From Electrical Impedance Tomography to Network Tomography

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This talk will be divided into three parts

Part I. Tomography.
- -Introduction to the Radon transform
- -Some Applications of the Radon transform

Part II. Discrete analogues of tomography
- -Electrical Tomography (network of resistors)
- -Internet tomography (tree model)

Part III. Network tomography
- -Communication networks
  - -Weighted graph model

Part I. Tomography.

The Radon transform of a reasonable function $f(x)$, $x \in \mathbb{R}^n$ is defined to be

$$Rf(\alpha, p) = \int_{l_{\alpha p}} f(x)\,dx,$$

where $\alpha \in S^{n-1}$, S^{n-1} is the unit sphere in $\mathbb{R}^n$, $p \in \mathbb{R}$,

$l_{\alpha p} = \{ x \text{ such that } \alpha \cdot x = p \}$ is an hyperplane and dx is the Lebesgue measure on this hyperplane.

QUESTION: How can one recover $f(x)$ from $Rf(\alpha, p)$?

- *n=2 , CT scanners*. Original case studied by Radon, (1917)

- *n=3, (MRI) Minkowski, Fritz John (1934):* Relation with PDE

Inversion Formulae

If F_n stands for the n-dimensional Fourier transform then a standard inversion formula for the Radon transform in $\mathbb{R}^2$ is given by

$$f = F_2^{-1} F_1 (Rf)$$

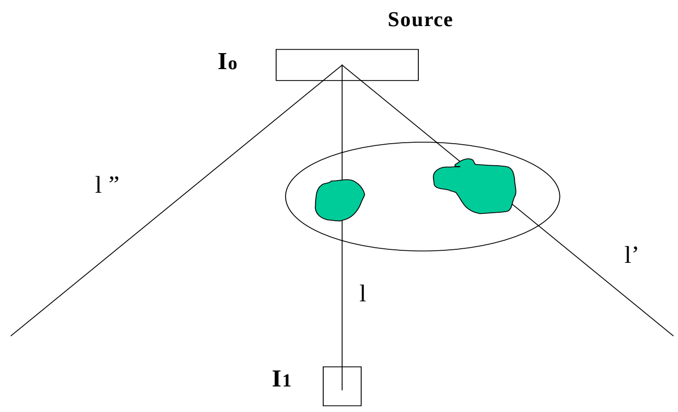
Definition. For $g : S^1 \times \mathbb{R} \to \mathbb{R}$ the *backprojection operator* R^* is defined by

$$R^* g(x) = \int_{S^1} g(\alpha, p) d\alpha, \quad p = \alpha \cdot x$$

One can introduce the operator Λ , square root of the Laplacian operator Δ, and we have the so called *backprojection inversion formula*

$$\Lambda (R^* R f)(x) = f(x)$$

CT Scans (X-ray transmission tomography $\mathbb{R}^2$)



Schematic CAT scanner

The setup consists of a detector and an X-ray beam source. A cross-section of the human body is scanned.

Let *f(x)* be the attenuation coefficient of the tissue at the point *x*. Let ζ be the straight line representing the beam, I_o the initial intensity of the beam, and I_1 its intensity after having traversed the body. It follows that

$$\frac{I_1}{I_o} = \exp\{-\int_{\zeta} f(x)dx\}$$

MRI in $\mathbb{R}^3$, ζ = plane

Electrical impedance tomography EIT

Finding the conductivity inside a plate by input-output current map

Electrical impedance tomography EIT

Let D be the unit disk in $\mathbb{R}^2$, $\beta > 0$ function on D, and let Ψ such that

$$\int_{\partial D} \Psi ds = 0$$

and u a solution of

$$\textbf{(NP)} \begin{cases} div(\beta grad u) = 0, \; in \quad D \\ \beta \frac{\partial u}{\partial n} = \Psi, \; on \quad \partial D \end{cases}$$

EIT Problem: Find β from the knowledge of Λ_β (*inverse conductivity problem).*

Nachman proved injectivity of the map $C: \beta \longrightarrow \Lambda_\beta$

Berenstein and *Casadio* formulated an approximate solution of this problem in terms of hyperbolic geometry in the hyperbolic disk and the corresponding hyperbolic Radon transform R_H

The input-output map $\Lambda_\beta : \Psi \rightarrow u$ (the Calderon map)

Approximate solution to the EIT problem

(Finding small cracks)

β_0 = Conductivity function of the material in normal conditions (Assume known)

β = Actual conductivity

Problem: How much does β deviate from β_0 ?

