On the negative spectrum of quantum graphs

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based on a joint work with
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Outline

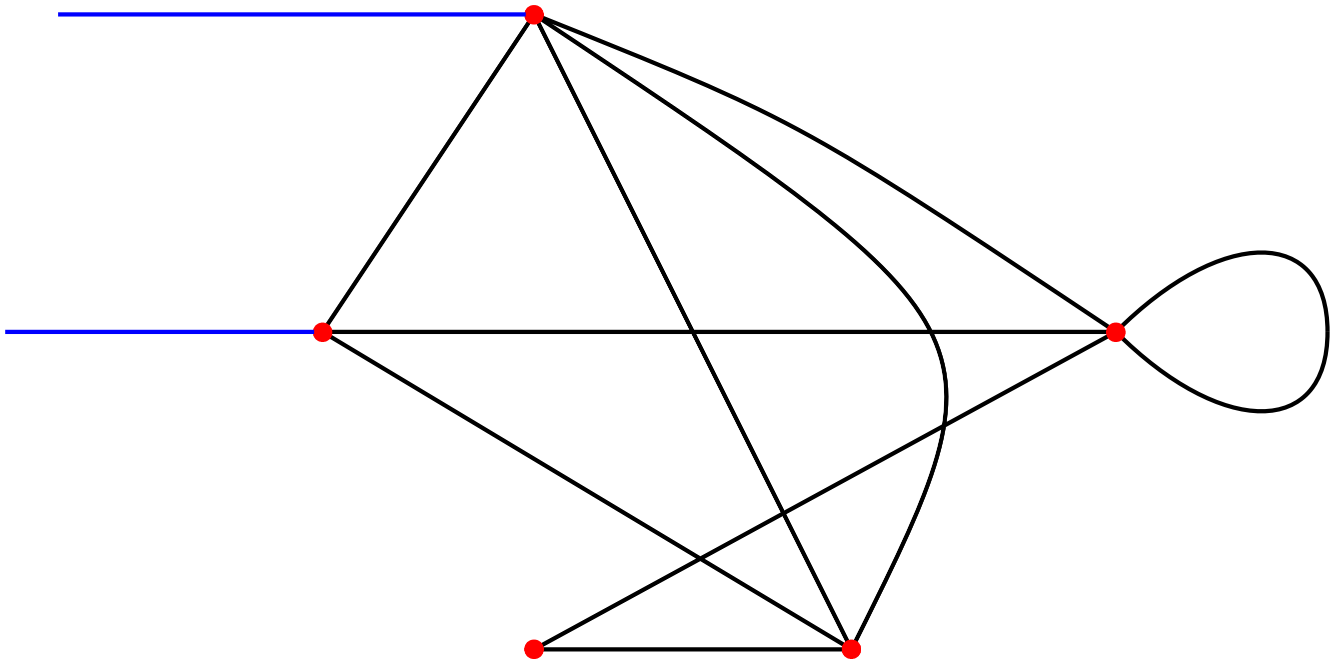
Self-Adjoint Laplace Operators

Number of Negative Eigenvalues

Lower Bounds on the Spectrum

Multiplicity of the Lowest Eigenvalue

Let $G = ({\color{red}V}, E)$ be a connected graph,
multiple edges and loops are allowed:



${\color{red}V}$	vertex set
$E = E_{\text{int}} \cup {\color{blue}E_{\text{ext}}}$	edge set

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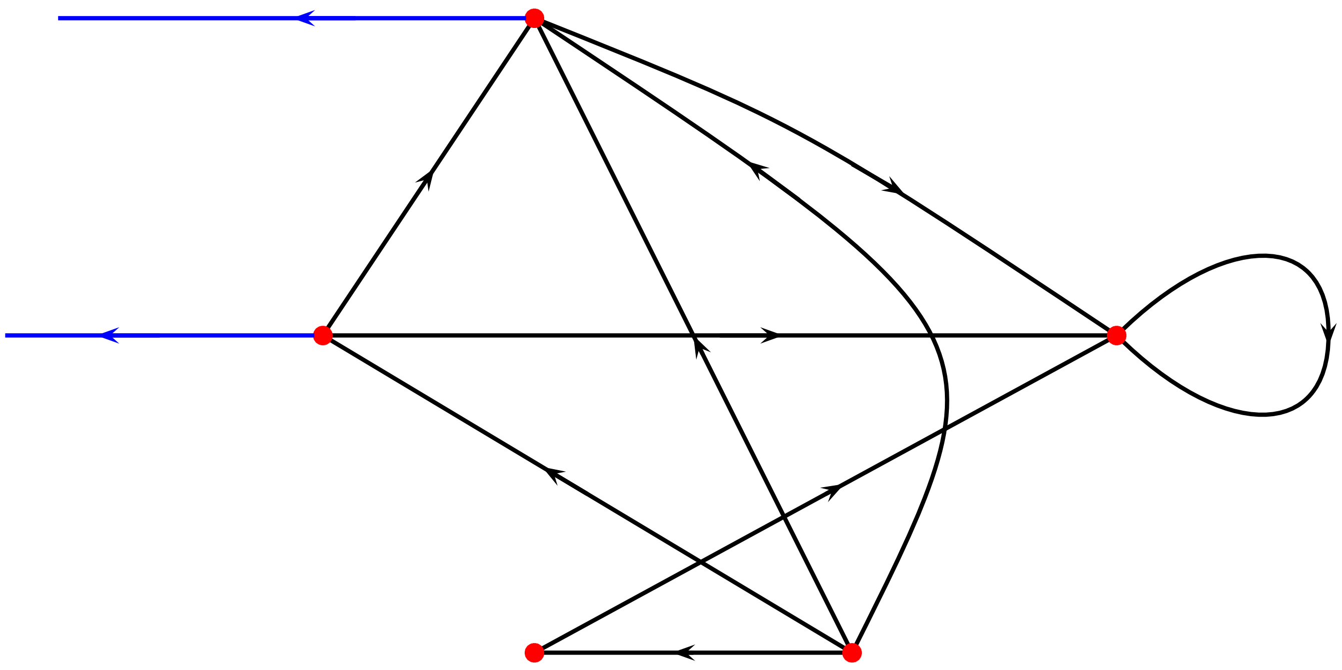

