High frequency scattering by convex polygons

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Joint work with **Simon Chandler-Wilde** (Reading), and also

- Steve Arden, Richard Silveira, Moss Mokgolele (Reading)
- Daan Huybrechs (Leuven)
- Markus Melenk (TU Wien)

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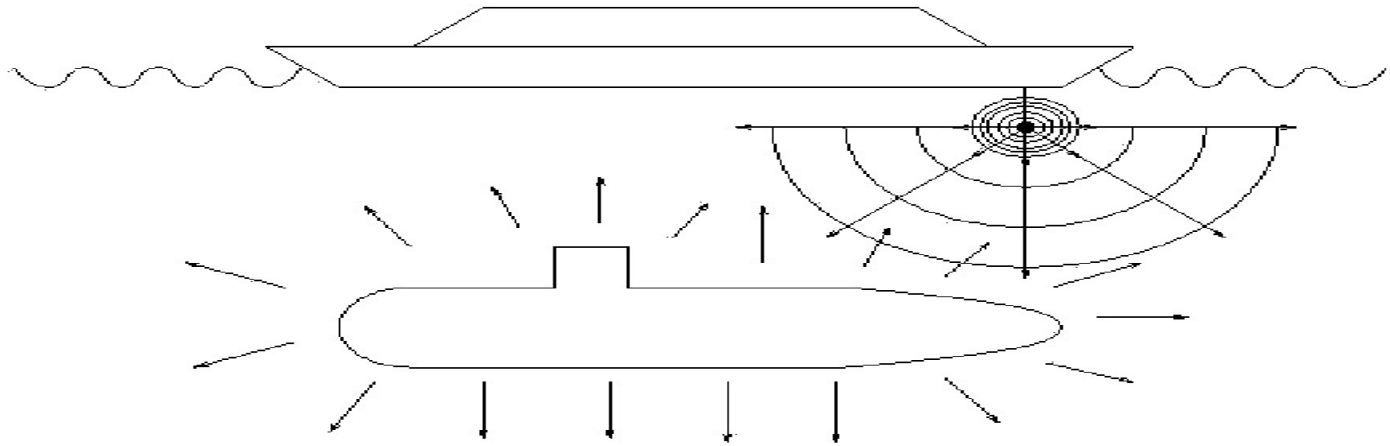
- Leverhulme Trust

Time harmonic acoustic scattering



With **time harmonic waves**, propagation of sound in a homogeneous media is mathematically described by the **Helmholtz equation**:

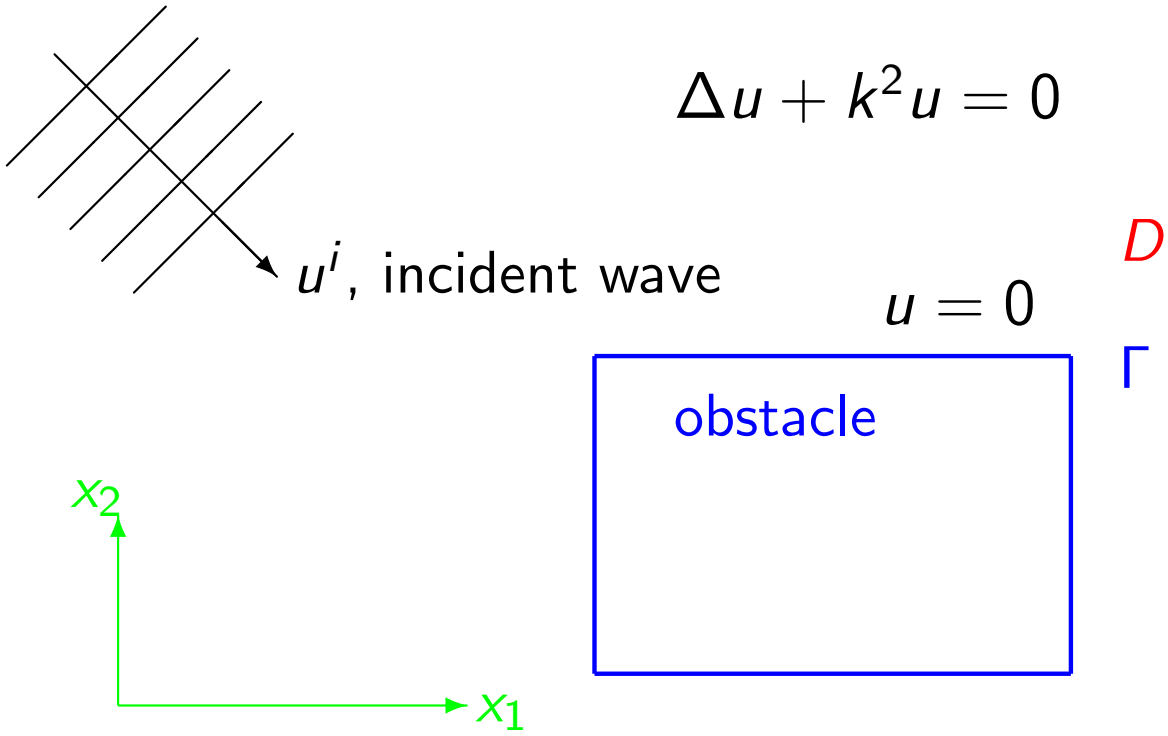
$$\Delta u + k^2 u = 0,$$

where the **wavenumber** k is proportional to frequency.

Difficulties at high frequencies

- Standard schemes for scattering problems become prohibitively expensive as $k \to \infty$.
- For standard **finite element** or **boundary element** methods, the usual requirement is ten elements per wavelength.
- The total number of degrees of freedom thus increases with at least $\mathcal{O}(k)$ in 2D, $\mathcal{O}(k^2)$ in 3D.
- This leads to large systems of equations to be solved when the wavelength is small.

Scattering by convex polygons



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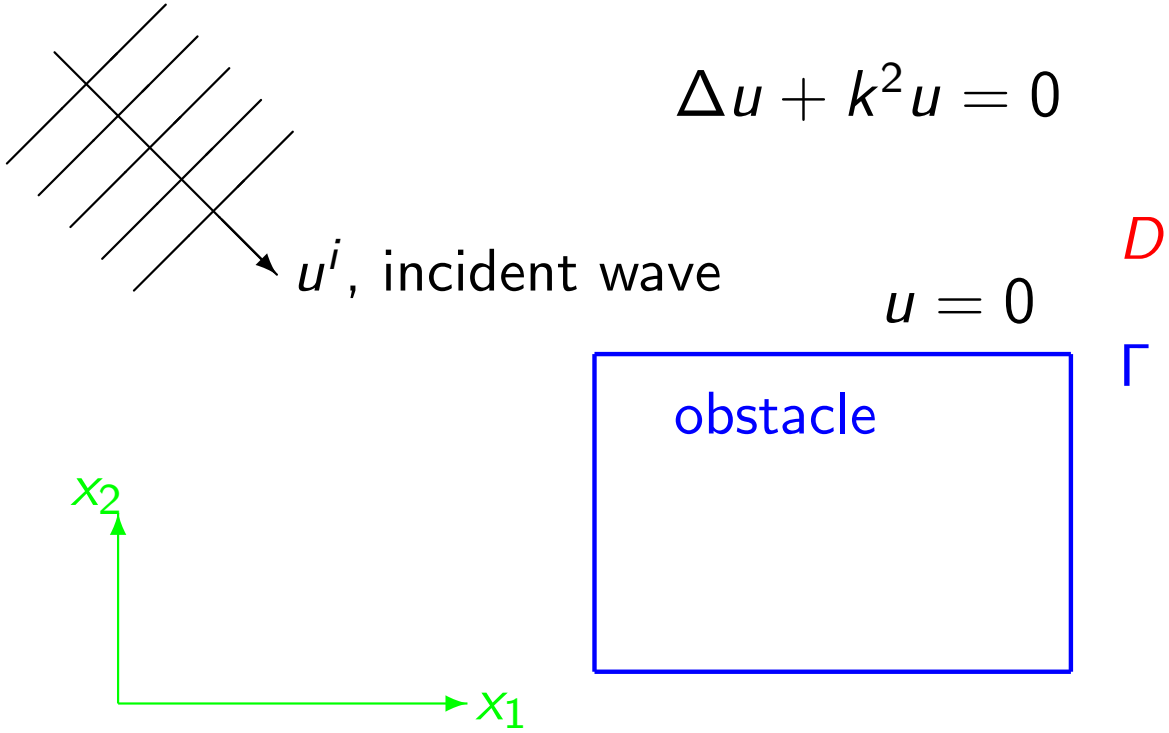

Green's representation theorem:

