Validated computation tool for the Perron-Frobenius eigenvalue using graph theory

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Introduction

Matrix Eigenvalue Problem

Find $\lambda \in \mathbb{C}$ and $x \neq 0$ such that

$$Ax = \lambda x$$

$A:$ real, large, nonnegative, sparse, non-symmetric

Aim: Enclosing of a special eigenvalue
(Perron-Frobenius eigenvalue)

We want to provide a numerical computational tool:

Input

nonnegative matrix

Output

rigorous upper and lower bounds
of Perron-Frobenius eigenvalue

- Self-validating methods
- Numerical verification
- Computer-assisted proofs
- Numerical computations with guaranteed accuracy

■ It guarantees the existence (and uniqueness) of the solutions

■ It guarantees the residual between exact solution and approximate solution

■ The errors occurred in a computing process in a computer are assured.

INTLAB - Interval Laboratory,
the Matlab toolbox for verified computations
(S. M. Rump, http://www.ti3.tu-harburg.de/rump/intlab/)

VERIFYEIG Verification of eigencluster near (lamda,xs)

[L,X] = VerifyEig(A,lambda,xs,B)

An eigenvalue cluster near lambda is enclosed, where xs(:,i), i=1:k is an approximation to the corresponding invariant subspace. If B is specified, the generalized eigenproblem A*x = lambda*B*x is treated.

On output, the complex interval L contains (at least) k eigenvalues of A, and the complex $n \times k$ matrix X includes a base for the corresponding invariant subspace.

Ex. Maximum eigenvalue of A (real symmetric matrix)

$\tilde{\lambda}$: approximate maximum eigenvalue

$$\mu \equiv (1+\delta)\tilde{\lambda}, \quad \delta > 0$$

$$X \equiv \mu I - A, \ \tilde{C}\tilde{C}^T \approx X$$

$$Y \equiv \tilde{C}\tilde{C}^T - X$$

$$\rho \equiv \max_{1 \le k \le N} \sum_{j=1}^{N} \left| (Y)_{kj} \right|$$

⇨ $\lambda_{\max}(A) \le \tilde{\lambda} + \rho$

Matrix eigenvalue problem ⟺ Numerical verification for differential equations

⇕

Infinite dimensional eigenvalue problem

$$Lu = \lambda u$$

L : Elliptic differential operator, e.g.
non-commutative harmonic oscillator
1-D Schroedinger operator etc.

Perron-Frobenius eigenvalue

Let A be a real $n \times n$ matrix and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A .

spectral radius $\rho(A) = \max_{1 \le i \le n} |\lambda_i|$

Simplicity of eigenvalue

λ is geometrically simple

$\Leftrightarrow$ corresponding eigenspace is one dimensional

λ is algebraically simple

$\Leftrightarrow$ λ is a simple root of the characteristic polynomial

A is reducible

$\Leftrightarrow$ ${}^{\exists}P$: permutation matrix s.t.

$$P^t A P = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$$ A_{11}, A_{22} : square matrices

(In case of $n = 1$ $\longrightarrow$ $A = O$ is reducible)

A is irreducible

$\Leftrightarrow$ A is not reducible

$\tilde{P}$: permutation matrix

$$A \xrightarrow{\tilde{P}} \tilde{P}^t A \tilde{P} = \begin{pmatrix} A_1 & * & \cdots & * \\ \vdots & A_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_m \end{pmatrix}$$

$A_1, A_2, \cdots, A_m$: Irreducible, nonnegative or 1×1 zero matrix

(irreducible components of A)

Perron-Frobenius Theorem

Let A be a nonnegative and irreducible matrix.
Then

$$\rho(A) > 0$$

is an eigenvalue of A, which is both geometrically and algebraically simple.

