Scattering on a decorated star-graph as a toy model for the spectral theory of automorphic functions

VLADIMIR GEYLER

Mordovian State University

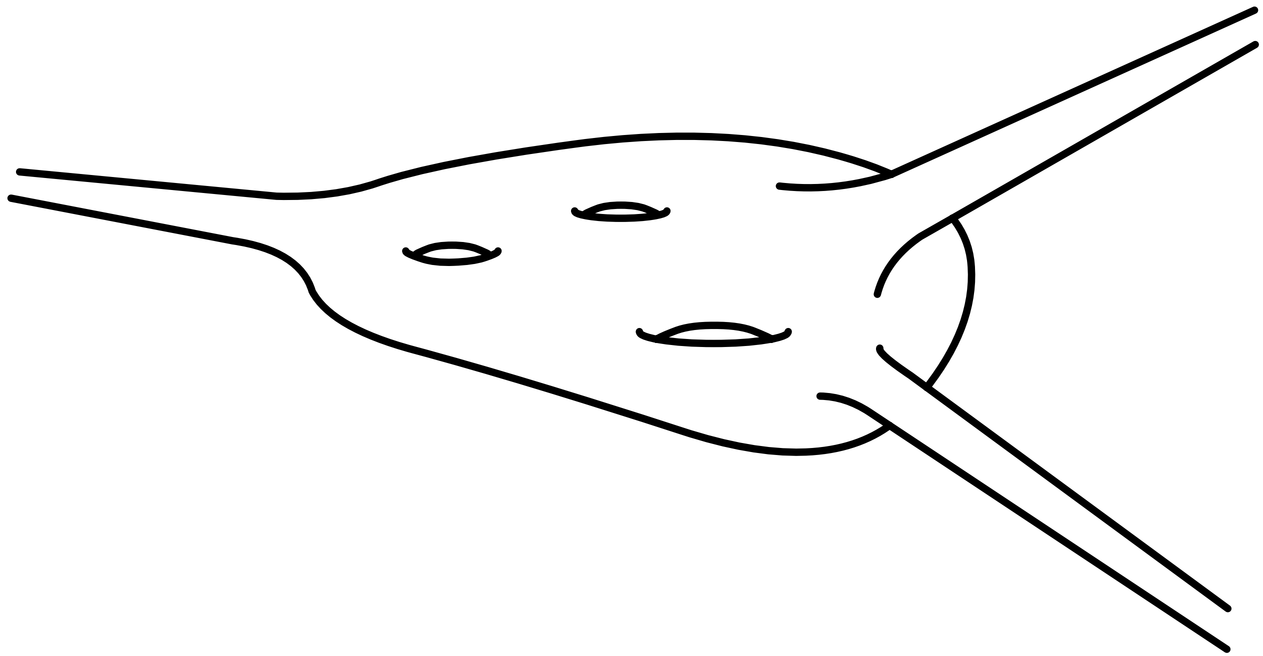
The talk is based on the joint results obtained in collaboration with J. BRÜNING (Humboldt-Universität zu Berlin) and partially presented in

- J. BRÜNING, V. GEYLER. Scattering on compact manifolds with infinitely thin horns. *J. Math. Phys.* **44** (2003), 371-405 (Prepr. mp_arc/02-233; math-ph/0205030).
- J. BRÜNING, V. GEYLER. Geometric scattering on compact Riemannian manifolds. *Doklady Mathematics*. **67** (2003), 275-278.
- J. BRÜNING, V. GEYLER. Geometric scattering on compact Riemannian manifolds and spectral theory of automorphic functions. Prepr. mp_arc/05-2.

Abstract

A compact RIEMANNIAN manifold of dimension d, $0 \leq d < 4$ with a finite number of semi-lines attached to the manifold is considered. It is shown that there is a deep analogy between the scattering and spectral properties of the SCHRÖDINGER operators on this hybrid manifold (=star graph decorated by a compact manifold) and those for the automorphic LAPLACIAN on RIEMANN surfaces with cusps. As an application, a relation between the scattering amplitude for hybrid manifolds with underlying compact RIEMANN surfaces of constant negative curvature and the SELBERG zeta-function for this surface is obtained.

