# The Method of Images on a Quantum Graph

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- $V$ vertices, $B$ bonds
- If $\ell$ is a bond, $\ell^+$ and $\ell^-$ are the associated directed bonds
- $L^2(G) = L^2[0, L_1] \times \cdots \times L^2[0, L_B]$
- $\mathbf{u} \in L^2(G) \times L^2(G)$ is a vector of functions on directed bonds if $u_\alpha(x) = u_{\bar{\alpha}}(L_{\bar{\alpha}} - x)$.

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  - $A$ and $B$ are square matrices such that
    - $(A, B)$ is of maximal rank, and
    - $AB^\dagger$ is self-adjoint.

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- This $S_{\alpha\beta}$ is zero for directed bonds $\alpha$ and $\beta$ that are not connected.

# Example: Kirchhoff conditions

- Kirchoff boundary conditions for a vertex are defined as follows:

$$u_\alpha(0) = \phi_i, \quad \forall \alpha \in S_i^+$$

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- If $N$ is the number of bounds going out of $i$, then the resulting $\sigma^{(i)}$ is

$$\sigma_{j,l}^{(i)} = \frac{2}{N} - \delta_{j,l}$$

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- But our main motivation is the vacuum energy (and its density). The vacuum energy is given as the regular part of

$$\frac{1}{2} \sum_{k_n} k_n e^{-k_n t}.$$

- With the cylinder kernel (defined later), we can find this vacuum energy and its density.

# The Method

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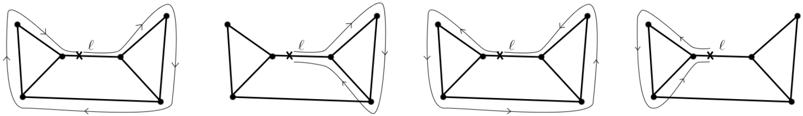
- $u$ lives on bond $\alpha$ (or rather, $u|_{[0,L_\alpha]}$) while $[K_{\beta\alpha} u]$ lives on bond $\beta$
- With this we trace out all closed paths that begin and end on point $x$, and propagate $u$ along this path with the above transformation. This will give a solution for that bond (in terms of the free space solution).

# The Method

4 possible types of paths that begin and end at point $x$ on bond $\ell$

1. $x \to \ell^+ \to \mathbf{p} \to \ell^+ \to x$: Periodic Path

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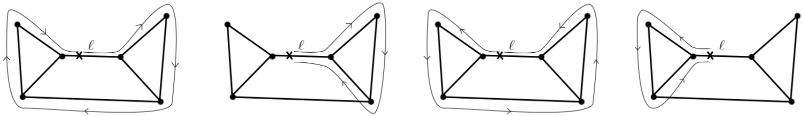


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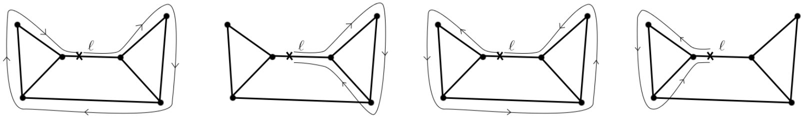


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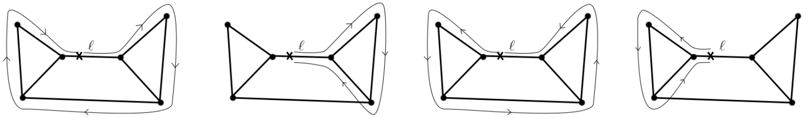

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- Propagate $u(x)$ around $\mathbf{p} = (\alpha_1, \ldots, \alpha_n)$:

$$u(x) \longmapsto K_{\alpha_n \alpha_{n-1}} \cdots K_{\alpha_3 \alpha_2} K_{\alpha_2 \alpha_1} u(x) := \Phi_{\alpha_1 \cdots \alpha_n} u(x)$$

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- Sum all the contributions over all paths going over $n$ bonds between $\beta$ and $\gamma$

$$K_n^{(\beta,\gamma)} u(x) := \sum_{\alpha_1} \cdots \sum_{\alpha_n} \Phi_{\beta \mathbf{p} \gamma} u(x), \quad n > 0$$

$$K_0^{(\beta,\gamma)} u(x) := K_{\gamma\beta} u(x)$$

# The Method

- After changing coordinates from bond $\ell^-$ to $\ell^+$ when necessary we obtain:

$$u^{\ell}_{\ell}(x) = u(x) + \sum_{n=0}^{\infty} [K_n^{(\ell^-,\ell^-)} u(L_\ell - z)]_{z=L_\ell - x} + [K_n^{(\ell^+,\ell^+)} u(x)] + [K_n^{(\ell^-,\ell^+)} u(L_\ell - x)] + [K_n^{(\ell^+,\ell^-)} u(z)]_{z=L_\ell - x}$$

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- (The coordinate of $\ell$ is that of $\ell^+$)