Moreover the corresponding eigenvector is positive.

$$\lambda_{\mathrm{PF}}(A) \equiv \rho(A):$$

Perron-Frobenius eigenvalue（P-F eigenvalue）

(Perron root, Perron eigenvalue, Frobenius eigenvalue)

Outline of proof

$$S = \left\{ x \in \mathbb{R}^n \mid x \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}$$ bounded, convex, closed, non-empty

$$Tx = \frac{1}{\theta(x)} Ax, \ \theta(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j > 0$$

$T : S \to S$ continuous

⇨ by Brouwer's fixed point theorem

$^{\exists} x_0 \in S$ such that $Tx_0 = x_0$

$$\lambda \equiv \theta(x_0) > 0 \quad \Rightarrow \quad Ax_0 = \lambda x_0$$

For a nonnegative matrix A with irreducible components $A_1, \cdots, A_m$

P-F eigenvalue of A

$$\Leftrightarrow \quad \lambda_{\mathrm{PF}}(A) \equiv \max_{1 \le i \le m} \lambda_{\mathrm{PF}}(A_i)$$

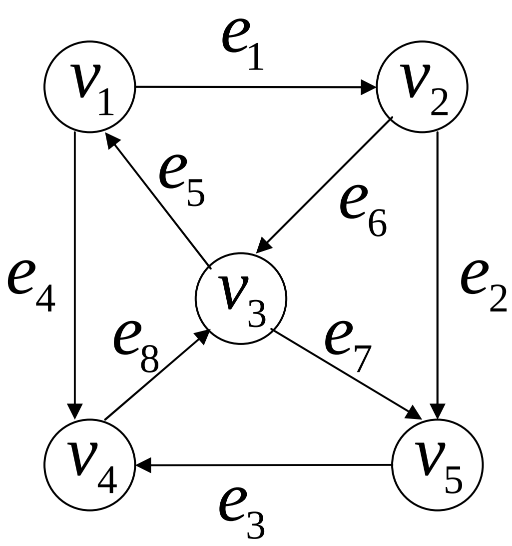
If μ is another eigenvalue of A, then of course

$$|\mu| \le \lambda_{\mathrm{PF}}(A)$$

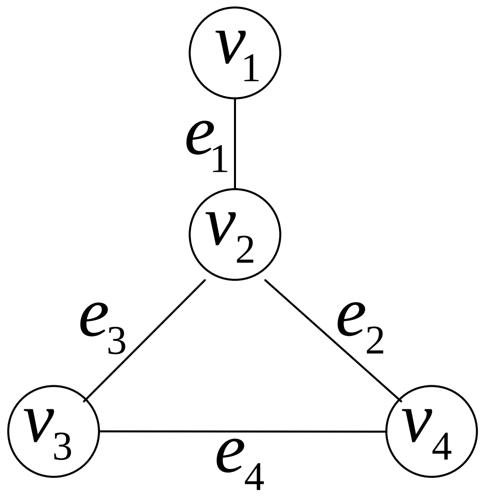
Matrix and graph

Graph $G = (V, E)$

V : a set of vertices, E : a set of edges $e = (v, w)$



directed graph



undirected graph

$v, w \in V$

path $p : v \to w$

$\Leftrightarrow$ sequence of vertices and edges leading from v to w

graph G is connected

$\Leftrightarrow$ there is a path between every pair of vertices

graph G is strongly connected

$\Leftrightarrow$ for each pair of vertices $v, w \in V$

there exist paths $p_1 : v \to w$ and $p_2 : w \to v$

Strong connectivity

Let $G=(V,E)$ be a directed graph and $v,w\in V$.

$v\sim w$ (equivalent)

$\Leftrightarrow$ there is a closed path $p:v\rightarrow v$ which contains w

Let the distinct equivalence classes under this equivalence relation be $V_i\ (i=1,\cdots,n)$ and let

$$G_i=(V_i,E_i),\quad E_i=\left\{(v,w)\in E\,|\,v,w\in V_i\right\}$$

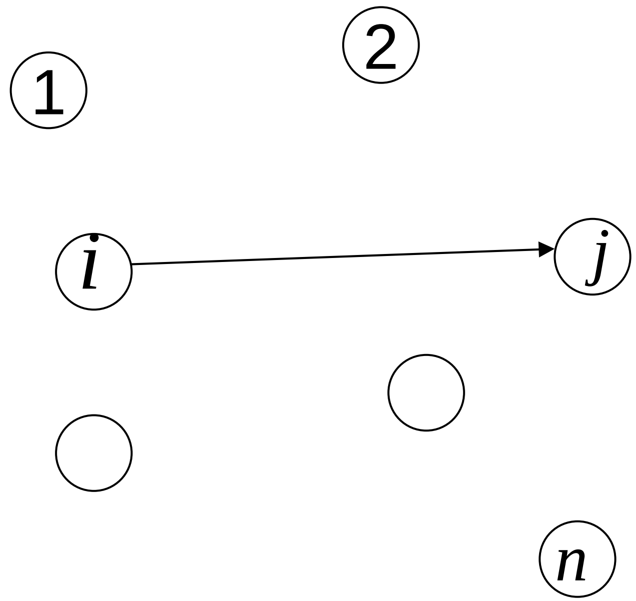
Then:

(i) Each G_i is strongly connected
(ii) No G_i is a proper subgraph of a strongly connected subgraph of G .

G_i : strongly connected components (SCC) of G

Matrix $A = (a_{ij})_{i,j=1,\cdots,n}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$



Associated graph $G = (V, E)$ given by

$$V = \{1, \cdots, n\},\ E = \{(i, j) \mid a_{ij} \neq 0\}$$

Irreducibility & strong connectivity

A is irreducible $\Longleftrightarrow$ G is strongly connected

$A_1, \cdots, A_m$: irreducible components of A

$\Updownarrow$

$G_1, \cdots, G_m$: strongly connected components of G

The connections $G_1, \cdots, G_m \Leftrightarrow A_1, \cdots, A_m$ and

P-F eigenvalue of a matrix A

$\Updownarrow$

P-F eigenvalues of irreducible components A_i

have been observed in

> Ishii, Y., Sands D.:
> Piecewise affine and projective maps in several dimensions, Ⅱ: Rigorous entropy computation, Preprint (2007).

and applied to compute the topological entropy of some piecewise affine homeomorphism of the plane

Tarjan's algorithm

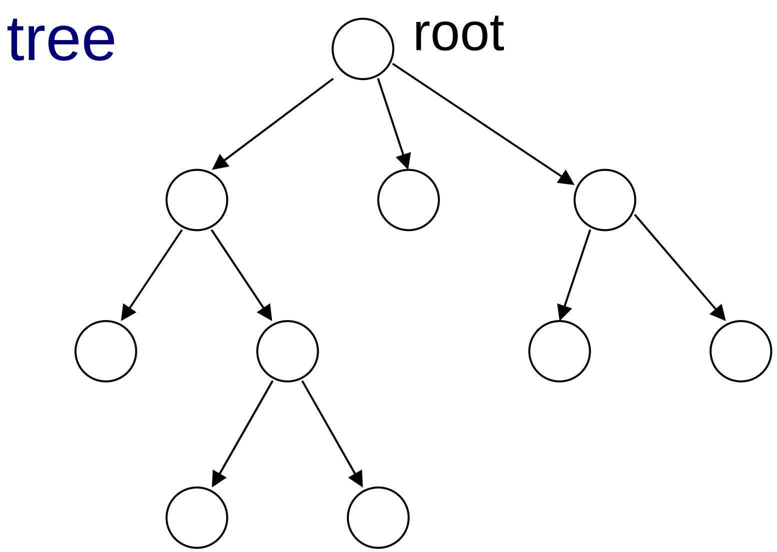
Tarjan, R.:
Depth-first search and linear graph algorithms,
SIAM J. Comput. 1 , no. 2, pp. 146--160 (1972).

- Efficient algorithm to find all the strongly connected components of any given finite directed graph G by using depth-first search
- Works correctly on any finite directed graph
- Requires $O(|V|,|E|)$ space and time

Depth First Search (DFS)

Consider the following choice rule on searching a graph:

When selecting an edge to traverse, always choose an edge emanating from the vertex most recently reached which still has unexplored edges.