Riemannian manifold with cusps=Horned manifold



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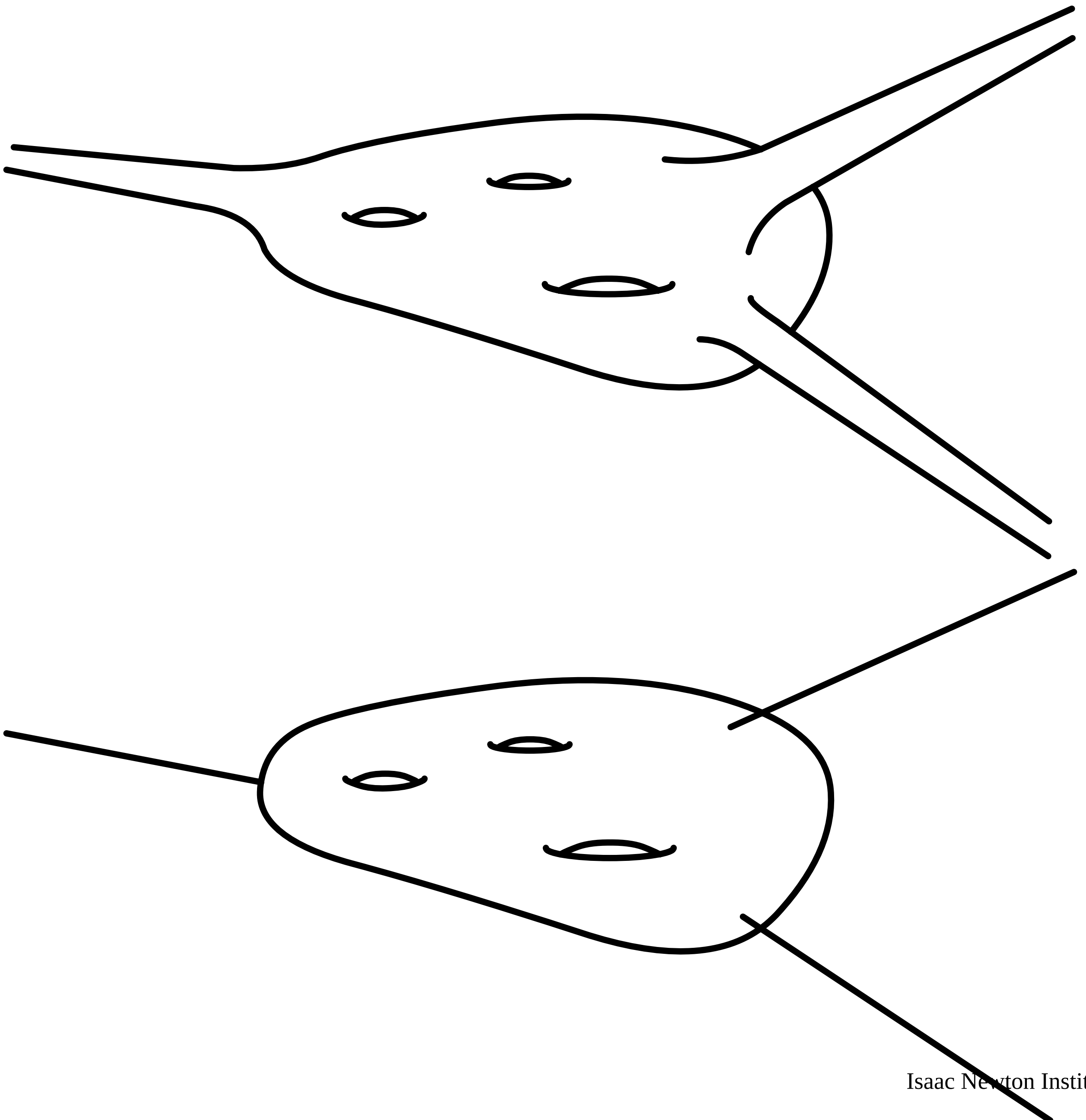

Motivation

"As MISHA GROMOV often explains in his lectures, Hyperbolic Geometry is visible from the infinity as one-dimensional one. Therefore, we may conclude, that for the discrete groups in 2D LOBACHEVSKI Plane with Noncompact Fundamental Domain of finite volume, Spectral Theory of the LAPLACE–BELTRAMI Operator should look in a sense 'similar' to the one on the graphs with k tails. We discussed this analogy with D. KAZHDAN, who pointed out to me that for arithmetic subgroups there are many discrete eigenvalues drown in the continuous spectrum. They disappear after non-arithmetic perturbation as PETER SARNAK pointed out. In the case of graphs with k tails, we have simplified version of this picture for operators with symmetry: exceptional eigenvalues disappear after generic nonsymmetric perturbation."