Assume β_0 constant equal to 1, then the deviation $\delta\beta$ is governed by

$$\beta = \mathbf{1} + \delta\beta, \text{ with } |\delta\beta| << \mathbf{1}$$

If no cracks on ∂D and U solution of (NP) for β =1,

$$\begin{cases} div(gradU) = 0, \; in \quad D \\ \frac{\partial U}{\partial n} = \Psi, \; on \quad \partial D \end{cases}$$

For u solution of (NP) for $\beta = 1+\delta\beta$,

$$\boldsymbol{u=U+\delta U}$$

The perturbation δU satisfies

$$\begin{cases} \Delta(\delta U) = -\langle grad\ \delta\beta, grad\ U\rangle \quad in \quad D \\ \frac{\partial U}{\partial n} = -(\delta\beta)\Psi \quad on \quad \partial D \end{cases}$$

With the only constraint

$$\int_{\partial D} \Psi ds = 0$$

Simplest input is a linear combination of dipoles $-\pi \frac{\partial}{\partial s}\delta_w$, δ_w the Dirac delta at ω in ∂D

The problem for the dipole (input) $-\pi\frac{\partial}{\partial s}\delta_w$ at ω now becomes

$$\begin{cases} \Delta U_w = 0 \quad in \quad D \\ \frac{\partial U_w}{\partial n} = -\pi\frac{\partial}{\partial s}\delta_w \quad on \quad \partial D \end{cases}$$

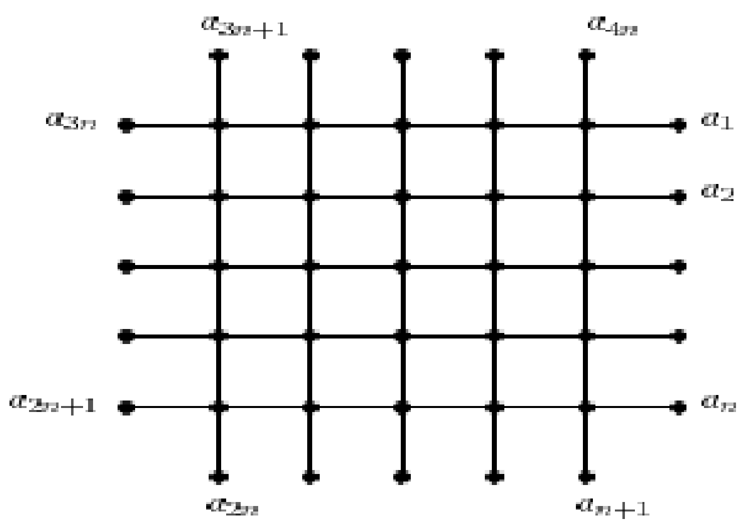
The solution U_w has level curves that are geodesics and thus the hyperbolic Radon transform appears naturally in this problem

Part II. Discrete analogue of EIT
Network Tomography

Consider a finite planar square network *G(V,E)*. The nodes of *V* are the integer lattice points $p=(i,j)$, $0\leq i\leq n+1$ and $0\leq j\leq n+1$, the corner points excluded

and *E* the set of edges. Let *intV*, interior of *V*, consisting of the nodes $p=(i,j)$ with $1\leq i\leq n$ and $1\leq j\leq n$. Let ∂G the boundary of *G*. If $\sigma\in$ E then σ

connects a pair of two adjacent nodes *p* and *q* and is denoted *pq*. Let *p* a node and denote *N(p)* the set of neighboring nodes of *p*



Definition. A *network of resistors* $\Gamma=\Gamma(V, E, \omega)$ is a network $G(V,E)$ together with a non-negative function $\omega : E \rightarrow \mathbb{R}^+$. For each edge *pq* in *E*, the number $\omega(pq)$ is called the *conductance* of *pq*, and $1/\omega(pq)$ is the *resistance* of *pq*. The function ω on *E* is called the *conductivity*.

Definition. Let $f : V \rightarrow \mathbb{R}$, and let $L_\omega f : intV \rightarrow \mathbb{R}$ a function defined by

$$L_\omega f(p) = \sum_{q \in N(p)} \omega(pq)(f(q) - f(p))$$

The function f is called ω-harmonic if $L_\omega f(p)=0 \ \forall p \in intV$

Let $\Phi(r)$ voltage applied at each boundary node *r*. Φ induces a voltage $f(p)$ at each $p \in intV$.

$$\mathbf{L_\omega f(p)=0 \ \forall p \in intV}. \text{ (Kirchhoff's law)}$$

On the boundary bV of V, f determines a *current* I_Φ by Ohm's law.