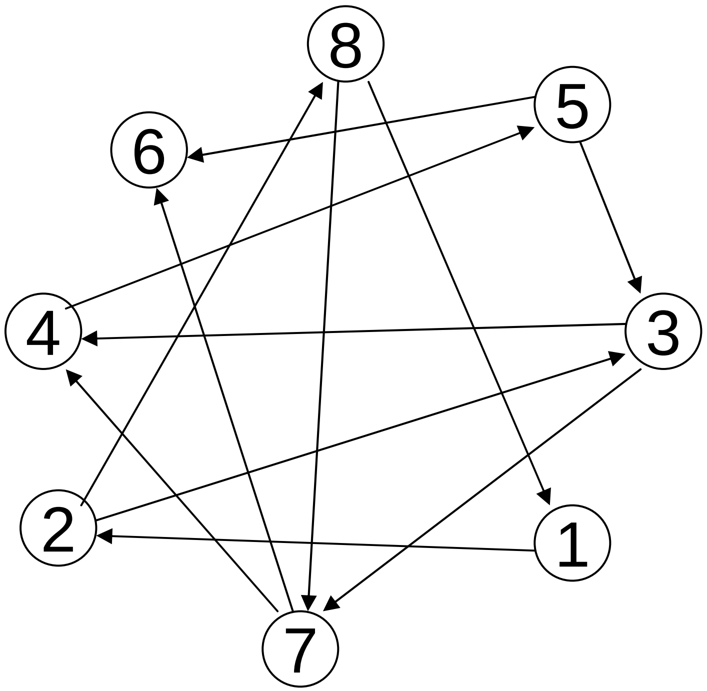
```
BEGIN
  INTEGER i ;
  PROCEDURE STRONGCONNECT(v)
    BEGIN
      LOWLINK(v) := NUMBER(v):= i := i+1;
      put v on stack of points;
      FOR w in the adjacency list of v DO
         BEGIN
           IF w is not yet numbered THEN
             BEGIN
               STRONGCONNECT(w);
               LOWLINK(v) := min(LOWLINK(v), LOWLINK(w);)
             END
           ELSE IF NUMBER(w) <  NUMBER(v) DO
             BEGIN
               if w is on stack of points THEN
                 LOWLINK(v) := min(LOWLINK(v), NUMBER(w));
             END
         END

      If(LOWLINK(v) = NUMBER(v)) THEN
        BEGIN
          start new strongly connected component;
          WHILE w on top of point stack satisfies NUMBER(w) <=  NUMBER(v) DO
            delete w from point stack and put w in current component;
        END
    END
   i:= 0;
   empty stack of points;
   FOR w a vertex IF w is not yet numbered
THEN STRONGCONNECT(w);
END
```

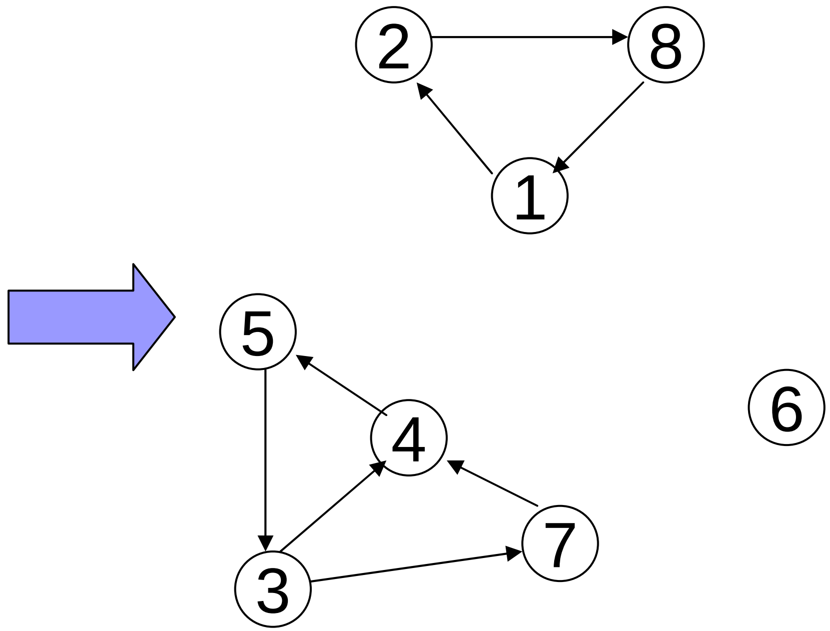
LOWLINK(v): the smallest vertex which is in the same strongly component as v .