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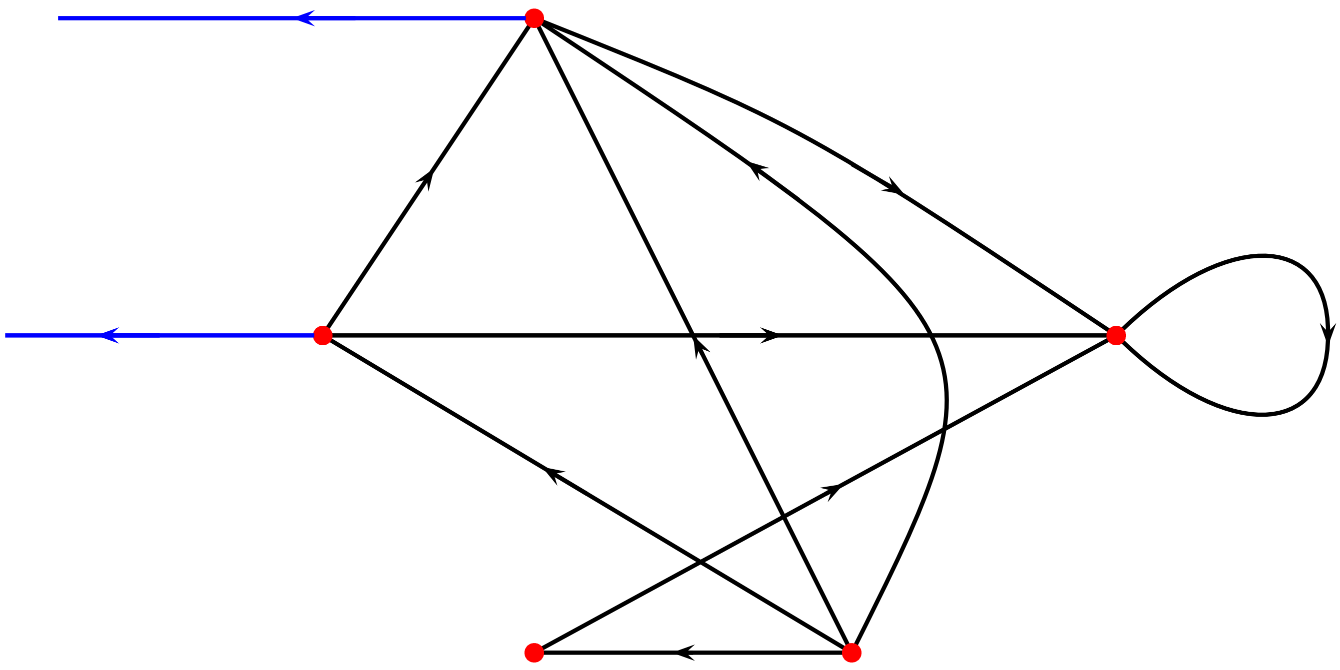

Each internal edge $j \in E_{\mathrm{int}}$ will be associated with a bounded open interval $I_j := (0, a_j)$, $a_j > 0$, according to its orientation.
Each external edge $j \in E_{\mathrm{ext}}$ will be associated with the semiline $I_j := (0, \infty)$.

- The Hilbert space $$\mathcal{H} = \bigoplus_{j\in E} L^2(I_j) = \bigoplus_{j\in E_{\text{int}}} L^2((0,a_j)) \oplus \bigoplus_{j\in E_{\text{ext}}} L^2((0,\infty))$$

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- For an arbitrary $\psi \in \mathcal{H}$ with $\psi_j \in H^2(I_j)$ (Sobolev space) set

$$\underline{\psi} = \begin{pmatrix} \{\psi_j(0)\}_{j\in E_{\text{ext}}} \\ \{\psi_j(0)\}_{j\in E_{\text{int}}} \\ \{\psi_j(a_j)\}_{j\in E_{\text{int}}} \end{pmatrix} \in \mathcal{K}, \qquad \underline{\psi}' = \begin{pmatrix} \{\psi_j'(0)\}_{j\in E_{\text{ext}}} \\ \{\psi_j'(0)\}_{j\in E_{\text{int}}} \\ \{-\psi_j'(a_j)\}_{j\in E_{\text{int}}} \end{pmatrix} \in \mathcal{K} \cong \mathbb{C}^{2|E_{\text{int}}|+|E_{\text{ext}}|}$$

and $[\psi] := \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \in \mathcal{K}_d := \mathcal{K} \oplus \mathcal{K}$ (space of boundary values).

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- Let Δ_0 be the Laplace operator in $\mathcal{H}$ defined on Sobolev functions in H^2 with the boundary conditions $[\psi] = 0$. It is a symmetric, closed, densely defined operator.