S. P. NOVIKOV. Schrödinger operators on graphs and symplectic geometry. *Fields Institute Commun.* **24** (1999), 397–413.

Hyperbolic plane

$$\mathbb{H}^2 = \{z = x + iy : x, y \in \mathbb{R},\ y > 0\}$$

$$ds^2 = y^{-2}(dx^2 + dy^2)$$

$$d\mu(z) = \frac{dxdy}{y^2}$$

$$r(z_1, z_2) = \cosh^{-1}\left(1 + \frac{|z_1 - z_2|^2}{2x_2y_2}\right)$$

Gauss curvature $K = -1$

$$-\Delta = -y^2\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)$$

$$PSL(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R}, ad-bc=1 \right\} / \{\pm 1\}$$

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad gz = \frac{az+b}{cz+d}$$

$$g\Delta = \Delta g$$

$$-\Delta f = s(1-s) f$$

$$\mathrm{spec}(-\Delta) = \mathrm{spec}_{\mathrm{ac}}(-\Delta) = [1/4, \infty)$$

$$E \in \mathrm{spec}(-\Delta), \quad E = s(1-s),\ s = \frac{1}{2} + ik,\ k \in \mathbb{R}_+$$

Riemann surfaces with cusps

$\Gamma \subset PSL(2,\mathbb{R})$ co-finite discrete subgroup

F its fundamental domain

$X = \Gamma \backslash \mathbb{H}^2$ RIEMANN surfaces with finitely many (n say) cusps

$-\Delta_X$ the self-adjoint operator in $L^2(X)$ generated by $-\Delta$

$$L^2(X) = L^2_{\text{cont}}(X) \oplus L^2_{\text{cusp}}(X) \oplus L^2_{\text{res}}(X)$$

Eisenstein series

$$\mathrm{spec}_{\mathrm{ac}}(-\Delta_X) = [1/4, +\infty) \quad \text{and } n\text{-fold degenerate}$$

$$-\Delta_X E_j(z;s) = s(1-s)E_j(z;s)\,, \quad s = \frac{1}{2} + ik,\, k > 0$$

$$E_j(z;s) = \sum_{g\in\Gamma'_j} \mathrm{Im}(gz)^s$$

$$E_j(z;s) = \delta_{jm} y^s + \varphi_{jm}(s) y^{1-s} + O(1) \quad \text{in } m\text{-th cusp}$$

$$\Phi(s) = [\varphi_{jm}(s)] \quad \text{is the scattering matrix,}$$

it is a meromorphic function of s in $\mathbb{C}$ with poles in

$$\{s \in \mathbb{C} : \mathrm{Re}\, s < \frac{1}{2}\} \bigcup (\frac{1}{2}, 1]$$

$f \in L^2(X)$

$$\hat{f}_j(k) = \int_X \overline{E_j\left(x, \frac{1}{2} + ik\right)} f(x)\, dx$$

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$\mathscr{F} : f \mapsto (\hat{f}_1, \ldots, \hat{f}_n)$ is a HILBERT isomorphism $L^2_{\text{cont}}(X)$ onto $L^2(\mathbb{R}^+) \otimes \mathbb{C}^2$ and $\mathscr{F}(-\Delta_X)\mathscr{F}^{-1}$ is the multiplication by k^2. In particular, the continuous spectrum of $-\Delta_X$ is n-fold degenerate.

Maaß forms

The Γ-invariant eigenfunctions of $-\Delta_X$ (MAAß forms).

$L^2_{\text{res}}(X)$ is finite dimensional and coincides with $L^2_{\text{dis}}(X)$; i.e. $\text{spec}_{\text{res}}(-\Delta_X)$, the spectrum of the restriction of $-\Delta_X$ to $L^2_{\text{res}}(X)$, coincides with $\text{spec}_{\text{dis}}(-\Delta_X)$. Moreover, $E \in \text{spec}_{\text{dis}}(-\Delta_X) \Leftrightarrow E = s(1-s)$, where s is a pole of $\Phi(s)$ in the interval $(\frac{1}{2}, 1]$

$\text{spec}_{\text{dis}}(-\Delta_X)$ is a finite subset of $[0, \frac{1}{4})$ and $0 \in \text{spec}_{\text{dis}}(-\Delta_X)$.