For each conductivity ω we define the linear Dirichlet-to-Neumann map Λ_ω by $\Lambda_\omega(\Phi) = I_\Phi$

Questions: *(i)* Is the map Λ_ω to ω one-to-one ?

(ii) Is it true that $\omega = \beta \Leftrightarrow \Lambda_\omega = \Lambda_\beta$? (uniqueness)

(iii) Is there a constructive algorithm to obtain ω from Λ_ω ?

(Curtis and Morrow)

Interesting questions

- Is it possible to extend these results to more general finite graphs?. For instance, is (ii) true?.

- What type of boundary measurements and associated probes we need to construct Λ_ω from the available boundary data and/or probes?

Internet tomography

Goal: Understanding a large network like internet

Typical problems: Traffic delay, link level parameter estimation, topology,

congestion in links, attacks

Definition. A tree T is a finite or countable collection of vertices $\{v_j \;\; j=0, 1,..\}$ and a collection of edges $e_{jk} = (v_j , v_k)$
Natural domain to visualize internet $\mathbb{H}^3$ **(Munzner).**

Locally, internet can be seen as part of a tree therefore natural domain is $\mathbb{H}^2$ **(Jonckheere E.-2004** experimentally**).** Hence, a way to study *locally* this kind of network can be done using the hyperbolic Radon transform on trees. *C. A. Berenstein et al.* [5, 6]

Data we need can be obtained using probes via measurements (sender-receiver)

Software: NS2 network simulator gives a number of other measurable quantities

Visualization in real hyperbolic space (Munzner)