Theorem. The operator Δ with domain $\mathrm{Dom}(\Delta)$ is a self-adjoint extension of the symmetric operator Δ_0 if and only if the subspace

$$\mathcal{M} := \{[\psi] \,|\, \psi \in \mathrm{Dom}(\Delta)\} \subset \mathcal{K}_d$$

is maximal isotropic (= complex Lagrange plane), that is, the Hermitian symplectic form

$$\omega([\varphi],[\psi]) := \langle [\varphi], \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} [\psi] \rangle_{\mathbb{C}}, \qquad \omega([\varphi],[\psi]) = -\overline{\omega([\psi],[\varphi])}$$

vanishes on $\mathcal{M}$ and $\mathcal{M}$ has a maximal dimension $(= |E_{\mathrm{ext}}| + 2|E_{\mathrm{int}}|)$. We will write $\Delta(\mathcal{M})$ for the corresponding extension.

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Similar to the real symplectic theory, $\mathcal{M}$ is maximal isotropic if and only if $\mathcal{M}^{\perp}$ is maximal isotropic. Therefore, there is a duality transformation $\Delta(\mathcal{M}) \mapsto \Delta(\mathcal{M}^{\perp})$. Under this transformation $\Delta_{\mathrm{Dirichlet}} \mapsto \Delta_{\mathrm{Neumann}}$ and $\Delta_{\mathrm{Neumann}} \mapsto \Delta_{\mathrm{Dirichlet}}$.

A subspace $\mathcal{M} \subset \mathcal{K}_d$ is maximal isotropic if and only if there are linear maps $A : \mathcal{K} \longrightarrow \mathcal{K}$, $B : \mathcal{K} \longrightarrow \mathcal{K}$ such that $(A \quad B) : \mathcal{K}_d \longrightarrow \mathcal{K}$ has maximal rank and AB^* is self-adjoint. Then

$$\mathcal{M} = \mathcal{M}(A,B) := \left\{ [\psi] = \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \middle| A\underline{\psi} + B\underline{\psi}' = 0 \right\}$$

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Remarks:

- A and B define the boundary conditions

$$[\psi] \in \mathcal{M}(A,B) \quad \Leftrightarrow \quad A\underline{\psi} + B\underline{\psi}' = 0.$$

- $\mathcal{M}(A,B) = \mathcal{M}(TA,TB)$ for any invertble T.

Theorem [V.K., Schrader (1999), (2000); Harmer (2000)]. *The following statements are equivalent:*

- *The Laplace operator* $\Delta(A,B)$ *is self-adjoint,*
- $(A \quad B)$ *has maximal rank and* $AB^* = BA^*$*,*
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- Cut all internal edges of the graph and replace each of them by an external edge. $S(k;A,B)$ is the scattering matrix for the resulting completely disjoint graph.

Self-Adjoint Laplace Operators: Resolvent

The resolvent $(-\Delta(\mathcal{M}) - k^2)^{-1}$ is the integral operator with the $(|E_{\text{int}}| + |E_{\text{ext}}|) \times (|E_{\text{int}}| + |E_{\text{ext}}|)$ matrix-valued integral kernel given by

$$\begin{aligned} r_{\mathcal{M}}(x,y;k) &= r^{(0)}(x,y,k) \\ &+ \frac{\mathrm{i}}{2k}\Phi(x,k)R_+(k)^{-1}[I - S(k;\mathcal{M})T(k)]^{-1}S(k;\mathcal{M})R_+(k)^{-1}\Phi(y,k)^T, \end{aligned}$$

where $\Phi(x,k) := \begin{pmatrix} \phi(x;k) & 0 & 0 \\ 0 & \phi_+(x;k) & \phi_-(x;k) \end{pmatrix}$,

$$R_+(k) := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}k\underline{a}} \end{pmatrix}, \qquad T(k) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}k\underline{a}} \\ 0 & \mathrm{e}^{\mathrm{i}k\underline{a}} & 0 \end{pmatrix},$$

with diagonal matrices $\phi(x;k) = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}kx_j}\}_{j\in E_{\text{ext}}}$, $\phi_\pm(x;k) = \mathrm{diag}\{\mathrm{e}^{\pm\mathrm{i}kx_j}\}_{j\in E_{\text{int}}}$, and

$$\mathrm{e}^{\mathrm{i}k\underline{a}} = \mathrm{diag}\{\mathrm{e}^{\pm\mathrm{i}ka_j}\}_{j\in E_{\text{int}}}, \qquad [r^{(0)}(x,y,k)]_{j,j'} = \mathrm{i}\delta_{j,j'}\frac{\mathrm{e}^{\mathrm{i}k|x_j-y_j|}}{2k}, \quad x_j, y_j \in I_j.$$

Self-Adjoint Laplace Operators

Assume: [Kuchment (2004)]

> P an orthogonal projection (possibly zero),
> L self-adjoint operator such that $LP^{\perp} = 0$.

Set $A = P^{\perp} + L$, $B = P$. Then $AB^* = L$ and $(A \quad B)$ has maximal rank. Hence, $\mathcal{M}(A, B)$ is maximal isotropic.

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Do we get all the maximal isotropic subspaces? Yes! Take an arbitrary unitary S_0. Split off its eigensubspace corresponding to the eigenvalue -1. Denote by $\widehat{S}_0$ the restriction of S onto the complimentary subspace. Take its Cayley transform $\widehat{L} = \mathrm{i}(I + \widehat{S}_0)^{-1}(I - \widehat{S}_0)$. Let L be the extension of $\widehat{L}$ to the whole of $\mathcal{K}$. Then we obtain

$$S(1; P^\perp + L, P) = -(P^\perp + L + \mathrm{i}P)^{-1}(P^\perp + L - \mathrm{i}P) = S_0,$$

where P is the orthogonal projection onto the eigensubspace of S_0 corresponding to the eigenvalue -1.