The corresponding eigenfunctions are the residues of the EISENSTEIN series (incomplete theta-series).

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E.g., for the principal congruence groups $\Gamma(N)$ $\mathrm{spec}_{\rm res}(-\Delta_X) = \{0\}$.

Maaß cusp forms

The spectrum of the restriction of $-\Delta_X$ to $L^2_{\text{cusp}}(X)$ coincides with $\text{spec}_{\text{pp}}(-\Delta_X) \bigcap [\frac{1}{4}, \infty)$ i.e., points from $\text{spec}_{\text{cusp}}$ are imbedded eigenvalues of finite multiplicity and can have only $+\infty$ as a limiting point. The corresponding Γ-invariant eigenfunctions are called MAAß cusp forms.

Schrödinger operator on decorated star-graphs

X_0 complete Riemannian manifold of dimension d with metric $g_{\mu\nu}$,

$g = \det(g_{\mu\nu})$,

$d\lambda$ Riemannian measure

$r(x,y)$ geodesic distance on X_0.

$\{q_1, \ldots, q_n\} \subset X_0$

X_j ($j = 1, \ldots, n$) copies of the half-line $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

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$X = (X_0 \sqcup X_1 \sqcup \ldots \sqcup X_n)/\{q_j = 0_j\}$

$\mathscr{H}_0 := L^2(X_0, d\lambda)$, $\mathscr{H}_j := L^2(X_j, dx)$.

$d\mu = d\lambda \oplus dx \oplus \ldots \oplus dx$

$\mathscr{H} \equiv L^2(X, d\mu) = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_n$.

Schrödinger operator on decorated star-graphs

$$H_0 = -g^{-1/2}(x)\left(\partial_\mu + i\mathscr{A}_\mu(x)\right) g^{1/2}(x) g^{\mu\nu}(x)\left(\partial_\nu + i\mathscr{A}_\nu(x)\right) + p(x)$$

$p, \mathscr{A}_\mu \in C^\infty(X_0)$

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$$H^0 = H_0 \oplus H_1 \oplus \dots \oplus H_n$$

“Restriction–Extension” Procedure

S is the restriction of H^0 to

$$\mathscr{D}(S) = \{ f \in \mathscr{D}(H^0) : f(q_1) = \ldots = f(q_n) = 0 \}$$

S has deficiency indices $(2n, 2n)$.

Any self-adjoint extension H of S is by definition the SCHRÖDINGER operator on X with the scalar potential p and the vector potential $(\mathscr{A}_\mu)$.

KREIN resolvent formula

H^0 a self-adjoint operator in a HILBERT space $\mathscr{H}$ with the resolvent $R^0(\zeta) = (H^0 - \zeta)^{-1}$

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$\mathscr{G}$, $\dim \mathscr{G} = n$. In $\mathscr{G} \oplus \mathscr{G}$ we consider the complex symplectic structure given by $[x|y] = \langle x_1|y_2 \rangle - \langle x_2|y_1 \rangle$.

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$\forall z, \zeta \in \mathrm{res}(H^0)$ $\gamma(z)$ is a bijection of $\mathscr{G}$ onto $\mathrm{Ker}(S^* - z)$

$$\gamma(z) = \gamma(\zeta) + (z - \zeta) R^0(z) \gamma(\zeta)$$

$$Q(z) - Q(\zeta) = (z - \zeta) \gamma^*(\bar{\zeta}) \gamma(z).$$

$\mathscr{L}$ set of all Lagrangian subspaces of $\mathscr{G} \oplus \mathscr{G}$ (in particular, if Λ is the graph of the densely defined operator L in $\mathscr{G}$, then $\Lambda \in \mathscr{L}$ iff L is self-adjoint)

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There is a bijection $H \leftrightarrow \Lambda$ between $\mathscr{S}\mathscr{A}$ and $\mathscr{L}$ given by the formula

$$(H - z)^{-1} = (H^0 - z)^{-1} - \gamma(z)[Q(z) - \Lambda]^{-1}\gamma^*(\overline{z}), \quad \operatorname{Im} z \neq 0$$