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D.$$

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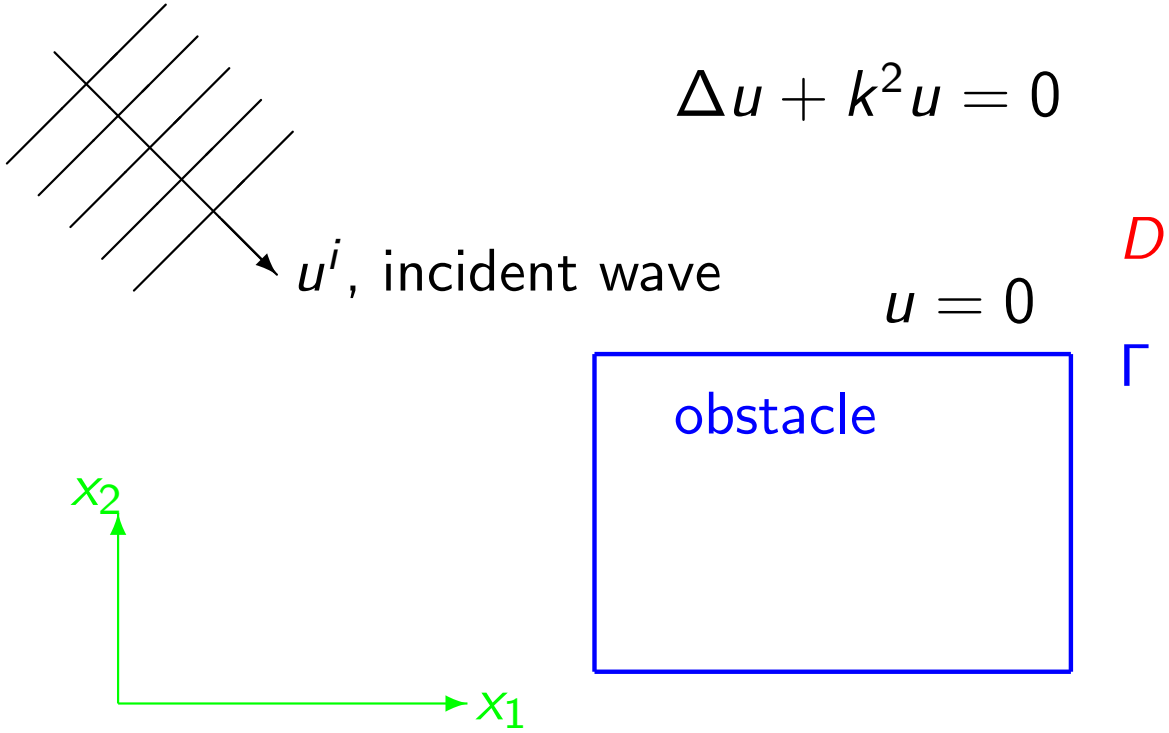

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$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) + \int_\Gamma \left(\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} + \mathrm{i}\eta \Phi(\mathbf{x},\mathbf{y}) \right) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\, \mathrm{d}s(\mathbf{y}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

Boundary element method

$$\frac{1}{2}\frac{\partial u}{\partial n}(x)+\int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)}+\mathrm{i}\eta\Phi(x,y)\right)\frac{\partial u}{\partial n}(y)ds(y)=f(x),\quad x\in\Gamma,$$

Conventional BEM

Galerkin method, approximating $\partial u/\partial n$ by a piecewise polynomial.

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Conventional BEM

Galerkin method, approximating $\partial u/\partial n$ by a piecewise polynomial.
Degrees of freedom must grow with O(k) to maintain accuracy.

Improved approximation space

$$\frac{1}{2}\frac{\partial u}{\partial n}(x)+\int_\Gamma \left(\frac{\partial \Phi(x,y)}{\partial n(x)}+\mathrm{i}\eta\Phi(x,y)\right)\frac{\partial u}{\partial n}(y)ds(y)=f(x),\quad x\in\Gamma,$$

For convex scatterers

Remove some of the oscillations by writing

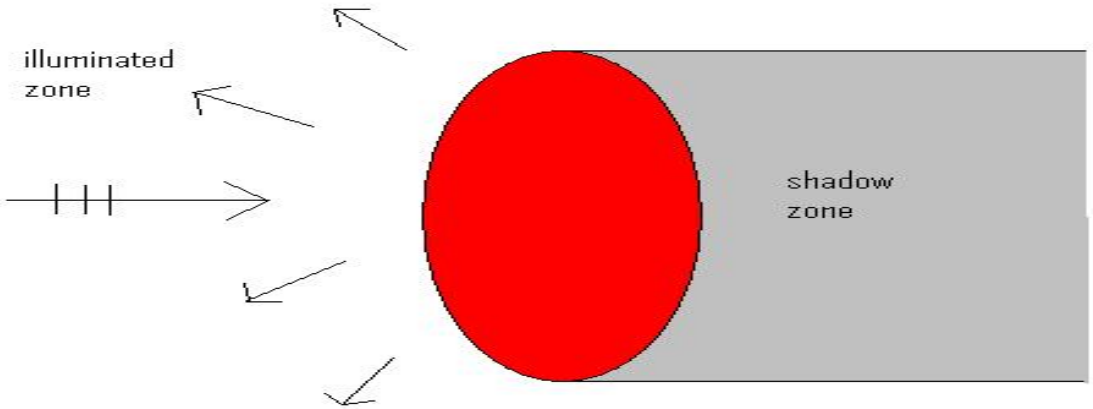
$$\frac{\partial u}{\partial n}(y)=\frac{\partial u^i}{\partial n}(y)\times F(y)$$

and approximate F by a conventional BEM
(e.g. Abboud et. al. 1994, Bruno et al. 2004, Domínguez, Graham and Smyshlyaev 2006, Huybrechs and Vandewalle 2006).

Improved approximation space

$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y)$$

For **smooth obstacles** this works well: equation holds with $F(y) \approx 2$ on the illuminated side (physical optics) and $F(y) \approx 0$ in the shadow zone (classical Kirchoff approximation).