- Radon transform in real hyperbolic spaces

trees $\longrightarrow$ $\mathbb{H}^2$

graphs $\longrightarrow$ $\mathbb{H}^3$

NETWORK CONNECTIVITY

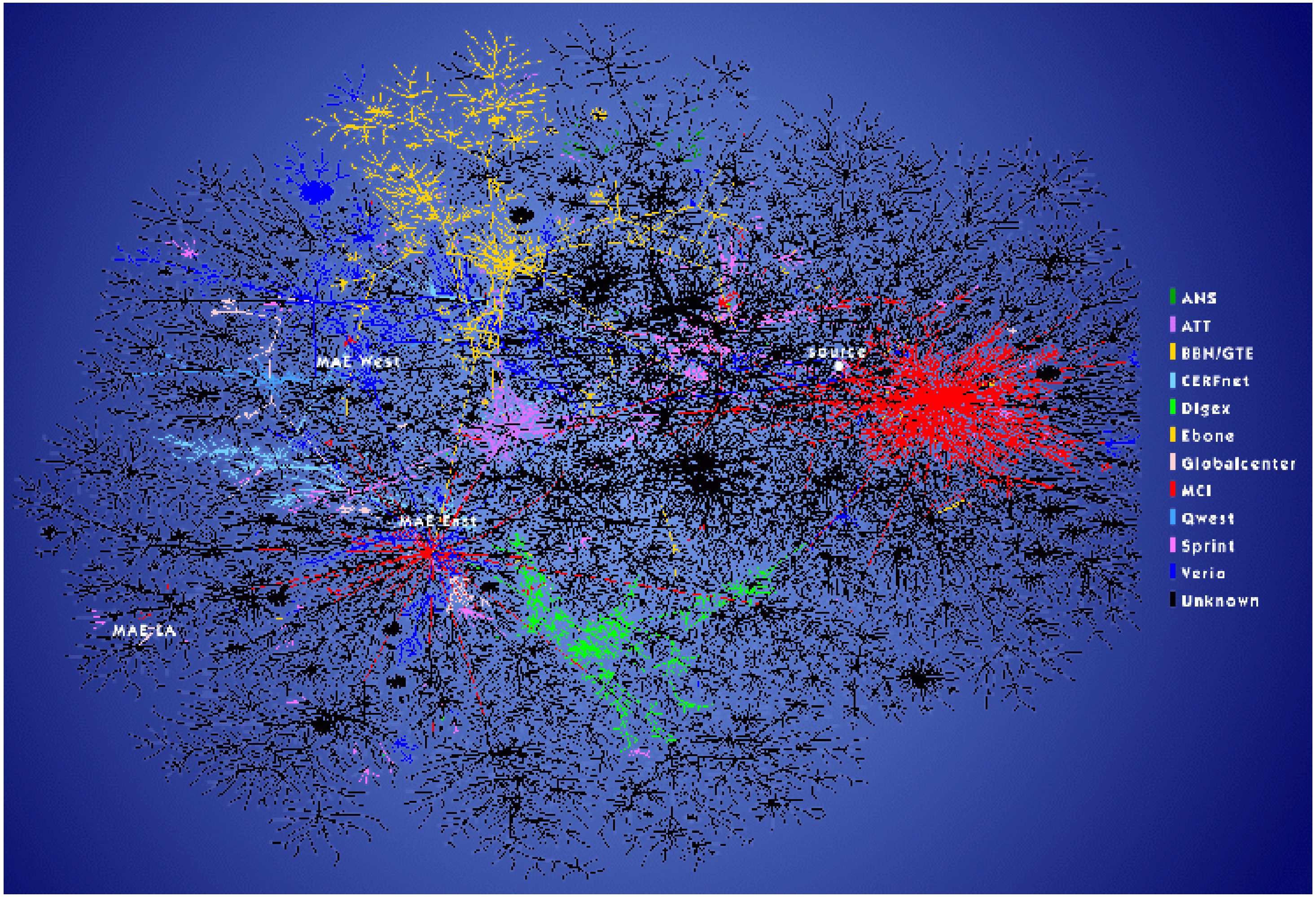
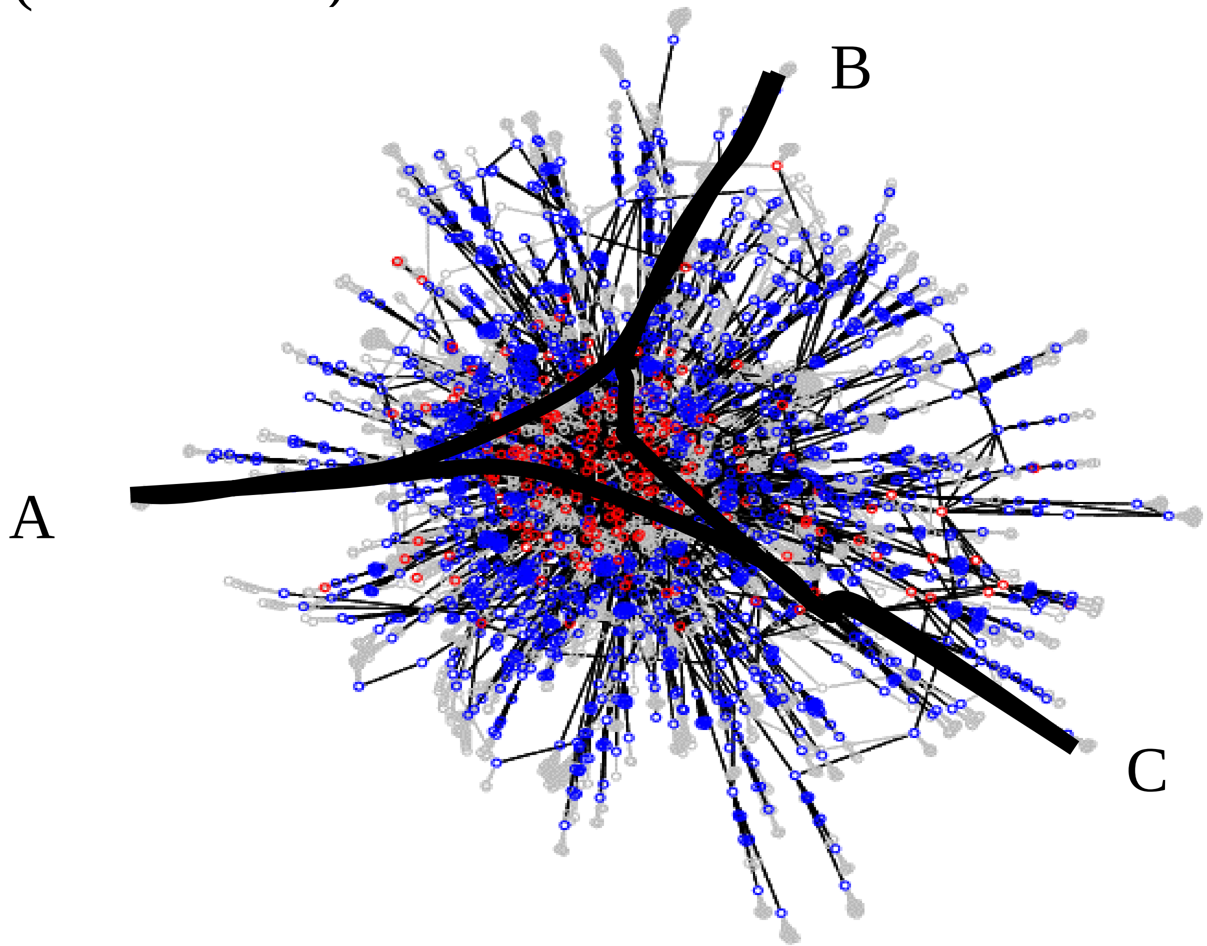
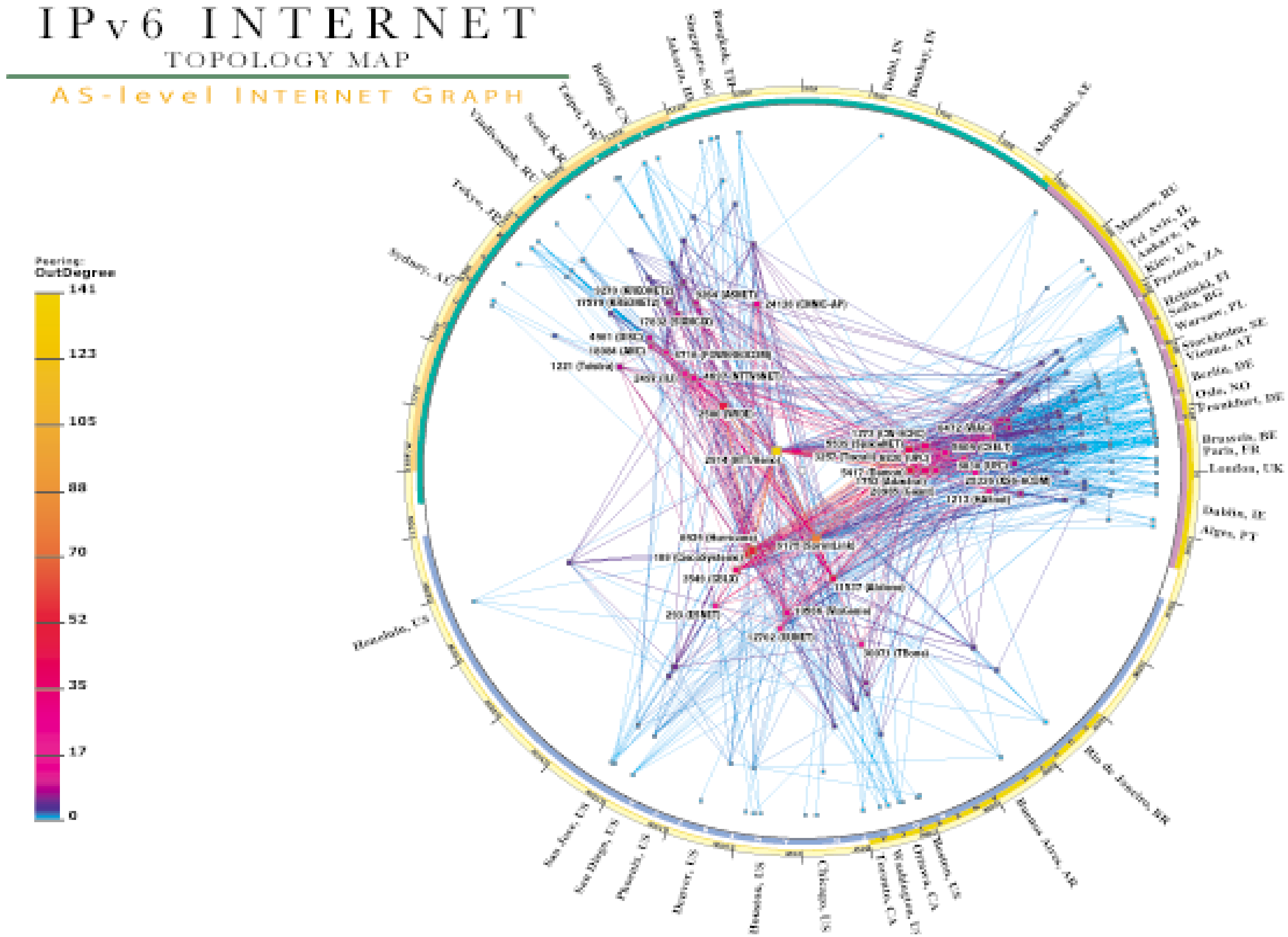