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H and H^0 are **disjoint** (i.e. $\mathscr{D}(H^0) \cap \mathscr{D}(H) = \{0\}$) iff Λ is a graph of a s.-a. operator

$$\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \ldots \oplus \mathscr{G}_n$$

$\mathscr{G}_0 = \mathbb{C}^n, \quad \mathscr{G}_j = \mathbb{C},\ j = 1, \ldots, n.$

$$\gamma(z) = \gamma_0(z) \oplus \gamma_1(z) \oplus \ldots \oplus \gamma_n(z)$$

$$Q(z) = Q_0(z) \oplus Q_1(z) \oplus \ldots \oplus Q_n(z)$$

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$$\gamma_j(z)(x) = \frac{1}{\sqrt{-z}} \exp(-\sqrt{-z}\,x)$$

$$Q_j(z) = \frac{1}{\sqrt{-z}}$$

Green function on a manifold

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Here $c_j(x,q)$ ($j=1,2,3$) is a continuous functions of x, and $c_j(q,q)=1$.

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$$R(x,q;z) = \begin{cases} o(r(x,q))\,, & \text{if } d=1; \\ o(1)\,, & \text{if } d=2 \text{ or } d=3\,. \end{cases}$$

$$\gamma_0(z)(\xi) = \sum_{j=1}^{n} \xi_j G_0(\cdot, q_j; z), \quad \xi \in \mathscr{G}_0 = \mathbb{C}^n$$

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$Q_0(z)$ is the $n \times n$-matrix:

$$[Q_0(z)]_{lm} = \begin{cases} G_0(q_l, q_m; z) & \text{if } l \neq m; \\ F_1(q_l, q_l; z) & \text{otherwise}. \end{cases}$$

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$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_{\text{sa}}$, $\mathscr{H}_{\text{sa}} = L^2(Y)$, $Y = X_1 \cup \ldots \cup X_n$;

$\mathscr{G} = \mathscr{G}_0 \oplus \mathscr{G}$sa, $\mathscr{G}_{\text{sa}} = \mathscr{G}_1 \oplus \ldots \oplus \mathscr{G}_n = \mathbb{C}^n$.

$$Q(z) = \begin{bmatrix} Q_0(z) & 0 \\ 0 & Q_{\text{sa}}(z) \end{bmatrix}$$

$Q_{\text{sa}}(z)(-z)^{-1/2}I$

$$L = \begin{bmatrix} B & A \\ A^* & C \end{bmatrix} \quad A = (\alpha_{lm}), \; B = (\beta_{lm}), \; C = (\gamma_{lm}), \; B = B^*, \; C = C^*$$

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$f = (f_0, f_1, \ldots, f_n) \in \mathscr{D}(S^*)$. Then $f_0 \in \mathscr{D}(S_0^*)$ and in a neighborhood of q_j

$$f_0(x) = a_j(f_0)F_0(x, q_j) + b_j(f_0) + R_j(x), \quad b_j(f_0) \in \mathbb{C}$$

$f \in \mathscr{D}(H^L)$ iff

$$\begin{aligned} b_j(f_0) &= \sum_{k=1}^{n} \beta_{jk} a_k(f_0) - \sum_{k=1}^{n} \alpha_{jk} f_k{}'(0), \\ f_j(0) &= \sum_{k=1}^{n} \bar{\alpha}_{kj} a_k(f_0) - \sum_{k=1}^{n} \gamma_{jk} f_k{}'(0), \quad j = 1, \ldots, n. \end{aligned}$$

An important remark

If $f_0 \in \mathscr{D}(S_0^*)$, $f_j \in H^1_{\mathrm{loc}}(\mathbb{R}^+)$ and the relations above are satisfied then Hf is defined in the sense of distributions.