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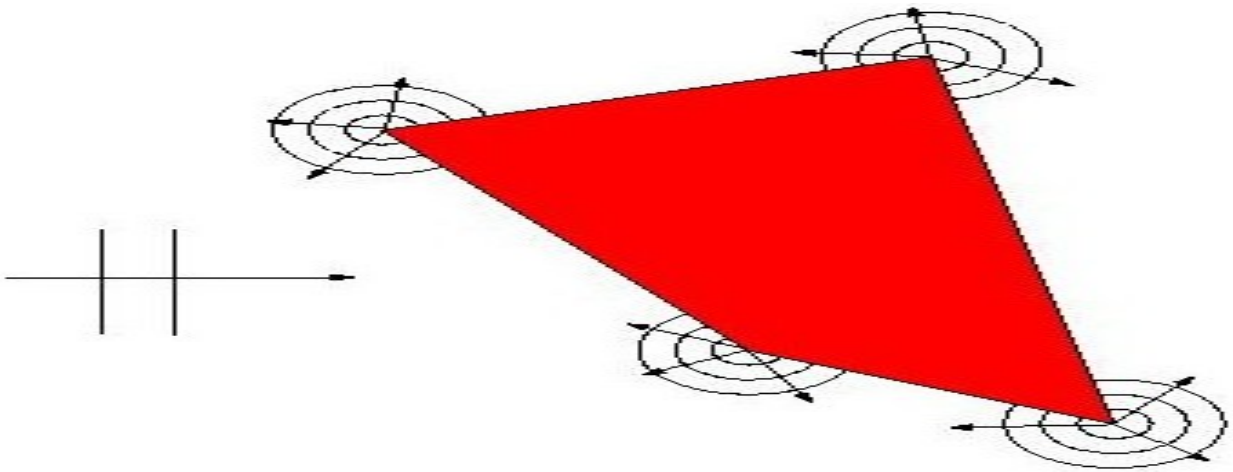
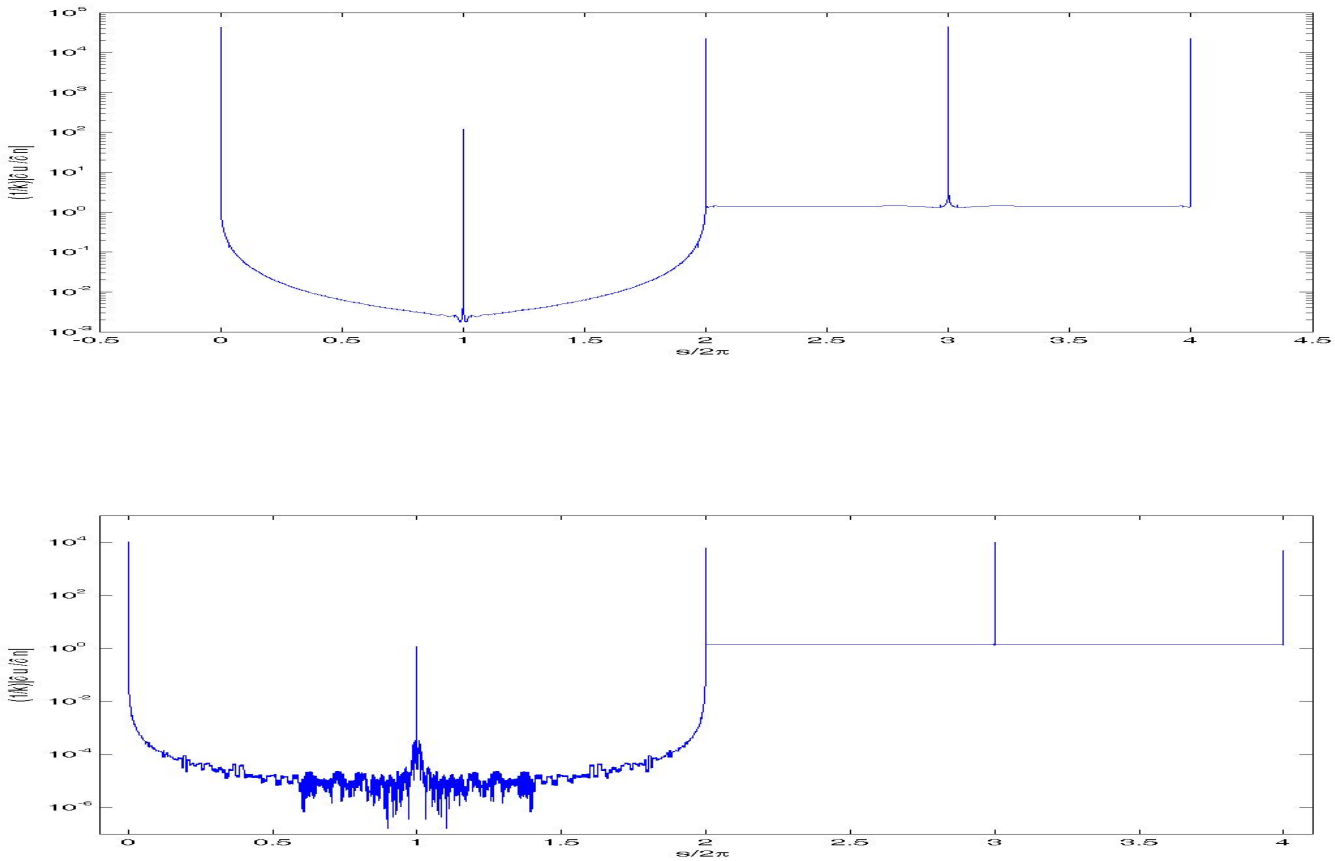
$$\frac{\partial u}{\partial n}(y) = \frac{\partial u^i}{\partial n}(y) \times F(y)$$

Not very effective for non-smooth obstacles

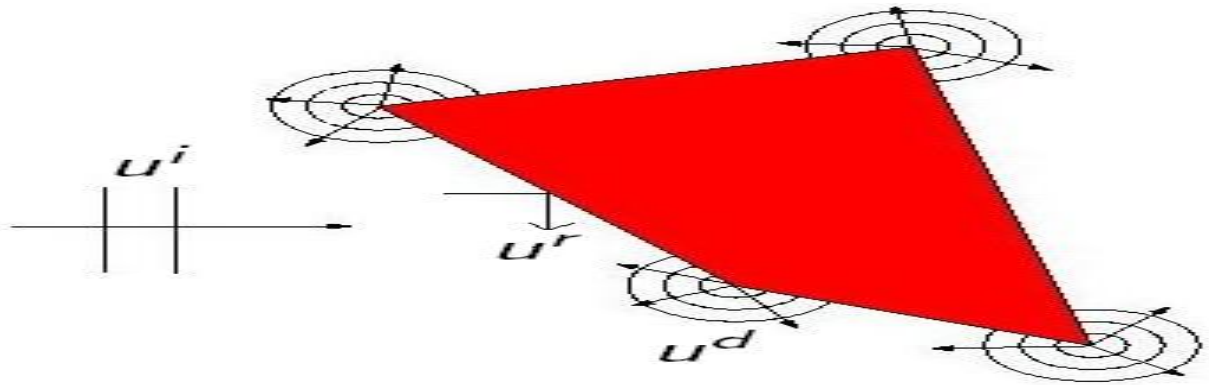


Scattering by a square, side length 2π

Plot of $\frac{1}{k}|\frac{\partial u}{\partial \mathbf{n}}|$ for $k = 10$ and $k = 640$



Step 1 - subtract off leading order behaviour as $k \to \infty$



On each side of polygon $u = u^i + u^r + u^d$.

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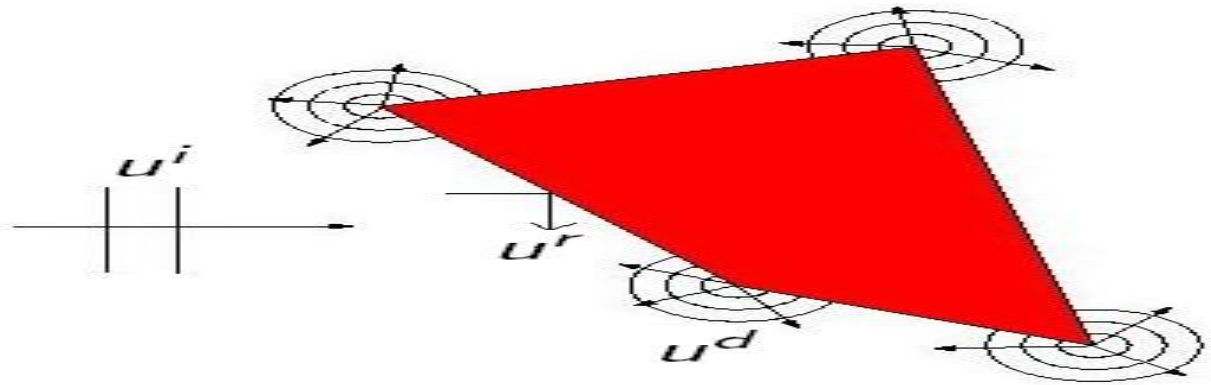

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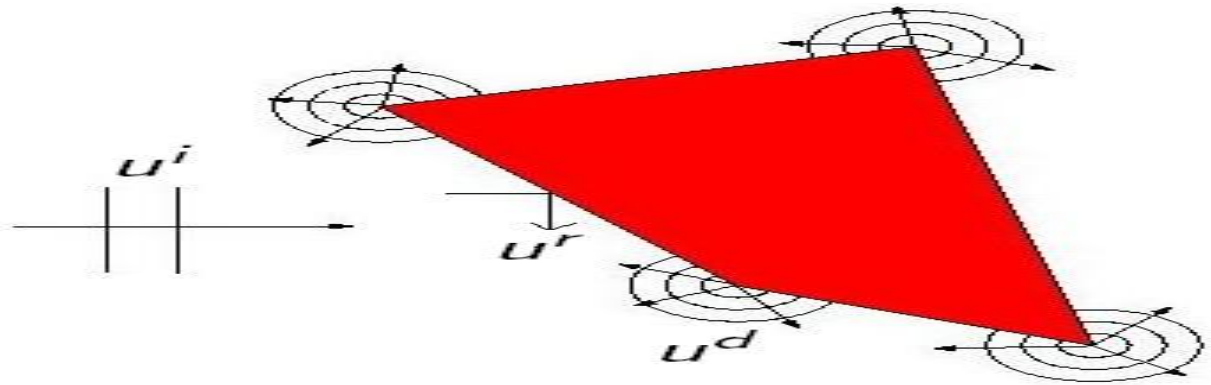