# The Method

- If $S$ is $k$-independent, we get the nice result:

$$\Phi_{\alpha_1,\ldots,\alpha_n} u(x) = \underbrace{S_{\alpha_1,\alpha_2} S_{\alpha_2,\alpha_3} \cdots S_{\alpha_{n-1},\alpha_n}}_{A_{\mathbf{p}}} u(\underbrace{L_{\alpha_1} + \cdots + L_{\alpha_n}}_{L_{\mathbf{p}}} + x)$$

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- This yields:

$$u_\ell^\ell(x) = u(x) + \sum_{\mathbf{p}} A_{\ell^+ \mathbf{p} \ell^-} u(L_{\mathbf{p}} + 2L_\ell - x) + A_{\ell^- \mathbf{p} \ell^+} u(-L_{\mathbf{p}} - x) + A_{\ell^- \mathbf{p} \ell^-} u(-L_{\mathbf{p}} - L_\ell + x) + A_{\ell^+ \mathbf{p} \ell^+} u(L_{\mathbf{p}} + L_\ell + x)$$

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- This gives $G^{\ell}_{\ell}(t; x, y)$.
- From this we can find the trace of the kernel by the formula:

$$\text{Tr}\, G(t) = \sum_{\ell=1}^{B} \int_0^{L_\ell} G^{\ell}_{\ell}(t; x, x) dx.$$

# Periodic Paths/Orbits

- Contribution from periodic paths:

$$PO := \sum_{\ell=1}^{B} \int_0^{L_\ell} \sum_{\mathbf{p}} A_{\ell^- \mathbf{p} \ell^-} G(t; |-L_{\mathbf{p}} - L_\ell + x - y|)|_{y=x} + A_{\ell^+ \mathbf{p} \ell^+} G(t; |L_{\mathbf{p}} + L_\ell + x - y|)|_{y=x} dx$$

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$$+ A_{\ell + \mathbf{p} \ell +} G(t; |L_{\mathbf{p}} + L_\ell + x - y|)|_{y=x} dx$$

- $$PO = \sum_{\ell=1}^{B} \sum_{\mathbf{p}} [A_{\ell - \mathbf{p} \ell -} + A_{\ell + \mathbf{p} \ell +}] G(t; L_{\mathbf{p}} + L_\ell) L_\ell$$

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# Bounce Paths

- Contribution for bounce paths:

$$BP := \sum_{\ell=1}^{B} \int_0^{L_\ell} \sum_{\mathbf{p}} A_{\ell^+ \mathbf{p} \ell^-} G(t; |L_{\mathbf{p}} + 2L_\ell - x - y|)|_{y=x} + A_{\ell^- \mathbf{p} \ell^+} G(t; |-L_{\mathbf{p}} - x - y|)|_{y=x} dx$$

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# Bounce Paths

- Lemma: *Under present assumptions,*

$$\sum_{\alpha_n=1}^{2B} A_{\bar{\alpha}_n \alpha_1 \cdots \alpha_n} = \delta_{\alpha_1, \overline{\alpha_{n-1}}} A_{\alpha_1 \cdots \alpha_{n-1}}.$$

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- Theorem:

$$BP = \frac{1}{2}\left[\sum_{\alpha} S_{\alpha,\bar{\alpha}}\right]\int_0^{\infty} G(t;x)dx$$

# Result

- *PO* and *BP* with the Weyl term then gives the result:

$$\mathrm{Tr}G(t) = LG(t;0) + \frac{1}{2}\left[\sum_{\alpha} S_{\alpha,\bar{\alpha}}\right]\int_0^{\infty} G(t;x)dx + \sum_p A_p \frac{L_p}{r_p} G(t;L_p)$$

# Trace Formula

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- Applying this we get the well known trace formula,

$$\sum_{k_n} \delta(k-k_n) = \frac{L}{\pi} + \frac{1}{2}\sum_{\alpha} S_{\alpha\bar{\alpha}}\delta(k) + \frac{1}{\pi}\sum_{p} A_p \frac{L_p}{r_p}\cos(kL_p)$$

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- Then putting it in our formula,

$$\mathrm{Tr}\ T(t) = \frac{L}{\pi t} + \frac{1}{4}\sum_{\alpha} S_{\alpha\bar{\alpha}} + \sum_{p} A_p \frac{L_p}{r_p}\frac{t/\pi}{t^2 + L_p^2}$$

# Vacuum Energy

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$$E_c = -\frac{1}{2\pi}\sum_p \frac{A_p}{L_p r_p}$$

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- As a consequence, it is continuous in the length of the legs.
- This answer can be obtained directly from trace formula, but this method offers a way to look at the vacuum energy density (which would be the regular part of $-\frac{1}{2}\frac{d}{dt}T^{\ell}_{\ell}(t;x,x)$)

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- Properties of this can shed light on other, more arbitrary, systems and their vacuum energy.
- This method, in particular, allows us to investigate vacuum energy density as well.