$\mathscr{D}^1(H)$

Point perturbation of H_0

$$(H^B - z)^{-1} = (H_0 - z)^{-1} - \gamma_0(z)[Q_0(z) - B]^{-1}\gamma^*(\bar{z})$$

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X_0 IS COMPACT

$$\mathscr{E}^0_0 \le \mathscr{E}^0_1 \le \ldots \le \mathscr{E}^0_m \le \mathscr{E}^0_{m+1} \le \ldots$$

$$0 = \mathscr{E}^0_0 < \mathscr{E}^0_1, \quad \text{if} \quad H_0 = -\Delta_{X_0}$$

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For the spectrum of H^B

$$\mathscr{E}^B_0 \le \mathscr{E}^B_1 \le \ldots \le \mathscr{E}^B_m \le \mathscr{E}^B_{m+1} \le \ldots$$

we have in the case $d > 1$: $\mathscr{E}^B_0 < \mathscr{E}^0_0$, and in any case $\mathscr{E}^0_m$ remains in the spectrum of H^B if $n < \deg(\mathscr{E}^0_m)$.

$$[Q(z)-L]^{-1}=\begin{bmatrix} U(z) & W(z) \\ W^*(\overline{z}) & V(z) \end{bmatrix}, \quad U(z)=U^*(\overline{z}), \quad V(z)=V^*(\overline{z})$$

according to the representation $\mathscr{G}=\mathscr{G}_0\oplus\mathscr{G}_{\mathrm{sa}}$

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Scattering solutions

Theorem 1.

$\exists$ discrete $Z_H \subset \mathbb{R}\ \forall\, k>0,\ k^2 \notin Z_H$ and $\forall\, j \in \{1,\ldots,n\}$ the SCHRÖDINGER equation

$$H f = k^2 f$$

has a unique solution $f = (f_0, f_1, \ldots, f_n) \in \mathscr{D}^1(H)$ of the form

(i) $f_0(x) = -2 \sum_{m=1}^{n} w_{mj}(k^2) G_0(x, q_m; k^2)$;

(ii) $f_l(x) = s_{lj}(k) \exp(ikx)$ for $l>0,\ l \neq j$;

(iii) $f_j(x) = \exp(-ikx) + s_{jj}(k)\exp(ikx)$,

and $[s_{jl}(k)] = \Sigma(k)$.

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Denote $f(x) = E^{(j)}(x,k)$

$$\mathscr{E}_{lm}(x,y;k) = \sum_{j=1}^{n} E_l^{(j)}(x,k)\overline{E_m^{(j)}(y,k)}, \quad 0 \leq l,m \leq n\,.$$

Main relation for the Eisenstein functions

Theorem 2.

$$\mathcal{E}(x,y;k) = -2ki[G(x,y;k^2+i0) - G(x,y;k^2-i0)]\ k>0\,,\ k^2 \notin Z_H$$

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Theorem 3. $\mathscr{F}$ is a HILBERT isomorphism of $\mathscr{H}_{\text{ac}}$ onto $L^2\left(\mathbb{R}^+, \frac{dk}{2\pi}\right)$ and $\mathscr{F} H \mathscr{F}^{-1}$ is the multiplicity by k^2. Therefore, the a.c.-spectrum of H is n-fold degenerate.

Decomposition theorem

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Suppose $H_0 \geq 0$

$\mathscr{H} \ominus \mathscr{H}_{\rm ac} = \mathscr{H}_{\rm cusp} \oplus \mathscr{H}_{\rm res}$ where the subspaces $\mathscr{H}_{\rm cusp}$, $\mathscr{H}_{\rm res}$ are H-invariant.

Theorem 4(1). $\mathscr{H}_{\rm cusp} \neq \{0\} \Rightarrow \mathrm{spec}_{\rm cusp}(H)$, the spectrum of H in $\mathscr{H}_{\rm cusp}$, is discrete and $\subset [0, \infty)$. For a generic choice of $\{q_1, \ldots, q_n\} \subset X_0$ and L, the set $\mathrm{spec}_{\rm cusp}(H)$ consists of all the eigenvalues $\mathscr{E}$ of H_0 which have in $\mathrm{spec}(H_0)$ multiplicity $m_0 > n$. The multiplicity m of $\mathscr{E}$ in $\mathrm{spec}(H)$ is $m_0 - n$. The corresponding eigenfunctions g have the form $g = (g_0, g_1, \ldots, g_n)$ where $g_j(0) = 0$ $(1 \leq j \leq n)$ and g_0 is an eigenfunction of H_0 vanishing at all the points q_j. Hence, if H_0 has only simple eigenvalues (this is a generic case), then $\mathscr{H}_{\rm cusp} = \{0\}$ and $\mathrm{spec}_{\rm cusp}(H) = \emptyset$. In any case, $\mathrm{spec}_{\rm cusp}(H) \subset \mathrm{spec}(H^B)$ and the multiplicity m allows the estimate $m_0 - 2n \leq m \leq m_0 + 2n$.