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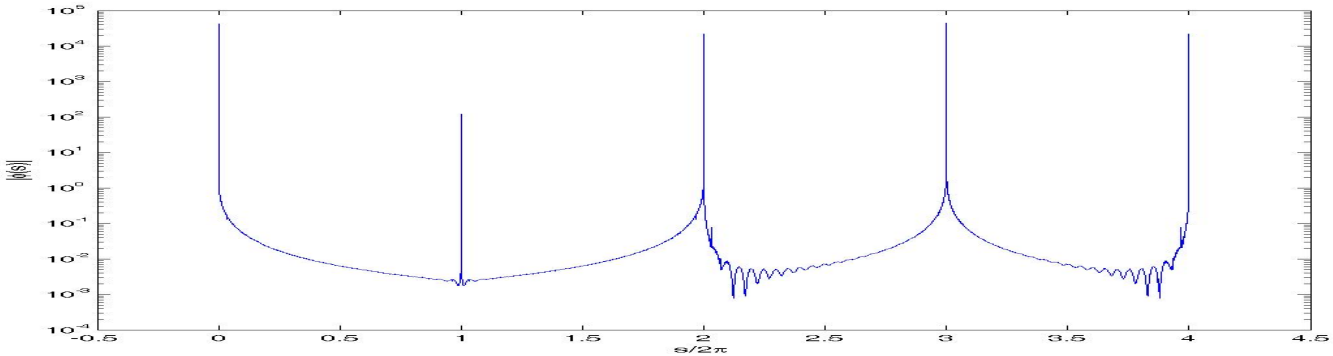
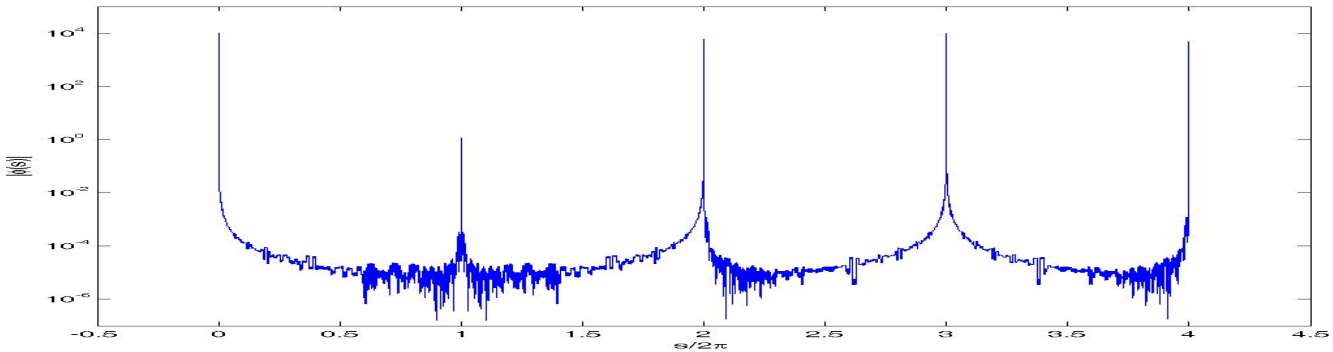
As $k \to \infty$, $\quad \dfrac{\partial u}{\partial \mathbf{n}} \sim \Psi := \begin{cases} \dfrac{\partial u^i}{\partial \mathbf{n}} + \dfrac{\partial u^r}{\partial \mathbf{n}} = 2\dfrac{\partial u^i}{\partial \mathbf{n}} & \text{on lit sides} \\ 0 & \text{on shadow sides.} \end{cases}$

Scattering by a square, side length 2π

Plot of $\phi := \frac{1}{k}\left(\frac{\partial u}{\partial \mathbf{n}} - \Psi\right)$ for $k = 10$ and $k = 640$





Step 2 - express ϕ as a product of oscillatory and non-oscillatory functions

On each side of the polygon we write

$$\phi(s) = \mathrm{e}^{\mathrm{i}ks} v^+(s) + \mathrm{e}^{-\mathrm{i}ks} v^-(s), \quad x(s) \in \Gamma.$$

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We can show that (if $s = 0$ represents a corner)

$$\Rightarrow k^{-m} |v^{+(m)}(s)| \leq C_m (ks)^{-1/2-m}, \quad \text{for } ks \geq 1,$$

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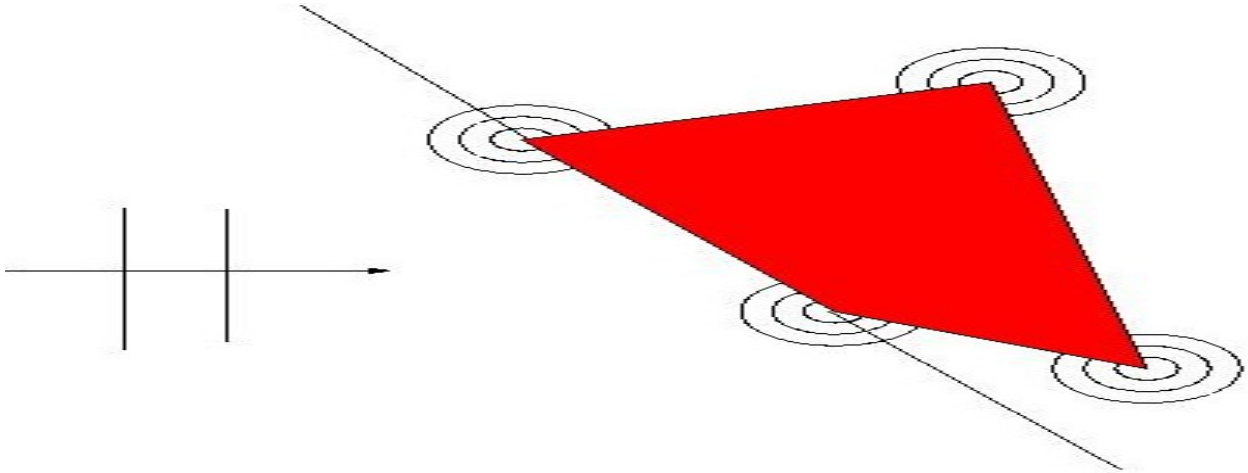
$$\Rightarrow k^{-m} |v^{+(m)}(s)| \leq C_m (ks)^{-1/2-m}, \quad \text{for } ks \geq 1,$$

and (by separation of variables local to corner)

$$\Rightarrow k^{-m} |v^{+(m)}(s)| \leq C_m (ks)^{-\alpha-m}, \quad \text{for } ks \leq 1,$$

where $\alpha < 1/2$ depends on the corner angle.

Proof that v_+ is non-oscillatory



We consider a related half-plane problem. Define $G(x, y) := \Phi(x, y) - \Phi(x, y')$. By Green's theorem,

$$u(x) = u^i(x) + u^r(x) + \int_{\partial\Omega\setminus\Gamma} \frac{\partial G(x,y)}{\partial n(y)} u(y) ds(y), \quad x \in \Omega,$$

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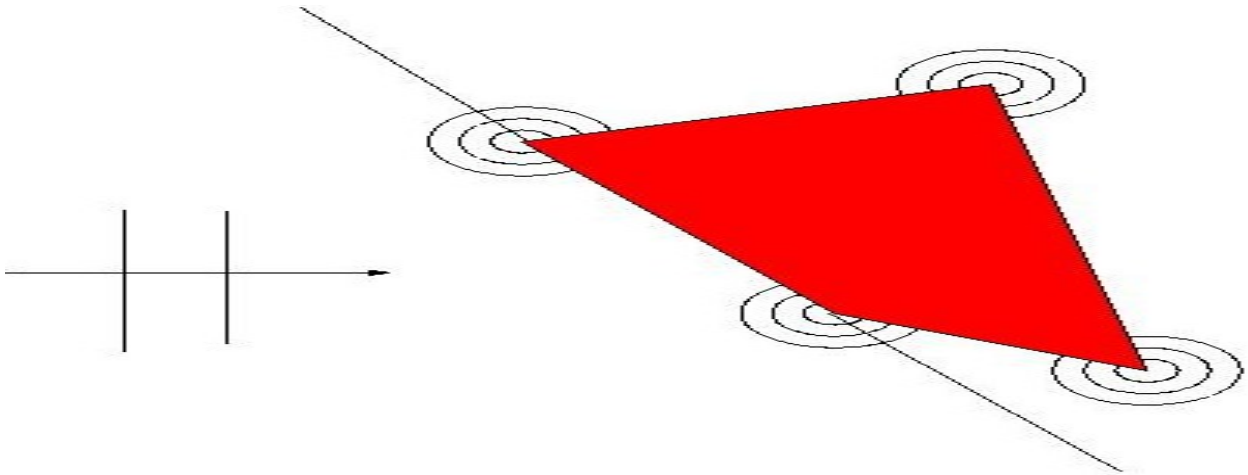

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$$\Rightarrow \frac{\partial u}{\partial n}(x) = 2\frac{\partial u^i}{\partial n}(x) + 2\int_{\partial\Omega\setminus\Gamma} \frac{\partial^2 \Phi(x,y)}{\partial n(x)\partial n(y)} u(y) ds(y), \quad x \in \gamma := \partial\Omega \cap \Gamma.$$