Figure 1: **Prototype two-dimensional image depicting global connectivity among ISPs as viewed from skitter host. Graph layout code provided by W. Cheswick and H. Burch (Lucent/Bell Laboratories).**

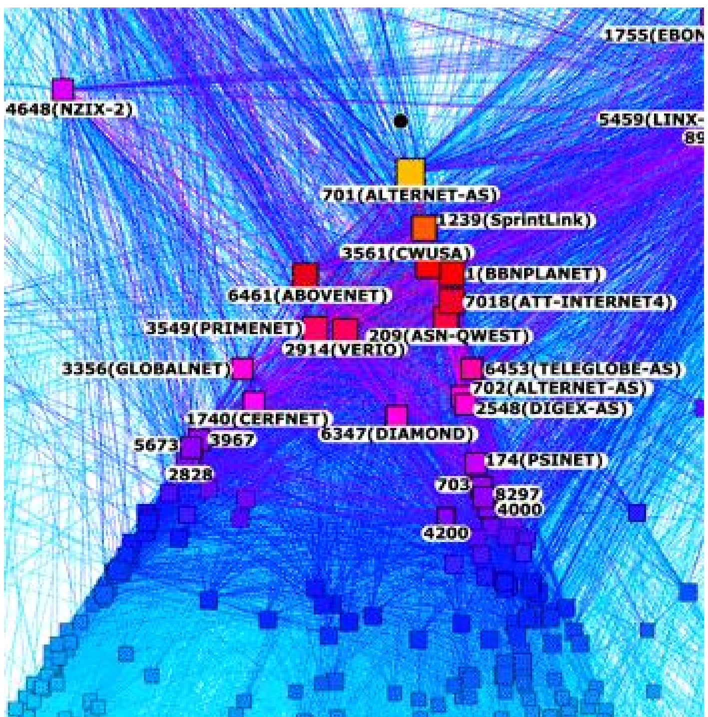
Taken from:http://www.nature.com/nature/webmatters/tomog/tomog.html, Authors are members of CAIDA organization.

Graphs with a highly connective core and long tendrils are hyperbolic because the sides of ΔABC are forced to go via the core (*Jonckheere*)





CAIDA Monitoring Tool: graphical representation of the Internet in hyperbolic space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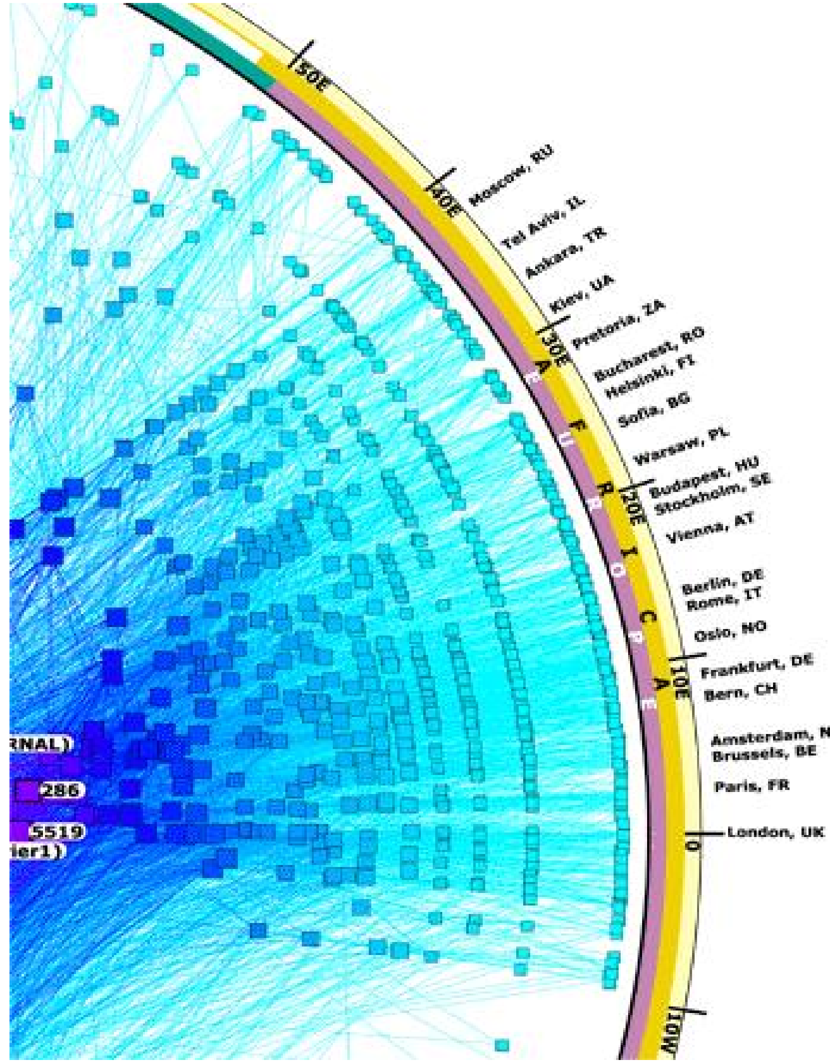
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Part III. Network Tomography

