Is there an Interesting Index Theory for Quantum Graphs?

Roth (1983): Kirchhoff BC

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} = \operatorname{Tr} K = \int_\Gamma K(t,x,x)\,dx$$

$$= \text{sum over closed orbits} = K_1 + K_2 + K_3\,.$$

1. Zero length: $K_1 = \dfrac{L}{\sqrt{4\pi t}}$ (Weyl's law).
2. Periodic: $K_2 = \dfrac{1}{\sqrt{4\pi t}} \sum_C \mathcal{A}(C) L(C_p) e^{-L(C)^2/4t}$.
3. Closed: "Ce calcul est un peu plus délicat."

$$K_3 = \frac{1}{2}(V-E) \qquad \text{(the } \textbf{only} \text{ constant term).}$$

Justin Wilson (2007): any real S-matrix

$$K_3 = \frac{1}{2} \sum_\alpha S_{\overline{\alpha}\alpha} \int_0^\infty K_0(t,x,0)\,dx$$

$$= \frac{1}{4} \sum_\alpha S_{\overline{\alpha}\alpha} \quad \text{(sum all bond backscatter amplitudes).}$$

Kirchhoff: $K_3 = \dfrac{1}{4} \sum_v \left(\dfrac{2}{N_v} - 1 \right) N_v = \dfrac{1}{2} V - \dfrac{1}{2} E.$

$V - E$ is

- an integer.
- "topological" (Euler characteristic of a 1-complex).

So what? …

Gilkey, *Asymptotic Formulae in Spectral Geometry*:
Interval example generalizes to:

! index theorem for DeRham complex of manifold with boundary.

? index interpretation of Roth–Wilson formula.

$H_{D,N} = -\frac{d^2}{dx^2}$ on $(0, L)$ with Dirichlet/Neumann BC.

$$\dim\ker H_D = 0, \quad \dim\ker H_N = 1.$$

$$\int_0^L K_{D,N}(t,x,x)\,dx = \frac{L}{\sqrt{4\pi t}} \mp \frac{1}{2}\,.$$

Therefore,

$$\dim\ker H_N - \dim\ker H_D = 1 = \operatorname{Tr} K_N - \operatorname{Tr} K_D\,. \quad (*)$$

Better: $\begin{cases} d\text{: functions (N)} \to \text{one-forms (D)}, \\ d^\dagger\text{: one-forms} \to \text{functions.} \end{cases}$

$$d + d^\dagger = \begin{pmatrix} 0 & d^\dagger \\ d & 0 \end{pmatrix} : \Lambda^0(0,L) \oplus \Lambda^1(0,L) \to \text{self.}$$

$$H_N = d^\dagger d, \qquad H_D \simeq \overline{d} d^\dagger.$$

$$\ker H_N = \ker d, \qquad \ker H_D = \ker d^\dagger.$$

Therefore, $\quad (*) = \operatorname{index} d\,!$

Extension to graphs

Kuchment (2004) "Quantum Graphs: I":
Self-adjoint BC on $F(v) = \{f_j(v)\}$ and $F'(v)$:

- Dirichlet part: $\quad P_v F = 0, \quad P_v =$ OG projector;
- Neumann part: $\quad (I - P_v)F' = -L_v F. \quad (L_v = 0).$

Form domain: $\quad \{f \in H^1(e)\text{: Dirichlet}\}$.
Operator domain: $\quad \{f \in H^2(e)\text{: both}\}$.

These provide natural "large" and "small" domains for $\frac{d}{dx}$!

Let $d = \frac{d}{dx}$ with Kirchhoff conditions —

- Dirichlet part: $f_j(v)$ independent of j;
- Neumann part: $\sum_j f'_j(v) = 0$.

Then $d^\dagger = -\frac{d}{dx}$ with "dual" conditions —

- Dirichlet part: $\sum_j f_j(v) = 0$;
- Neumann part: $f'_j(v)$ independent of j.

$$d^\dagger d = \text{Kirchhoff Laplacian}, \quad dd^\dagger = \text{dual Laplacian}$$

(outer operator on form domain, inner operator on operator domain).
Using one-forms removes ambiguity in direction of $\frac{d}{dx}$ in dual case! Using functions we would need

$$\sum \epsilon_j f_j(v) = 0, \qquad \epsilon_j f'_j(v) \text{ indep. of } j$$

for some arbitrary orientation of each edge,

$$\epsilon_j \equiv \begin{cases} +1 & \text{if } v = 0, \\ -1 & \text{if } v = L. \end{cases}$$

For dual Laplacian, $\frac{1}{4}\sum_\alpha S_{\overline{\alpha}\alpha} = -\frac{1}{2}(V-E)$.
Therefore,

Index theorem:

$$\operatorname{index} d = \dim\ker H_K - \dim\ker H_{\text{dual}} = V - E.$$

So what? Well,

$$\begin{aligned}\dim\ker H_K &= \text{number of components}\\ &= 1 \quad \text{for connected graph,}\end{aligned}$$

so

$$\begin{aligned}\dim\ker H_{\text{dual}} &= E - V + \text{number of components}\\ &= E - V + 1 \quad \text{for any connected graph}\\ &= \text{(e.g.)}\begin{cases}0 & \text{for a tree (e.g., a star),}\\ 1 & \text{for a cycle.}\end{cases}\end{aligned}$$

Open problem: Is this interesting?