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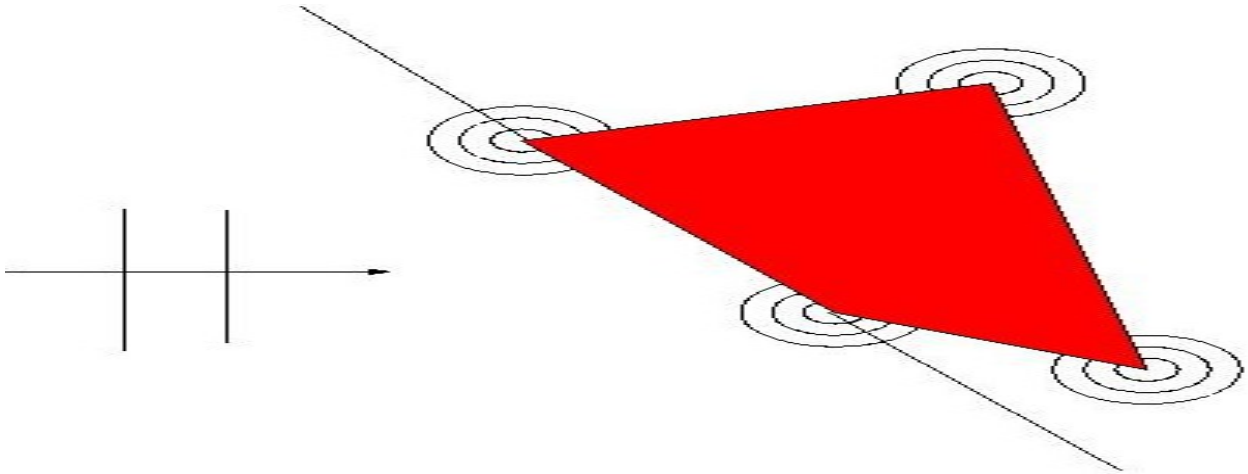

Hence

$$\phi(s) := \frac{1}{k}\left(\frac{\partial u}{\partial \mathbf{n}}(x(s)) - \Psi(s)\right) = \frac{\mathrm{i}}{2}\left[\mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s)\right]$$

where

$$v_+(s) := k\int_{-\infty}^{0} \mu(k|s-t|)\mathrm{e}^{-\mathrm{i}kt}u(x(t))\,\mathrm{d}t$$

and $\mu(z) := \mathrm{e}^{-\mathrm{i}z}H_1^{(1)}(z)/z$.

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$$v_+(s) := k \int_{-\infty}^{0} \mu(k|s-t|) \mathrm{e}^{-\mathrm{i}kt} u(x(t)) \, \mathrm{d}t$$

and $\mu(z) := \mathrm{e}^{-\mathrm{i}z} H_1^{(1)}(z)/z$, which is **non-oscillatory**, in that

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$$\Rightarrow v_+^{(m)}(s) \leq C_m (k^m (ks)^{-1/2-m}) \text{ for } ks \geq 1.$$

h version approximation scheme

Thus approximate

$$\phi(s) = \mathrm{e}^{\mathrm{i}ks} v^+(s) + \mathrm{e}^{-\mathrm{i}ks} v^-(s), \quad \mathbf{x}(s) \in \Gamma,$$

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where V^+ and V^- are piecewise polynomials on graded meshes.

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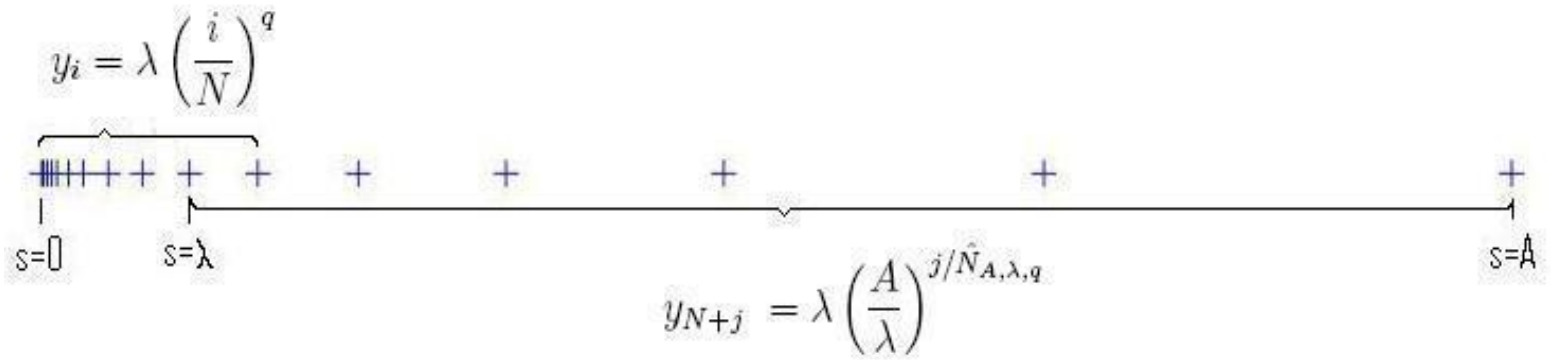

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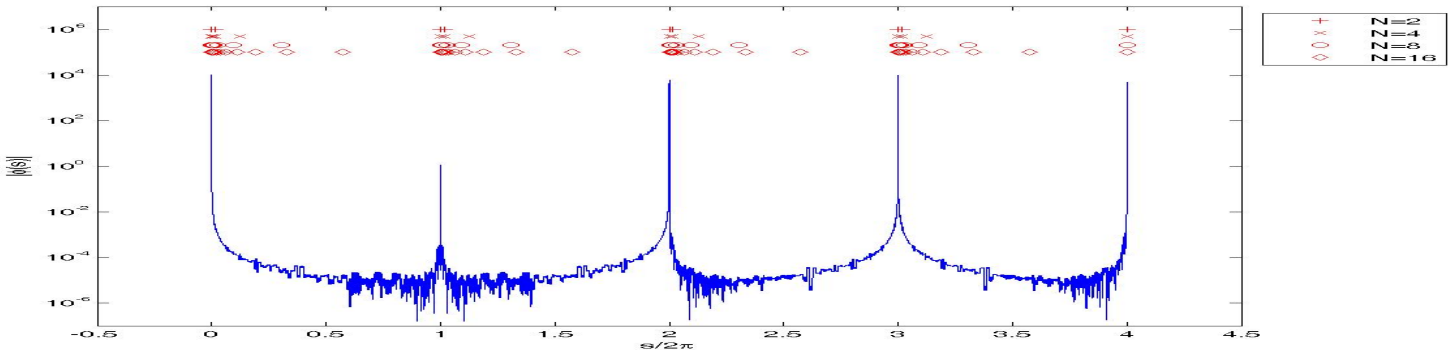

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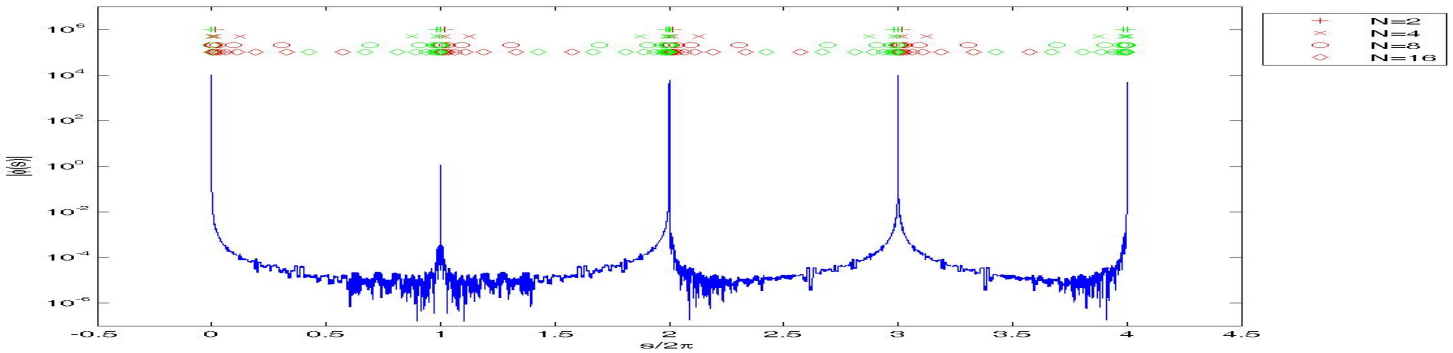