For any maximal isotropic subspace $\mathcal{M}$ there are unique $P_{\mathcal{M}}$ and $L_{\mathcal{M}}$ such that

$$\mathcal{M} = \left\{ [\psi] = \begin{pmatrix} \psi \\ \underline{\psi}' \end{pmatrix} \,\middle|\, (P^{\perp} + L)\underline{\psi} + P\underline{\psi}' = 0 \right\}.$$

What happens with $L_{\mathcal{M}}$ under the duality transformation $\mathcal{M} \mapsto \mathcal{M}^{\perp}$?

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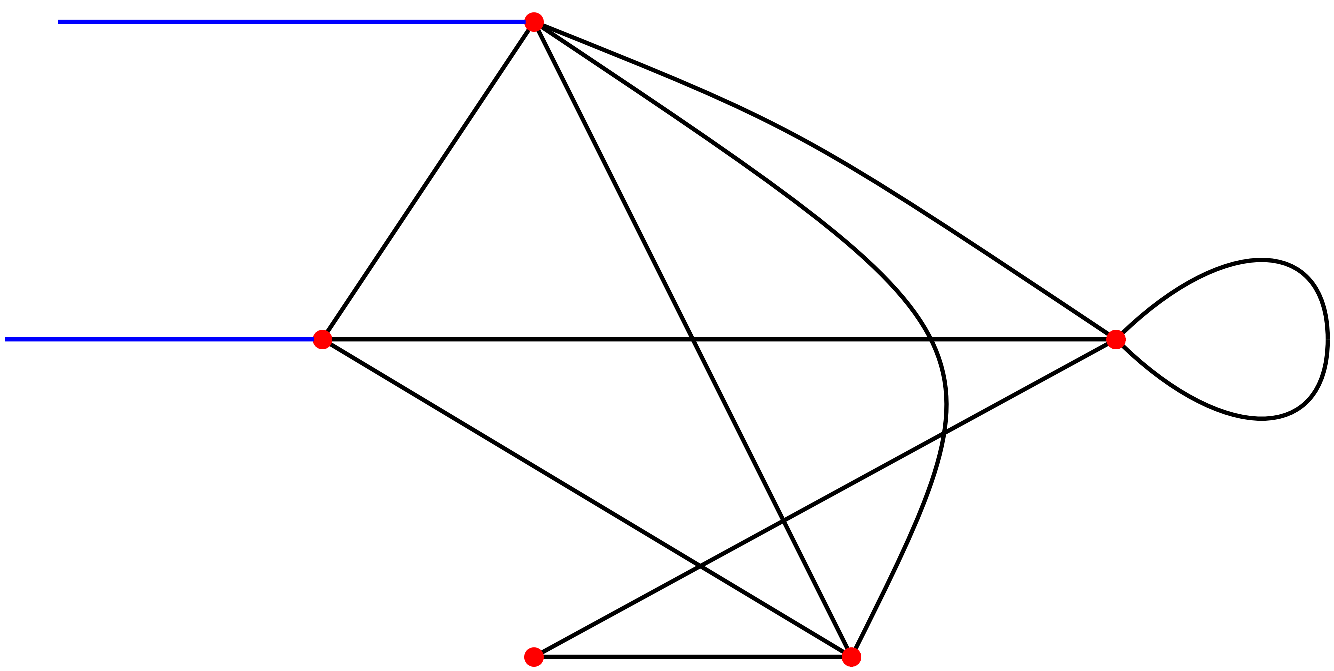
We know that $\mathcal{M}(A,B)^{\perp} = \mathcal{M}(-B,A)$. Hence, the duality transformation sends AB^* to $(-B)A^* = -BA^* = -AB^*$.

Take

$$A = P_{\mathcal{M}}^{\perp} + L_{\mathcal{M}}, \quad B = P_{\mathcal{M}} \quad \Rightarrow \quad AB^* = L_{\mathcal{M}}.$$

Thus, the duality transformation sends $L_{\mathcal{M}}$ to $-L_{\mathcal{M}}$.

The boundary conditions are local, if $S(k;\mathcal{M})$ can be written in the block diagonal form $S(k;\mathcal{M}) = \bigoplus_{v\in V} S(k;\mathcal{M}_v)$ such that each block corresponds to a particular vertex.