• Communication networks

-Weighted graph model

Networks: ATM, Internet, highways, phone,

Typical network problems:

- Failure of some nodes
  (Topology configuration)
- Congestion in links
  (Link-level parameter estimation)

Goal: Obtain some information of the inner structure of network

Network $\longleftrightarrow$ $G(V, E)$ a finite planar connected graph with $\partial G \neq \phi$

$\omega : E \rightarrow \mathbb{R}^+$, $G(V, E, \omega)$ weighted graph

weight $\omega(x, y)$ $\longleftrightarrow$ total traffic between endpoints x and y of the edge

Calculus on weighted graphs

Definitions: The degree of a node x in $G(V, E, \omega)$ is defined by

$$d_\omega x = \sum_{y \in V} \omega(x, y).$$

The Laplacian operator corresponding to this weight ω is defined by

$$\Delta_\omega f(x) = \sum_{y \in V} [f(x) - f(y)] \cdot \frac{\omega(x,y)}{d_\omega x}, \; x \in V$$

The integration of a function $f : G \rightarrow \mathbb{R}$ on a graph $G = G(V, E)$ is defined by

$$\int_G f d_\omega = \sum_{x \in V} f(x) d_\omega x$$

A graph $S=S(V', E')$ is a subgraph of $G(E, V)$ if $V' \subset V$ and $E' \subset E$

For a subgraph S of G, the boundary of S, ∂S, is defined by

$$\partial S = \{z \in V \mid z \notin S \, and \, z \sim y \;\; for \, some \, y \in S\}$$

the inner boundary of S, $inn\partial S$, is defined by

$$inn\partial S = \{z \in S \mid z \sim y \;\; for \, some \, y \in S\}$$

$\overline{S}$ is the graph whose edges and nodes are in $S \cup \partial S$

the (outward) normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is defined by

$$\frac{\partial f}{\partial n_\omega}(z) = \sum_{y \in S}[f(z) - f(y)] \cdot \frac{\omega(z,y)}{d'_\omega z}$$

where $d'_\omega x = \sum_{y \in S} \omega(z, y)$.

Weighted graph model —— two kinds of disruptions

1-Edge ceases to exist $\Rightarrow$ Topology changes (F. Chung)

Problem 1: Determine the topology

(F. Chung)

2-Increase of traffic $\Rightarrow$

- same topology
- weights ω remain same or increase

Problem 2: Determine ω

CAN PROBLEMS 1 AND 2 BE SOLVED SIMULTANEOUSLY?

Problem 1: Determine the topology of the graph.

(F. Chung)

Problem 2. Determine the weights ω

First, we need ω to be distinguished from any other weight β

We appeal to the following theorem of *C. Berenstein* and *S. Chung- 2003*

TEOREMA [2] ***(** *Berenstein-Chung)* **)**

Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$ *and*

$f_1, f_2 : \overline{S} \to \mathbb{R}$ *be functions satisfying that for j=1, 2,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, \; x \in S \\ \frac{\partial f}{\partial n_{\omega_j}}(z) = \Phi(z), \; z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases}$$

for any given function $\Phi : \partial S \to \mathbb{R}$ *with* $\int_{\partial S} \Phi = 0$ *and a given constant K with*

$K > m_{0,}$ *where*

$$m_o = \max_{j=1,2} \left| m_j \right| \cdot vol(S, w_j), m_j = \min_{z \in \partial S} f_j(z), j = 1,2 \text{ and } vol(S, w_j) = \sum_{x \in S} d_{\omega_j} x$$

If we assume that

(i) $\omega_1(z, y) = \omega_2(z, y)$ *on* $\partial S \times \mathrm{Int}\,(\partial S)$

(ii) $f_1|_{\partial S} = f_2|_{\partial S}$,

then we have

$$f_1 \equiv f_2$$

and

$$\omega_1(x, y) = \omega_2(x, y)$$

for all x and y in $\overline{S}$

Berenstein-Chung (uniqueness theorem)

$\Downarrow$

Dirichlet-to-Neumann map Λ_ω determines ω uniquely

$\Downarrow$

weight ω can be computed from knowledge of Dirichlet data for convenient choices of the input Neumann data in a "similar" way to the one for resistors networks

EIT ⟷ Neumann-to-Dirichlet problem

↑ continuous setting

Internet tomography ⟷ Neumann-to-Dirichlet problem in graphs

CURRENT WORK

Based on the EIT approach, find the conductivity (weight) ω (output)

where

$$\frac{\partial f}{\partial n_\omega}(z) \ , \qquad f\,|_{\partial S} \quad \text{and} \quad \omega|_{\partial S \times int \partial S}$$

inputs (given or measured).