Decomposition theorem

Theorem 4(2). $d_{\text{res}} \equiv \dim \mathscr{H}_{\text{res}} \leq 2n$; $d_{\text{res}} \geq 1$ if $H_0 = -\Delta_{X_0}$ and $d > 1$. For a generic choice of L, the number $\mathscr{E} = (ik)^2$, $k > 0$, belongs to $\text{spec}_{\text{res}}(H)$ iff ik is a pole of $\Sigma(k)$. In this case, the corresponding eigenfunctions are residues of the meromorphic continuation of the EISENSTEIN functions $E^{(j)}(x,k)$. In any case, if g is an eigenfunction for $\mathscr{E} \in \text{spec}_{\text{res}}(H)$, then $g = (g_0, g_1, \ldots, g_n)$ where g_0 is a linear combination of the functions $g_0(x) = G_0(x, q_j; \mathscr{E})$ and $g_j(x) = c_j \exp(-\sqrt{-\mathscr{E}}x)$ for $1 \leq j \leq n$.

Decomposition Theorem above shows that there is a deep analogy

(1) between the functions $E^{(j)}(x,k)$ and the EISENSTEIN series;

(2) between eigenfunctions from the item (1) of Theorem 4 and the MAAß cusp forms;

(3) between eigenfunctions from the item (2) of Theorem 4 and the MAAß forms which are represented by incomplete theta-series.

Smilansky spectral duality

Suppose the set $\{q_1, \ldots, q_n\}$ is generic. Let $k > 0$. Then $k^2 \in \text{spec}(H_0)$ iff 1 is an eigenvalue of $\Sigma(k)$.

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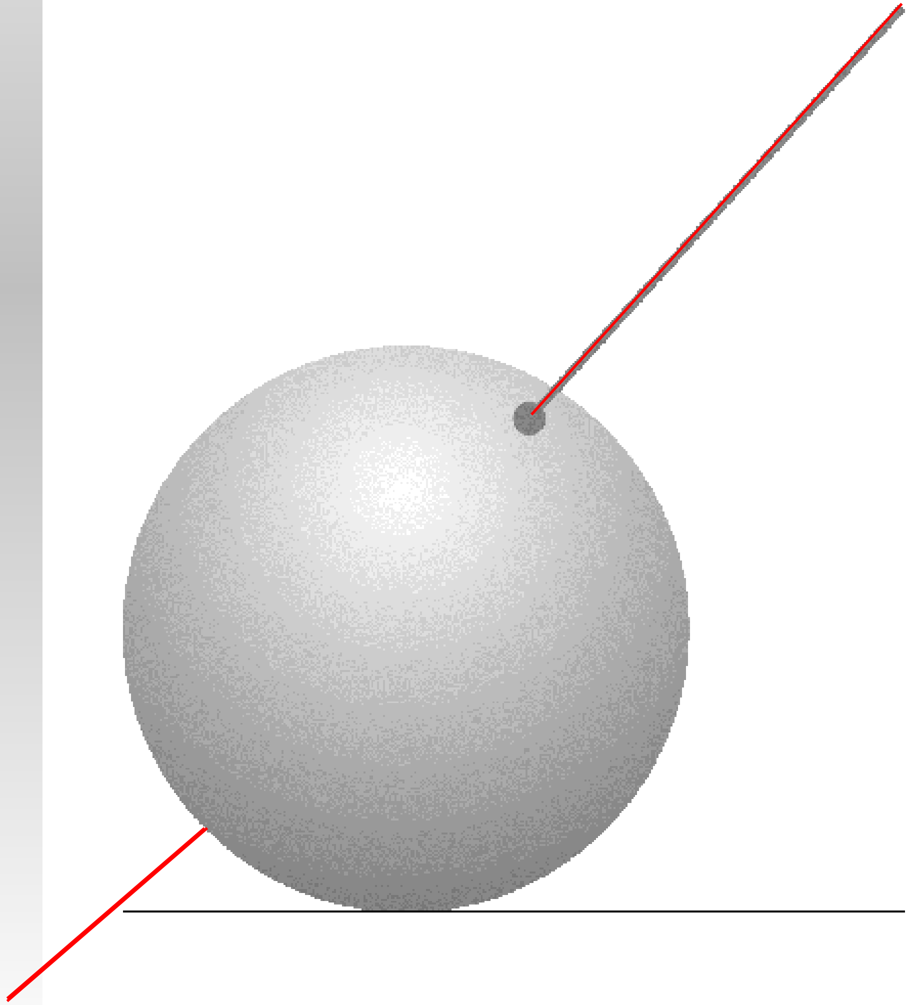