Local Boundary Conditions

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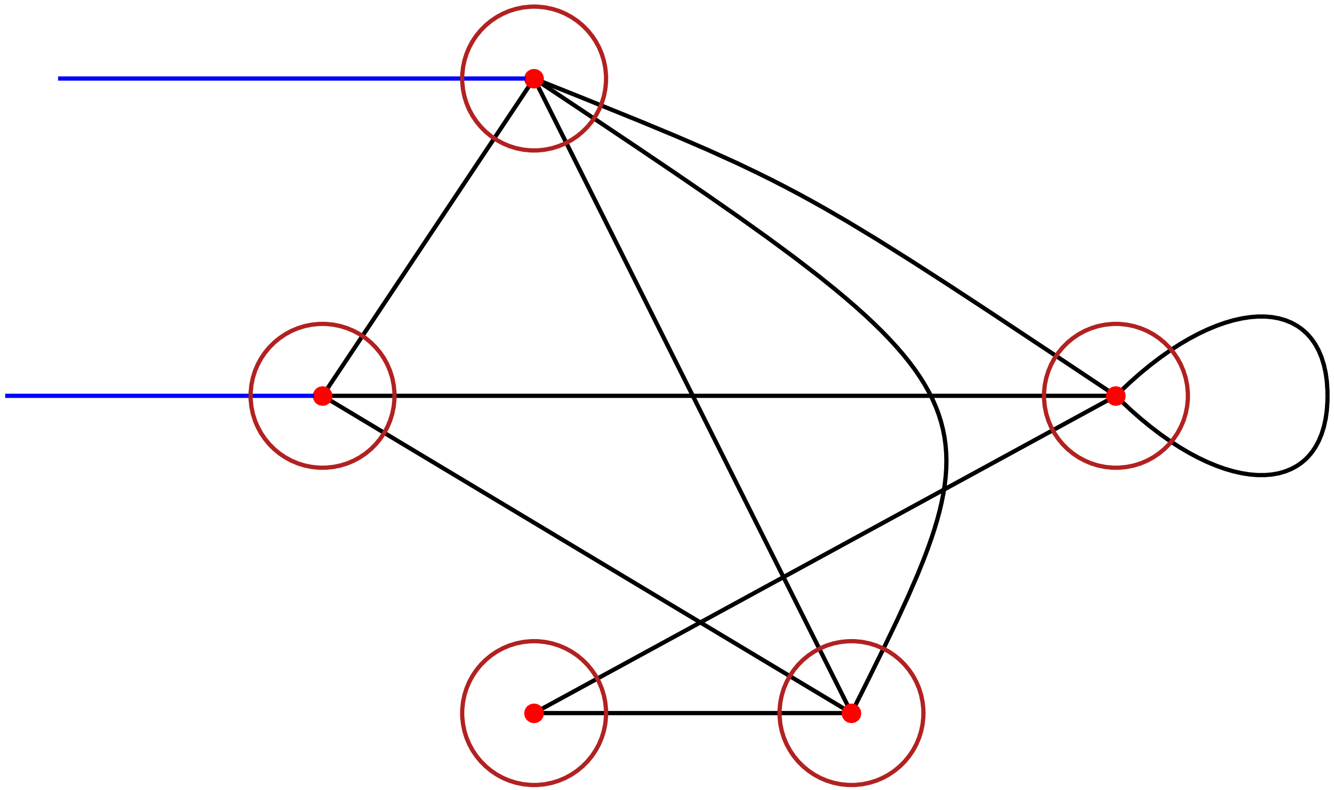
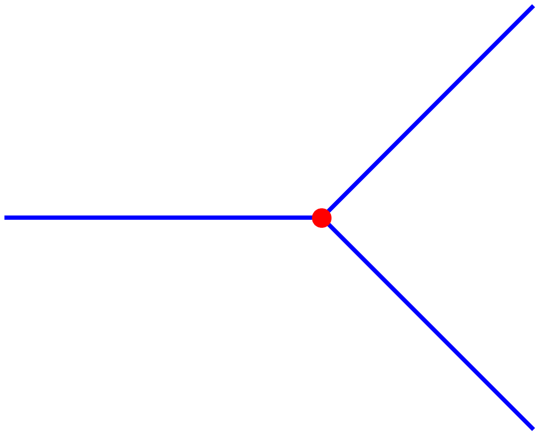


Let G be a “star graph”, that is a graph with a single vertex and no internal edges:



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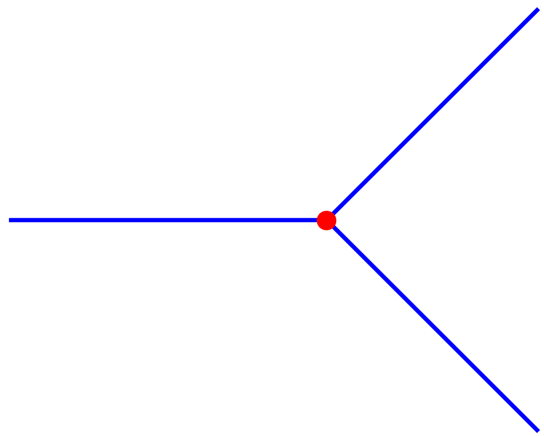

Claim: The number of negative eigenvalues of $-\Delta(A, B)$ (counting multiplicity) equals the number of positive eigenvalues of AB^* (counting multiplicity).

Remark: By Sylvester’s Inertia Law the number of positive eigenvalues of $(TA)(TB)^* = TAB^*T^*$ does not depend on T whenever T is invertible. Thus, AB^* and L have an equal number of positive eigenvalues.

Number of Negative Eigenvalues

Proof of $N_-(-\Delta(A,B)) = N_+(AB^*)$: With the ansatz $\psi_j(x) = \chi_j e^{-\varkappa x}$ we obtain that ψ is an eigenfunction of $-\Delta(A,B)$ with the eigenvalue $-\varkappa^2 < 0$ if and only if

$$(A - \varkappa B)\chi = 0.$$

The number of negative eigenvalues of $-\Delta(A,B)$ equals the number of positive zeroes of the function

$$\begin{aligned} f(\varkappa) &:= \det(A - \varkappa B) = \det(T(P^\perp + L - \varkappa P)) \\ &= \det T \det(P^\perp + L - \varkappa P) = \det T \det \begin{pmatrix} P^\perp & 0 \\ 0 & PLP - \varkappa P \end{pmatrix} \\ &= \det T \det(L - \varkappa). \end{aligned}$$

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For arbitrary graphs we have the inequality $N_-(-\Delta(A,B)) \leq N_+(AB^*)$.