v is the root of some strongly connected component of G

$\Leftrightarrow$ LOWLINK(v) = v



graph G



SCCs G_1, G_2, G_3

MATLAB Code for Tarjan's Algorithm

```
% Function to find all strongly connected components
function [G,N,L,c,k,sccl,stack]=scc(v,G,N,L,c,k,sccl,stack)
UNDEFINED=0; c=c+1;  k=k+1;
N(v)=c;   % Visit number for the vertex v
L(v)=c;   %  LOWLINK(v)
stack(k)=v;
for w=list(v)  % w is in the adjacency list of v
   if N(w)==UNDEFINED
      [G,N,L,c,k,sccl,stack]=scc(w,G,N,L,c,k,sccl,stack);
      L(v)=min(L(v),L(w));
   elseif N(w)<N(v)
     for z=stack
        if w==z
           L(v)=min(L(v),N(w));
        end
     end
   end
end
if L(v)==N(v)
   sccl=sccl+1; j=1; m=1;
   for t=stack
      if (t>0 & N(t)>=N(v))
         G(sccl,j)=t;     %SCC component
         j=j+1; stack(m)=-1;
      end
      m=m+1;
   end
end
```

Enclosing of Perron-Frobenius eigenvalue

Let A be a nonnegative matrix.

For any vector $x=(x_i)_{i=1}^n,\ \ x_i>0\ \ (i=1,\cdots,n)$

we have

$$\min_{1\le i\le n}\frac{(Ax)_i}{x_i}\le \lambda_{\mathrm{PF}}(A)\le \max_{1\le i\le n}\frac{(Ax)_i}{x_i} \qquad (1)$$

Finding a good vector x $\longrightarrow$ Getting sharp bounds for $\lambda_{\mathrm{PF}}(A)$

Algorithm of validated computation for P-F eigenvalue

nonnegative matrix A $\longrightarrow$ graph G

↓

Strongly connected components of G

↓

Irreducible components of A

↓

P-F eigenvalue enclosing of each irreducible component

↓

P-F eigenvalue enclosing of A

Numerical examples

Example 1 $n \times n$ tridiagonal matrix

$$A = \begin{pmatrix} 1 & 0 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 0 & 1 \end{pmatrix}$$

There are two eigenvalues (±) which have the same maximum absolute value

⟶ Power method is not valid to obtain good approximate eigenvector

$n = 1024$

Apply (1) for A itself

$$\longrightarrow \lambda_{\mathrm{PF}}(A) \in [1.0, 1.999999998409] \quad \text{(86.494 s)}$$

Apply (1) for each irreducible components of A (15.663 s)

$$\longrightarrow \lambda_{\mathrm{PF}}(A) \in [1.999990569206479, 1.999990569210389]$$

MATLAB Version 7.0.1 (R14)
on DELL Inspiron 5150 (Intel Pentium4 2.8GHz)

Example 2

$$x(k) = 0 \quad \text{or} \quad 1 \quad (k = 1, \cdots, n)$$

$$A = \begin{pmatrix} x(1) & x(2) & 0 & \cdots & & & \cdots & 0 \\ 0 & 0 & x(3) & x(4) & 0 & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & & x(n/2-1) & x(n/2) \\ x(n/2+1) & x(n/2+2) & 0 & & & & \cdots & 0 \\ 0 & 0 & x(n/2+3) & x(n/2+4) & 0 & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & & x(n-1) & x(n) \end{pmatrix}$$

Results

$n = 1024, \quad b = 0.1$

Lozi map

$$L_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|x| + by \\ x \end{pmatrix}$$

$a \in \mathbb{R}, \quad b \in \mathbb{R} \setminus \{0\}$: parameters

a	SCC	$\lambda_{\mathrm{PF}}(A)$	Time (s)
1.8	369	[1.65913882323117, 1.65913882323130]	3.074
2.0	41	[1.94945261576407, 1.94945261576415]	3.956
2.062	1	[1.99999999999996, 2.00000000000026]	7.251
2.1	1	[1.99999999999996, 2.00000000000026]	7.301
3.0	1	[1.99999999999996, 2.00000000000026]	7.151

※ $a \geq 2.06 \Rightarrow \lambda_{\mathrm{PF}}(A) = 2$

Conclusion

We have developed a tool for rigorous computation of Perron-Frobenius eigenvalue for nonnegative matrix.
(This program can be used to judge the irreducibility of a matrix)

Improvements

- Making use of the sparsity
- Parallel computing

We are planning to release our tool in near future.

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