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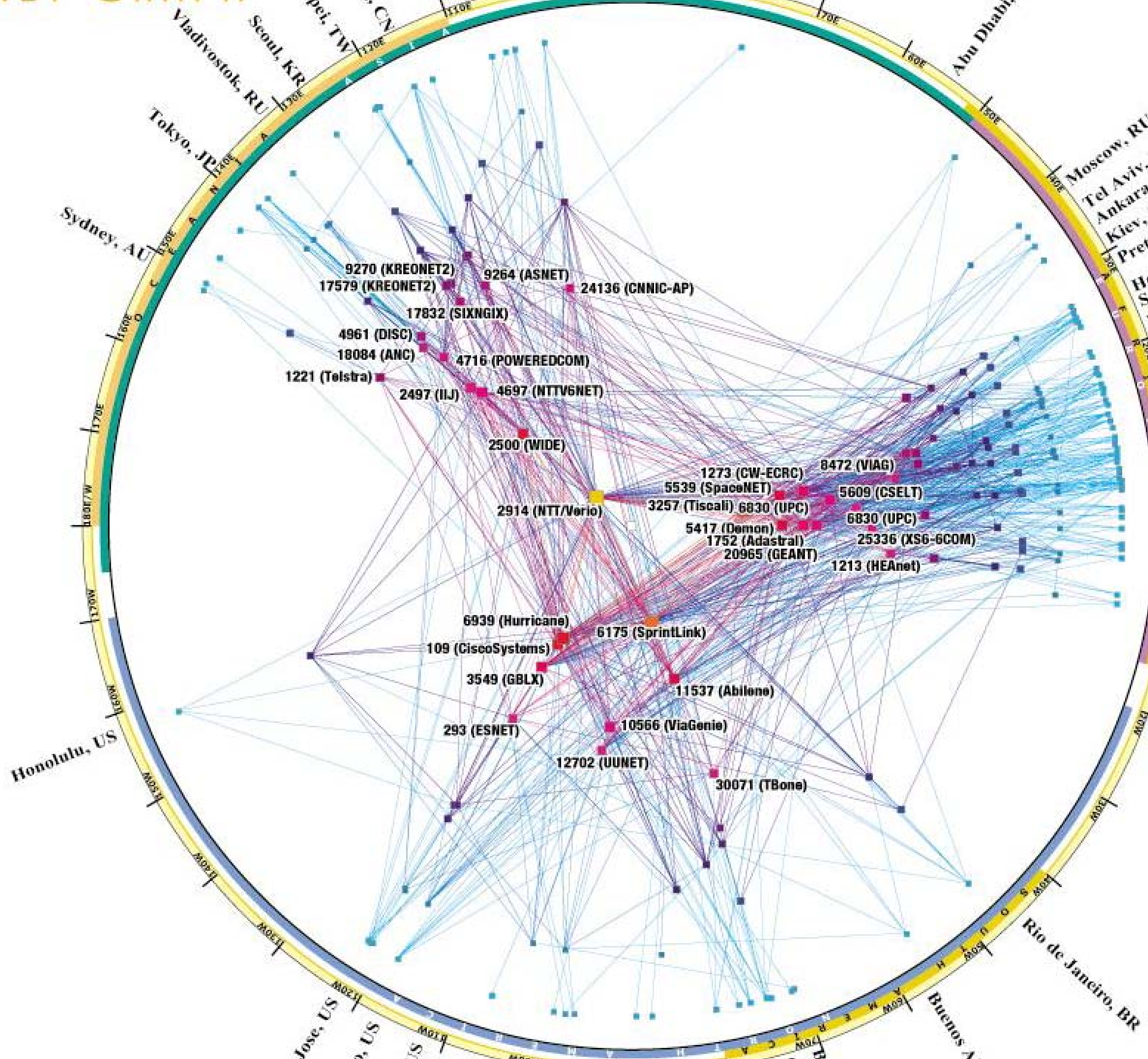

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