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Proof: Compute the quadratic form of $-\Delta(A,B)$:

$$\langle \psi, -\Delta(A,B)\psi \rangle_{\mathcal{H}} = \sum_{j \in \mathcal{E}} \langle \psi_j', \psi_j' \rangle_{L^2(I_j)} + \langle [\psi], Q[\psi] \rangle_{\mathcal{K}_d}, \qquad [\psi] := \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix},$$

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It is straightforward to verify that

$$P_{\mathcal{M}} = \begin{pmatrix} -B^* \\ A^* \end{pmatrix} (AA^* + BB^*)^{-1} (-B \quad A)$$

is the orthogonal projection onto $\mathcal{M} = \left\{ [\psi] = \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \middle| A\underline{\psi} + B\underline{\psi}' = 0 \right\}$.

Positivity

Since $\psi \in \mathcal{H}$ belongs to the domain of $-\Delta(A,B)$, we have $P_{\mathcal{M}}[\psi] = [\psi]$. Therefore,

$$\langle [\psi], Q[\psi] \rangle_{\mathcal{K}_d} = \langle [\psi], P_{\mathcal{M}} Q P_{\mathcal{M}} [\psi] \rangle_{\mathcal{K}_d}$$

with

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Putting all together we obtain

$$\langle \psi, -\Delta(A,B)\psi \rangle_{\mathcal{H}} = \sum_{j \in \mathcal{E}} \langle \psi_j', \psi_j' \rangle_{L^2(I_j)} + \langle [\psi], P_{\mathcal{M}} Q P_{\mathcal{M}} [\psi] \rangle_{\mathcal{K}_d} \geq 0.$$

$\square$

A Special Class of Positive Operators

Let $\mathcal{M} = \mathcal{M}(A, B)$ be a maximal isotropic subspace. The following conditions are equivalent:

- $S(k; \mathcal{M})$ is k-independent,
- $S(k; \mathcal{M})$ is self-adjoint for some $k > 0$,
- $S(k; \mathcal{M})^2 = I$ for some $k > 0$,
- $AB^* = 0$.

If $S(k; \mathcal{M})$ is k-independent, then $S(k; \mathcal{M}^\perp) = -S(k; \mathcal{M})$ is k-independent, too.

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Examples:

- Dirichlet $S(\mathcal{M}_D) = -I$ and Neumann $S(\mathcal{M}_N) = I$,
- "standard" $[S(\mathcal{M}_{\mathrm{st}})]_{j,j'} = \frac{2}{\deg(v)} - \delta_{j,j'}$ and "co-standard" $[S(\mathcal{M}_{\mathrm{st}}^\perp)]_{j,j'} = -\frac{2}{\deg(v)} + \delta_{j,j'}$,
- all magnetic perturbations of the standard boundary conditions $U^* S(\mathcal{M}_{\mathrm{st}})U$, U depends on magnetic fluxes.

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- $S(k;\mathcal{M})$ is k-independent,
- $S(k;\mathcal{M})$ is self-adjoint for some $k > 0$,
- $S(k;\mathcal{M})^2 = I$ for some $k > 0$,
- $AB^* = 0$.

If $S(k;\mathcal{M})$ is k-independent, then $S(k;\mathcal{M}^\perp) = -S(k;\mathcal{M})$ is k-independent, too.

Examples:

- Dirichlet $S(\mathcal{M}_D) = -I$ and Neumann $S(\mathcal{M}_N) = I$,
- "standard" $[S(\mathcal{M}_{\mathrm{st}})]_{j,j'} = \frac{2}{\deg(v)} - \delta_{j,j'}$ and
  "co-standard" $[S(\mathcal{M}_{\mathrm{st}}^\perp)]_{j,j'} = -\frac{2}{\deg(v)} + \delta_{j,j'}$,
- all magnetic perturbations of the standard boundary conditions $U^* S(\mathcal{M}_{\mathrm{st}}) U$, U depends on magnetic fluxes.

Remark: Standard and Dirichlet are the only two boundary conditions with the property that all functions in $\mathrm{Dom}(\Delta(\mathcal{M}))$ are continuous.

Theorem [V.K., Potthoff, Schrader (in preparation)]. *The following statements are equivalent:*

- *The Laplace operator* $-\Delta(A,B)$ *is m-accretive, that is,*

$$\mathrm{Re}\langle\psi, -\Delta(A,B)\psi\rangle \geq 0 \qquad \textit{for all} \qquad \psi \in \mathrm{Dom}(-\Delta(A,B)),$$

- $(A \quad B)$ *has maximal rank and* $\mathrm{Re}\, AB^* \leq 0$*,*
- $S(\mathrm{i}\varkappa; A, B) := -(A - \varkappa B)^{-1}(A + \varkappa B)$ *is contractive for some (and hence for all)* $\varkappa > 0$*.*

A Generalization of Positivity

Theorem [V.K., Potthoff, Schrader (in preparation)]. *The following statements are equivalent:*

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- $S(\mathrm{i}\varkappa; A,B) := -(A - \varkappa B)^{-1}(A + \varkappa B)$ *is contractive for some (and hence for all)* $\varkappa > 0$*.*

The following statements are equivalent:

- *The Laplace operator* $-\Delta(A,B)$ *is m-dissipative, that is,*

$$\mathrm{Im}\langle\psi, -\Delta(A,B)\psi\rangle \geq 0 \qquad \text{for all} \qquad \psi \in \mathrm{Dom}(-\Delta(A,B)),$$

- $(A \quad B)$ *has maximal rank and* $\mathrm{Im}\, AB^* \leq 0$*,*
- $S(-k; A,B) := -(A - \mathrm{i}kB)^{-1}(A + \mathrm{i}kB)$ *is contractive for some (and hence for all)* $k > 0$*.*

- For star graphs $\lambda_{\text{min}} = -\|L_+\|^2$, its multiplicity equal to the multiplicity of the largest eigenvalue of L. It is a square of the biggest positive solution of the equation

$$\det(A - \varkappa B) = 0.$$

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- For general graphs the situation is much more interesting. Peter Kuchment (2004) proved the following bound:

$$\lambda_{\text{min}} \geq -4\|L\|(|E_{\text{ext}}| + 2|E_{\text{int}}|) \max\left\{2\|L\|, \max_{i \in E_{\text{int}}} \frac{1}{a_i}\right\},$$

For large graphs the right hand side goes to $-\infty$.

