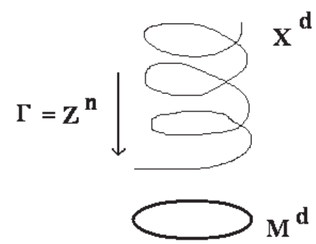
  A. Brudnyi, V. Kaimanovich, V. Lin,
  M. Zaidenberg, ...

- *Harmonic functions on coverings of manifolds, discrete groups and graphs*

  E. Dynkin, H. Furstenberg, V. Kaimanovich, F. Ledrappier, R. Lyons, G. Margulis, L. Sallof-Coste, D. Sullivan, N. Varopoulos, ...

3. **The main set-up**

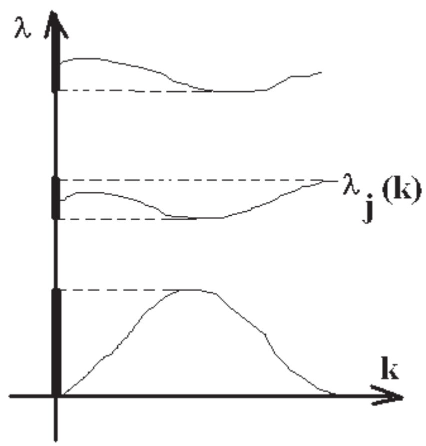
Abelian covering of a compact (Riemannian manifold, complex manifold, graph) $X^d \overset{\Gamma}{\mapsto} M^d$. L - periodic elliptic on X.



E.g., X is a crystal lattice in terms of T. Sunada.
Polynomially growing solution $Lu = 0$ (on nilpotent covering) $\Rightarrow 0 \in \sigma(L)$.

Floquet theory.

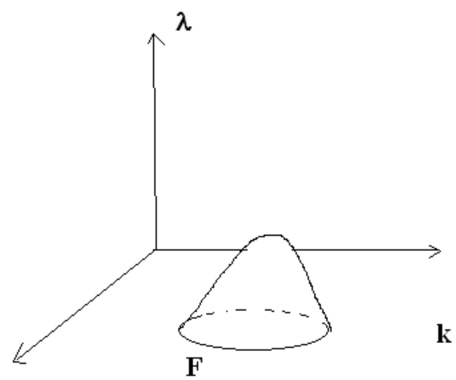
Quasimomentum $k \in \mathbb{R}^n$ (i.e., *character* $e^{ik\cdot\gamma}, \gamma \in \Gamma$).
Floquet-Bloch solutions $u(\gamma x) = e^{ik\cdot\gamma}u(x)$.
Floquet operators $L(k)$ in linear vector bundles on M.
Dispersion relation $\lambda(k)$.



Spectrum of $L =$ range of $\lambda(k)$.

Fermi surface F_L:

zero level set of the dispersion relation.
$0 \in \sigma(L) \Leftrightarrow F_L \neq \emptyset$



"Normally" inside the spectrum $\#F_L = \infty$, at the spectral edges $\#F_L < \infty$.

4. **The Liouville theorem**

Theorem 5. *(Liouville)*
The following statements are equivalent:

(a) $dim\, V_N(L) < \infty$ *for some* $N \geq 0$

(b) $dim\, V_N(L) < \infty$ *for all* $N \geq 0$

(c) $\# F_L < \infty$

This holds for overdetermined elliptic systems as well. E.g.,

Theorem 6. *(holomorphic Liouville)*
On any abelian covering of a compact complex manifold $dim\, V_N(\overline{\partial}) < \infty$ *for all* $N \geq 0$

Indeed, if $u(\gamma z) = e^{ik\cdot\gamma}u(z)$, then $|u(z)|$ is periodic and thus, by maximum principle, $u(z)$ constant. Hence, $F_{\overline{\partial}} = \{0\}$.

5. **Dimension count**

At an edge of the spectrum (not necessarily at its bottom) assume that 0 is a simple eigenvalue and $F_L = k_0$. Taylor expansion $\lambda(k) = \sum_{l \geq l_0} \lambda_l (k - k_0)$.

Theorem 7. *(quantitative Liouville)*
$$dim\, V_N(L) = \binom{n+N}{N} - \binom{n+N-l_0}{N-l_0}.$$

Remarks:

♣ Graphen operators give an example of Liouville theorem inside the spectrum (Olaf Post & P.K., preprint)

♣ Dimension of the manifold does not explicitly enter the answers. Gromov's ideology.

♣ Gromov&Shubin - Riemann-Roch theorems for elliptic operators

♣ Dispersion relations at internal edges not well understood.

♣ Non-degeneracy of the leading Taylor term λ_{l_0} is not needed.

♣ Homogenization at internal spectral edges.

6. **Some references**

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