Approximation error

Theorem

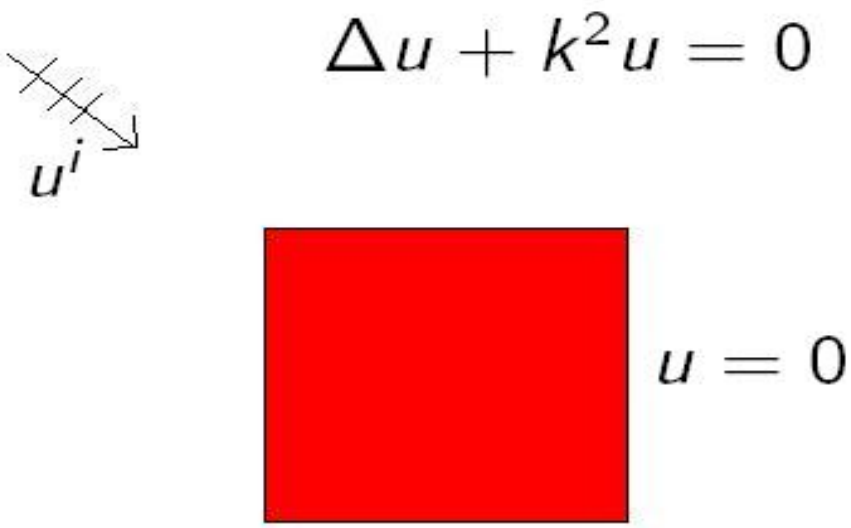
If V^+ is the best L_2 approximation from the approximation space,

$$k^{1/2}\|v^+ - V^+\|_2 \leq C_p \frac{n^{1/2}(1 + \log^{1/2}(kL/n))}{N^{p+1}}.$$

- $N \propto$ degrees of freedom
- $p =$polynomial degree
- $L =$ max side length
- $n =$ number of sides of polygon

Numerical results for Galerkin scheme

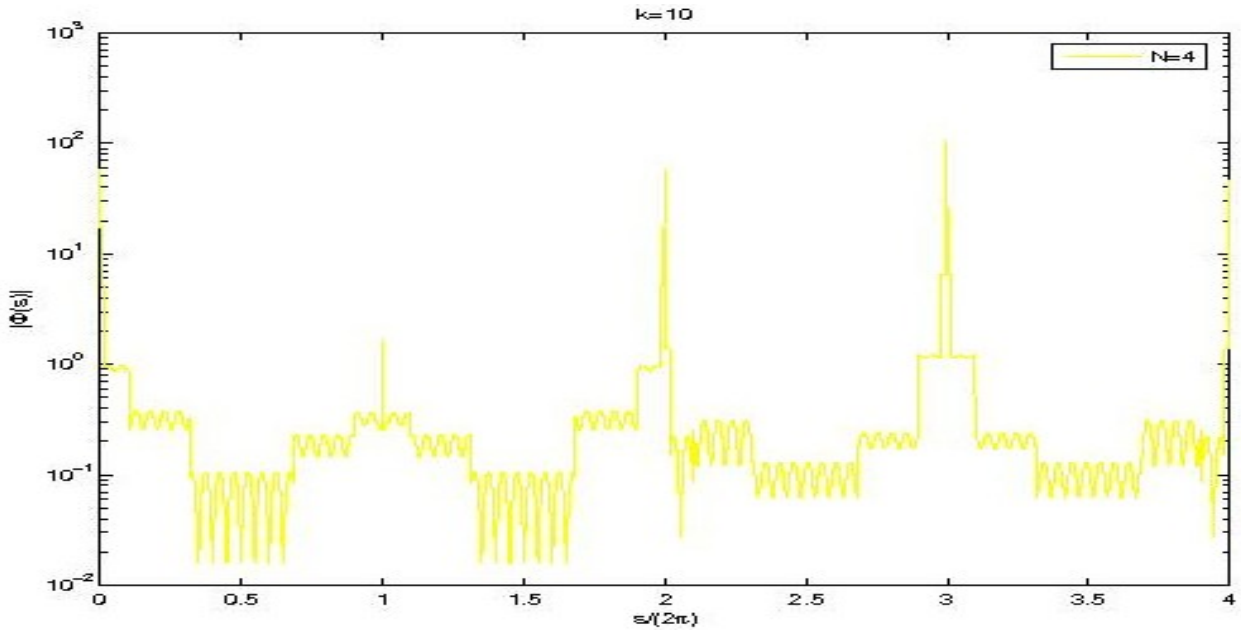
Example: scattering by a square of side length 2π, so each side is k wavelengths long.



VIDEO FOR THE CASE $k = 5$

Numerical results (scattering by a square)

Solution minus P.O. approximation;



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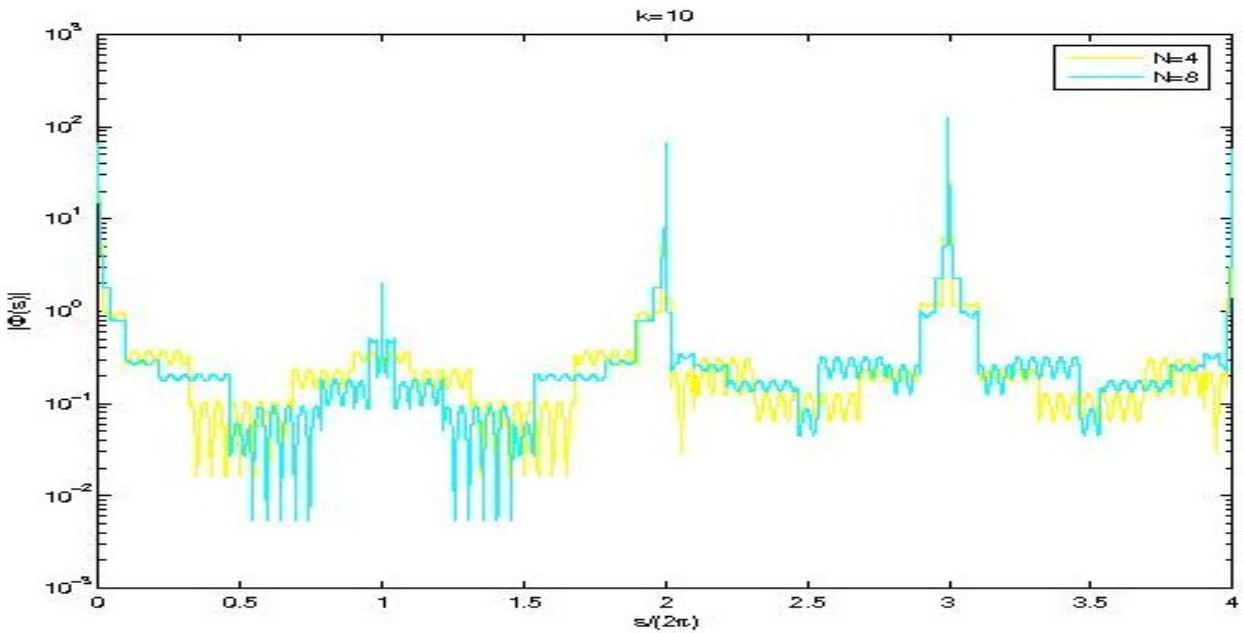

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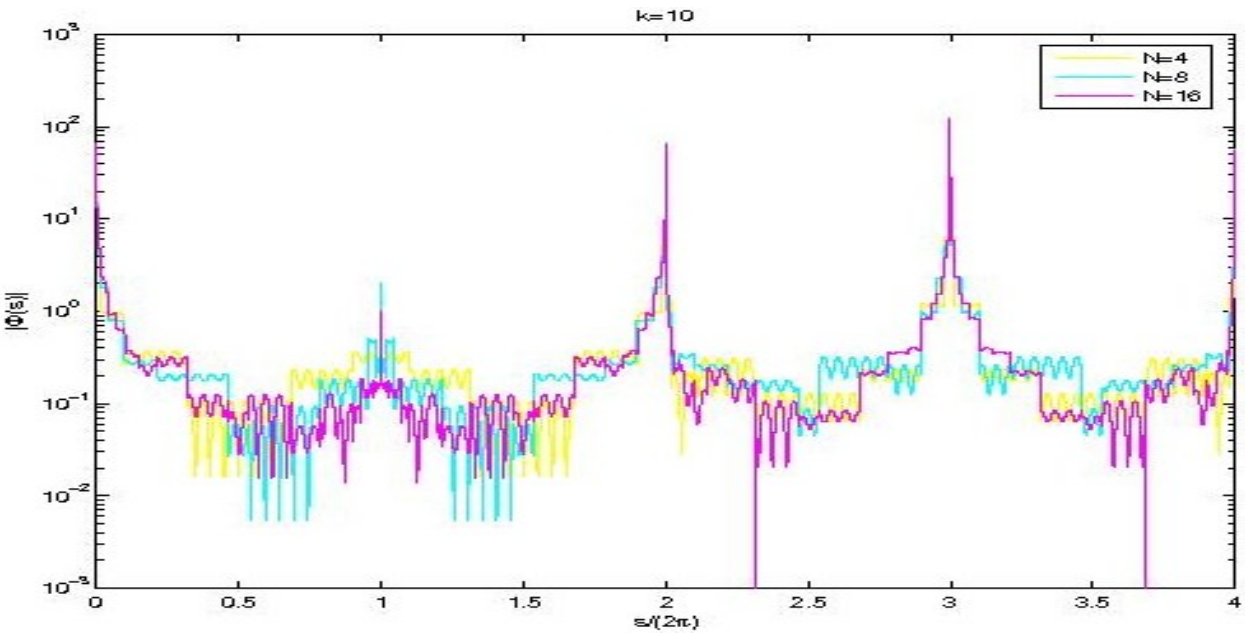