Lower Bounds on the Spectrum

Theorem [V.K., Schrader (2006)]. Assume that $E_{\text{int}} \neq \emptyset$. Then

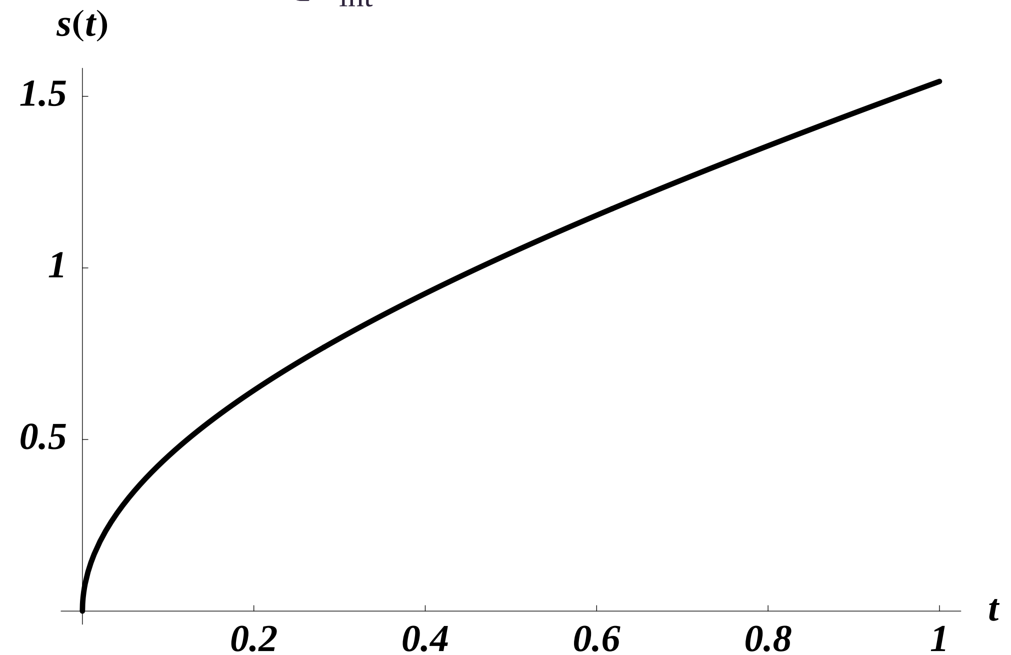
$$\lambda_{\min} \geq -s_a(\|L_+\|)^2,$$

where L_+ denotes the positive part of the operator L and $s_a(t)$ is the unique nonnegative solution of the equation

$$s \cdot \tanh\left(\frac{as}{2}\right) = t, \qquad a := \min_{i \in E_{\text{int}}} a_i.$$

Properies of $s_a(t)$:

- $s_a(0) = 0$,
- $s_a(t) \sim t$ for $t \to \infty$,
- $s_a(t) \sim \sqrt{\frac{2t}{a}}$ for $a \to 0$.

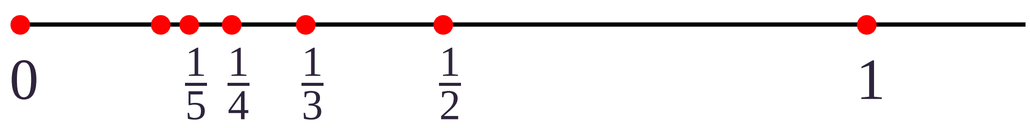


Lower Bounds on the Spectrum: An Example

Does $s_a(t) \sim \sqrt{\frac{2t}{a}}$ for $a \longrightarrow 0$ corresponds to the correct behaviour of the lowest eigenvalue?

Example: Consider the operator

$$H_n = -\frac{d^2}{dx^2} - \sum_{k=1}^{n} \delta(x - 1/k), \qquad \varkappa\psi(0) + \psi'(0) = 0, \quad \varkappa > 0 \qquad \text{on} \qquad L^2(0,\infty).$$



For $\psi(x) := \sqrt{2\varkappa}\mathrm{e}^{-\varkappa x}$ compute the quadratic form

$$\langle \psi, H_n \psi \rangle = \int_0^\infty \psi'(x)^2 dx - \sum_{k=1}^{n} \psi(1/k)^2 = \varkappa^2 - 2\varkappa \sum_{k=1}^{n} \mathrm{e}^{-2\varkappa/k}.$$

For $n \longrightarrow \infty$ the sum goes to $+\infty$. Thus, $\lambda_{\text{min}}(H_n) \underset{n\to\infty}{\longrightarrow} -\infty$.

Proof of $\lambda_{\textbf{min}} \geq -s_a(\|L_+\|)^2$. Let $\varkappa_0 > 0$ be the biggest solution of the equation $\det(A - \varkappa B) = 0$. Obviously,

$$\varkappa_0 := \|L_+\| = \text{the biggest positive eigenvalue of } L$$

Estimate

$$\|S(\mathrm{i}\varkappa; A, B)\| = \left\| -P_{\mathrm{Ker}B} - P^{\perp}_{\mathrm{Ker}B} \frac{L + \varkappa}{L - \varkappa} P^{\perp}_{\mathrm{Ker}B} \right\| = \frac{\varkappa + \varkappa_0}{\varkappa - \varkappa_0} > 1$$

for all $\varkappa > \varkappa_0$.