A relation to the Selberg zeta-function

X_0 compact RIEMANN surface of constant curvature $K = -1$, $X_0 = \Gamma \setminus \mathbb{H}^2$;
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The SELBERG zeta-function

$$\mathsf{Z}_X(s) = \prod_{p \in \mathscr{P}} \prod_{k=0}^{\infty} \left(1 - e^{-\tau(p)(s+k)}\right)$$

$\mathscr{P}$ is complete set of primitive geodesics, $\tau(p)$ is the length of p

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$$\rho_n = \sqrt{\mathscr{E}_n^0 - \frac{1}{4}}$$

A relation to the Selberg zeta-function

ELSTRODT–CARTIER–VOROS

$$\sum_{n=0}^{\infty} \frac{d}{ds^2} \frac{1}{\rho_n^2 + s^2} = (2-2g) \frac{d}{ds^2} \psi\left(\frac{1}{2}+s\right) + \left(\frac{d}{ds^2}\right)^2 \log \mathsf{Z}_X\left(\frac{1}{2}+s\right)$$

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$$\frac{\mathsf{Z}'(s)}{\mathsf{Z}(s)} = 2(2s-1)(g-1) \times$$

$$\left(\psi(s) + C_E - \ln 2 + \frac{\sigma_0^{1-n}}{2(g-1)n} \int_{X_0} \dots \int_{X_0} \mathrm{Tr}\, Q_0(s(1-s); \mathbf{q})\, d\mathbf{q} \right)$$

$C_E = -\psi(1)$ Euler constant, $\mathbf{q} = (q_1, \dots, q_n)$,

$$\sigma_0 \equiv \mathrm{Area}(X_0) = 4\pi(g-1)\,,$$

A relation to the Selberg zeta-function

CAYLEY transform for a unitary operator U

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$$\left.\int\limits_{X_0}\dots\int\limits_{X_0} \mathrm{Tr}\left[B - \sqrt{s(1-s)}A(\mathrm{Ca}(\Sigma(\sqrt{s(1-s)};\mathbf{q})) + \sqrt{s(1-s)}C)^{-1}A^*\right]d\mathbf{q}\right)$$

Selberg trace formula for point perturbations

It is convenient to represent the trace formula in the form given by J. ELSTRODT

$$\mathrm{Tr}[R_0(s(1-s)) - R_0(a(1-a))] =$$

$$2(1-g)(\psi(s)-\psi(a)) + \frac{1}{2s-1}\frac{\mathsf{Z}'(s)}{\mathsf{Z}(s)} - \frac{1}{2a-1}\frac{\mathsf{Z}'(a)}{\mathsf{Z}(a)}$$

Selberg trace formula for point perturbations

$$\mathrm{Tr}\left[R_0(x(1-x))\right]_{x=a}^{x=s} = 2(1-g)\psi(x)\Big|_{x=a}^{x=s} +$$

$$+\left[\frac{1}{2x-1}\left(\ln \mathrm{Z}(x)\right)'\right]_{x=a}^{x=s}$$

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Then for the point perturbation of $H_0 = -\Delta_{X_0}$ defined by B

$$\mathrm{Tr}\left[R(x(1-x))\right]_{x=a}^{x=s} = 2(1-g)\psi(x)\Big|_{x=a}^{x=s} +$$
$$\left[\frac{1}{2x-1}\left(\ln\left(Z(x)\det[Q(x(1-x)-B]\right)\right)'\right]_{x=a}^{x=s}.$$

Conclusion

The main results reinforce the GROMOV–NOVIKOV thesis concerning relations between Hyperbolic Geometry on infinity and One-Dimensional Geometry. Namely, the spectral theory of the LAPLACE–BELTRAMI operator on a compact manifold with *n* attached semi-axes (infinitely thin horns) completely looks like the spectral theory of the automorphic LAPLACIAN for a FUCHSIAN group of the first kind with *n* cusps.