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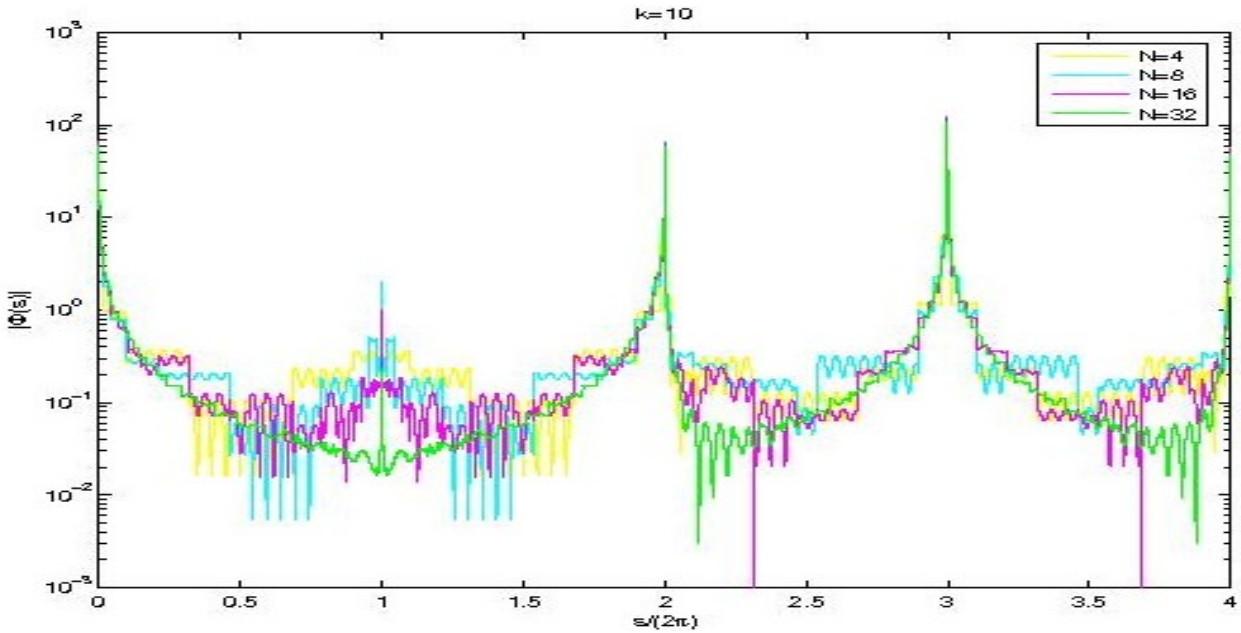

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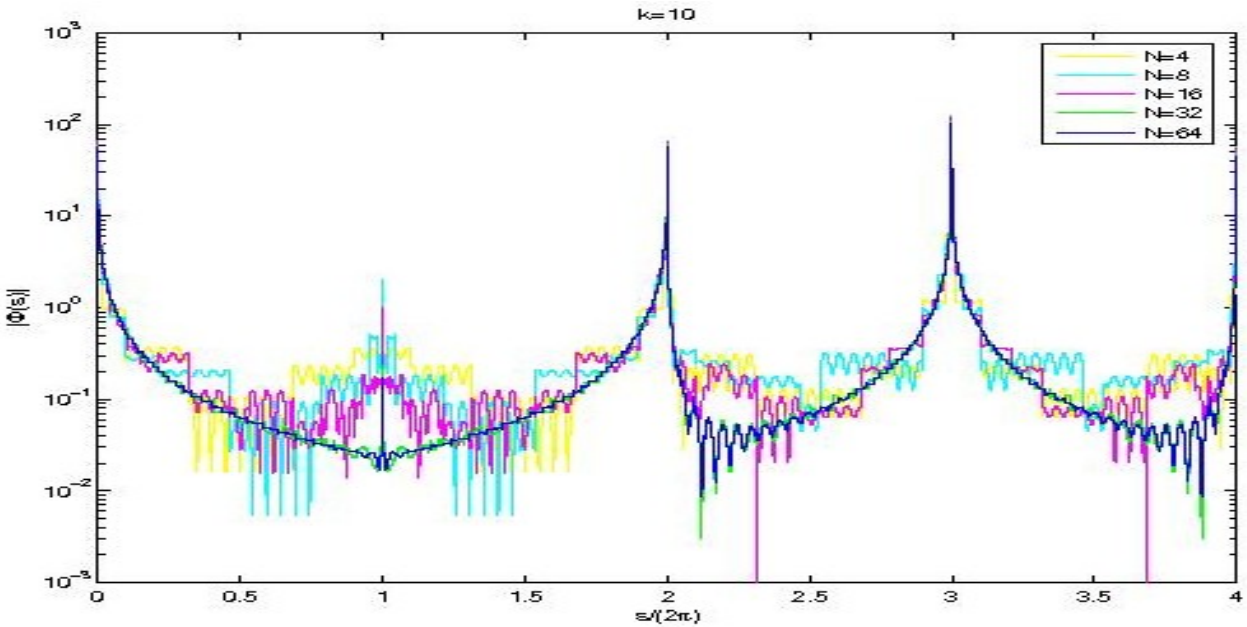

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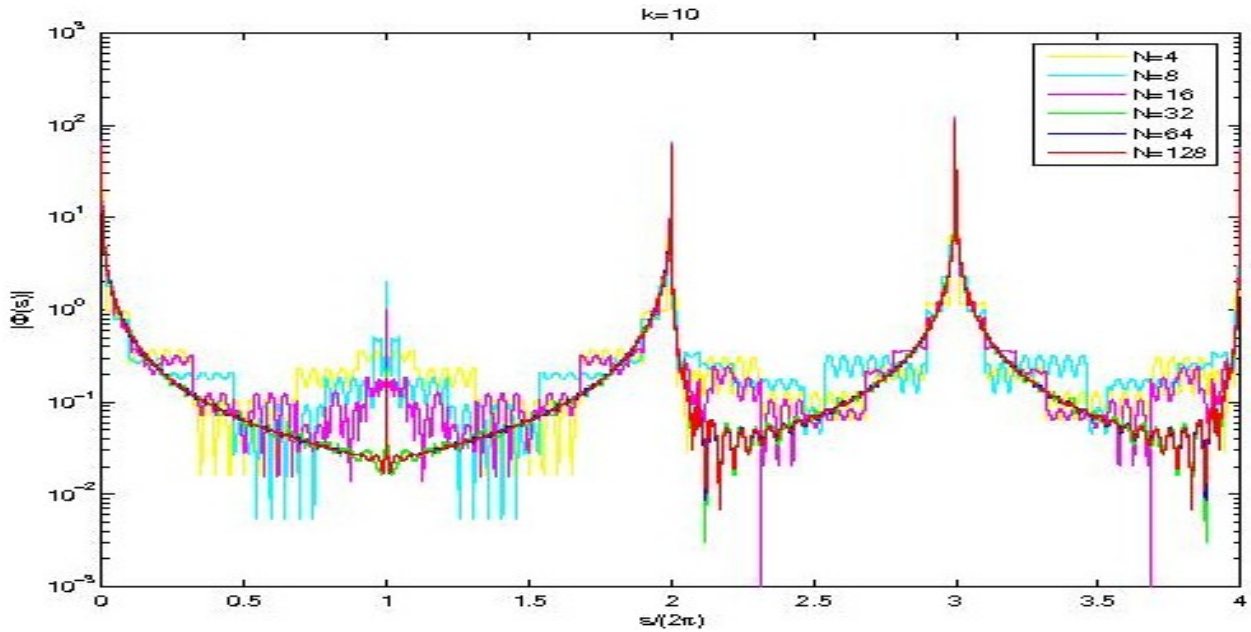