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for all $\varkappa > \varkappa_0$.The number $-\varkappa_+^2$ is the lowest eigenvalue of $-\Delta(A, B)$ if $\varkappa_+$ is the biggest solution of the equation

$$\det[I - S(\mathrm{i}\varkappa; A, B) T(\mathrm{i}\varkappa)] = 0 \qquad \text{with} \qquad T(\mathrm{i}\varkappa) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\varkappa \underline{a}} \\ 0 & \mathrm{e}^{-\varkappa \underline{a}} & 0 \end{pmatrix}$$

with $\mathrm{e}^{-\varkappa \underline{a}} := \operatorname{diag}(\mathrm{e}^{-\varkappa a_j})_{j \in E_{\mathrm{int}}}$.

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with $\mathrm{e}^{-\varkappa \underline{a}} := \mathrm{diag}(\mathrm{e}^{-\varkappa a_j})_{j \in E_{\mathrm{int}}}$. Therefore,

$$\|S(\mathrm{i}\varkappa_+; A, B) T(\mathrm{i}\varkappa_+)\| \geq 1$$

Proof of $\lambda_{\min} \geq -s_a(\|L_+\|)^2$. Let $\varkappa_0 > 0$ be the biggest solution of the equation $\det(A - \varkappa B) = 0$. Obviously,

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Estimate

$$\|S(\mathrm{i}\varkappa; A, B)\| = \left\| -P_{\mathrm{Ker} B} - P_{\mathrm{Ker} B}^{\perp} \frac{L + \varkappa}{L - \varkappa} P_{\mathrm{Ker} B}^{\perp} \right\| = \frac{\varkappa + \varkappa_0}{\varkappa - \varkappa_0} > 1$$

for all $\varkappa > \varkappa_0$.The number $-\varkappa_+^2$ is the lowest eigenvalue of $-\Delta(A, B)$ if $\varkappa_+$ is the biggest solution of the equation

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with $\mathrm{e}^{-\varkappa \underline{a}} := \mathrm{diag}(\mathrm{e}^{-\varkappa a_j})_{j \in E_{\mathrm{int}}}$. Therefore,

$$\|S(\mathrm{i}\varkappa_+; A, B)T(\mathrm{i}\varkappa_+)\| \geq 1 \Rightarrow \|S(\mathrm{i}\varkappa_+; A, B)\| \|T(\mathrm{i}\varkappa_+)\| \geq 1.$$

Lower Bounds on the Spectrum: The Proof

Thus,

$$\varkappa_+ < \varkappa_1 := \inf\{\varkappa > \varkappa_0 \,|\, \|S(\mathrm{i}\varkappa; A, B)\| \|T(\mathrm{i}\varkappa)\| < 1\}.$$

Since

$$\|S(\mathrm{i}\varkappa; A, B)\| \|T(\mathrm{i}\varkappa)\| = \frac{\varkappa + \varkappa_0}{\varkappa - \varkappa_0} \mathrm{e}^{-a\varkappa} \qquad \text{for all } \varkappa > \varkappa_0,$$

$\varkappa_1$ is not larger than the unique positive solution of the equation

$$\frac{\varkappa + \varkappa_0}{\varkappa - \varkappa_0} \mathrm{e}^{-a\varkappa} = 1$$

Lower Bounds on the Spectrum: The Proof

Thus,

$$\varkappa_+ < \varkappa_1 := \inf\{\varkappa > \varkappa_0 | \|S(\mathrm{i}\varkappa; A, B)\| \|T(\mathrm{i}\varkappa)\| < 1\}.$$

Since

$$\|S(\mathrm{i}\varkappa; A, B)\| \|T(\mathrm{i}\varkappa)\| = \frac{\varkappa + \varkappa_0}{\varkappa - \varkappa_0} \mathrm{e}^{-a\varkappa} \qquad \text{for all } \varkappa > \varkappa_0,$$

$\varkappa_1$ is not larger than the unique positive solution of the equation

$$\frac{\varkappa + \varkappa_0}{\varkappa - \varkappa_0} \mathrm{e}^{-a\varkappa} = 1 \quad \Leftrightarrow \quad \varkappa = s_a(\varkappa_0).$$

We are done!

Lower Bounds on the Spectrum: A Remark

Consider an arbitrary finite graph. Keep all but one edge length (say a_0) fixed and let $a_0 \to 0$. Then $a := \min_{i \in E_{\text{int}}} a_i \to 0$, but the lowest eigenvalue remains finite. The bound

$$\lambda_{\text{min}} \geq -s_a(\|L_+\|)^2,$$

becomes meaningless in this limit:

$$\lambda_{\text{min}} \geq -\lim_{a_0 \to 0} s_a(\|L_+\|)^2 = -\infty.$$

Quite unsatisfactory!

Assume that the boundary conditions are local, that is, $S(k;\mathcal{M}) = \bigoplus_{v\in V} S(k;\mathcal{M}_v)$ such that each block corresponds to a particular vertex. Assume, in addition,

$$I + S(\mathrm{i}\varkappa, \mathcal{M}_v) > 0 \qquad \text{(componentwise)}$$

for all vertices $v \in V$ and all sufficiently large $\varkappa > 0$. Then the eigenvalue $\lambda_{\min}$ is simple.

Proof is by using the semigroup technique.

Thank You For Your Attention!

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