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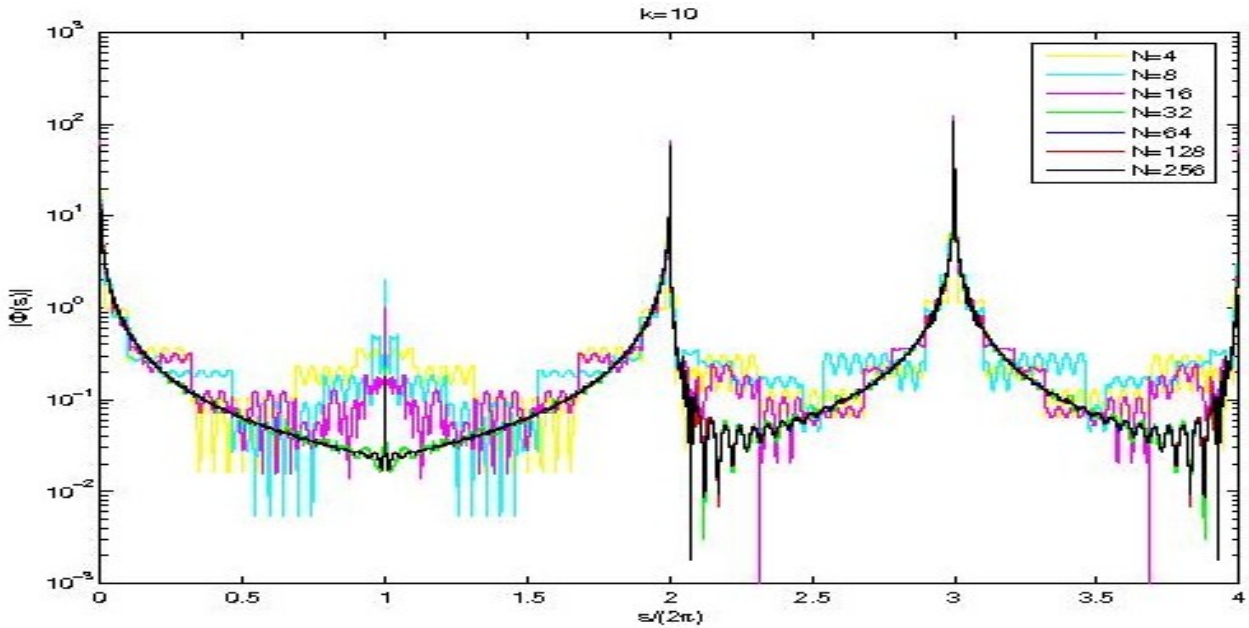

Relative errors, $k = 5$

k	N	dof	$k^{1/2}\|\varphi_N - \varphi_{256}\|_2$	$\frac{\|\varphi_N - \varphi_{256}\|_2}{\|\varphi_{256}\|_2}$	EOC
5	8	88	5.7339×10^{-1}	2.4426×10^{-1}	
	16	176	3.7454×10^{-1}	1.5955×10^{-1}	0.6
	32	360	1.6176×10^{-1}	6.8909×10^{-2}	0.9
	64	712	7.7267×10^{-2}	3.2916×10^{-2}	1.0
	128	1416	3.3541×10^{-2}	1.4289×10^{-2}	1.0

Relative errors, $k = 320$

k	N	dof	$k^{1/2}\|\varphi_N - \varphi_{256}\|_2$	$\frac{\|\varphi_N - \varphi_{256}\|_2}{\|\varphi_{256}\|_2}$	EOC
320	8	120	7.0765×10^{-1}	3.6736×10^{-1}	
	16	240	5.9792×10^{-1}	3.1040×10^{-1}	0.2
	32	472	1.9668×10^{-1}	1.0211×10^{-1}	0.9
	64	944	7.5808×10^{-2}	3.9354×10^{-2}	1.1
	128	1888	4.8814×10^{-2}	2.5341×10^{-2}	1.0

Relative L_2 errors, various k, $N = 64$ (h version)

k	M_N	$\|\phi - \phi_N\|_2 / \|\phi\|_2$	COND
5	712	3.2916×10^{-2}	2.4×10^{2}
10	752	2.8702×10^{-2}	8.4×10^{1}
20	792	1.6914×10^{-1}	5.1×10^{3}
40	824	5.9856×10^{-2}	1.2×10^{3}
80	864	3.4801×10^{-2}	2.7×10^{3}
160	904	3.4570×10^{-2}	1.4×10^{3}
320	944	3.9354×10^{-2}	1.1×10^{3}
640	984	3.5693×10^{-2}	1.5×10^{3}

First outstanding question - discrete operator bounds?

Galerkin scheme is

$$(I + P_{N_G}\mathcal{K})\varphi_{N_G} = P_{N_G}F,$$

where P_{N_G} is the orthogonal projection. We have shown that for $N \geq N^*(k)$, the operator $(I + P_{N_G}\mathcal{K}) : L_2(\Gamma) \mapsto V_N$ is bijective with bounded inverse

$$\|(I + P_{N_G}\mathcal{K})^{-1}\|_2 \leq C_s(k),$$

First outstanding question - discrete operator bounds?

Galerkin scheme is

$$(I + P_{N_G}\mathcal{K})\varphi_{N_G} = P_{N_G}F,$$

where P_{N_G} is the orthogonal projection. We have shown that for $N \geq N^*(k)$, the operator $(I + P_{N_G}\mathcal{K}) : L_2(\Gamma) \mapsto V_N$ is bijective with bounded inverse

$$\|(I + P_{N_G}\mathcal{K})^{-1}\|_2 \leq C_s(k),$$

but the dependence of $N^*(k)$ and $C_s(k)$ on k is not clear.

Second outstanding question - fully discrete schemes?

Galerkin: cost of evaluating numerically many integrals of the form

$$\int_{-b}^{-a}\int_{c}^{d}\left[H_0^{(1)}(k\sqrt{s^2+t^2})+\frac{\mathrm{i}tH_1^{(1)}(k\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\right]\mathrm{e}^{\mathrm{i}k(\sigma_j t-\sigma_m s)}\,\mathrm{d}t\,\mathrm{d}s.$$

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Collocation: only need to evaluate numerically integrals of form

$$\int_{a}^{b}\left[H_0^{(1)}(k\sqrt{s_m^2+t^2})+\frac{\mathrm{i}tH_1^{(1)}(k\sqrt{s_m^2+t^2})}{\sqrt{s_m^2+t^2}}\right]\mathrm{e}^{\mathrm{i}k\sigma_j t}\,\mathrm{d}t.$$

Collocation method easier to implement

Third outstanding question - well conditioned collocation scheme?

To solve

$$\phi(s) + \mathcal{K}\phi(s) = F(s),$$

seek $\phi_{N_C} \in V_N$ such that

$$(I + P_{N_C}\mathcal{K})\varphi_{N_C} = P_{N_C}F,$$

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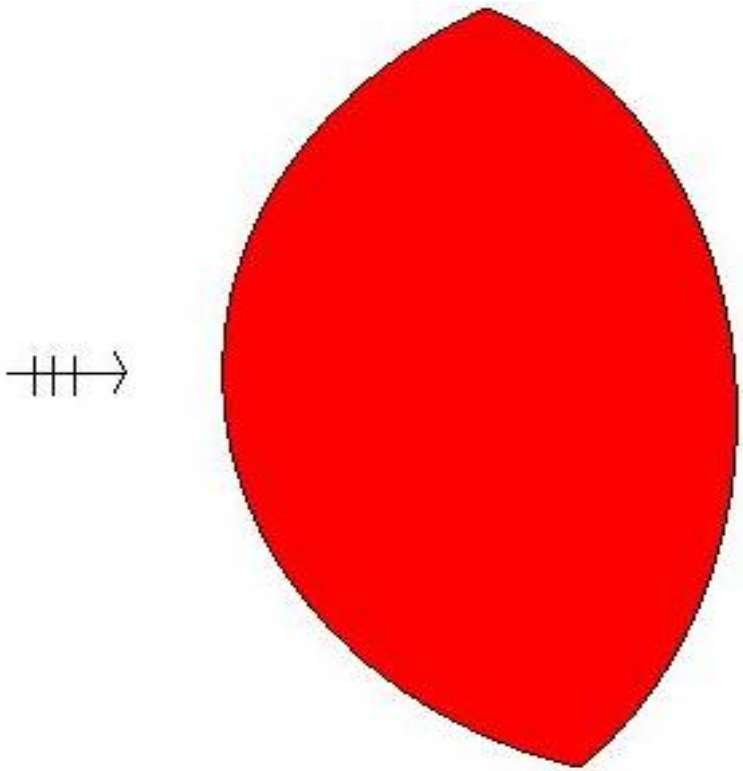
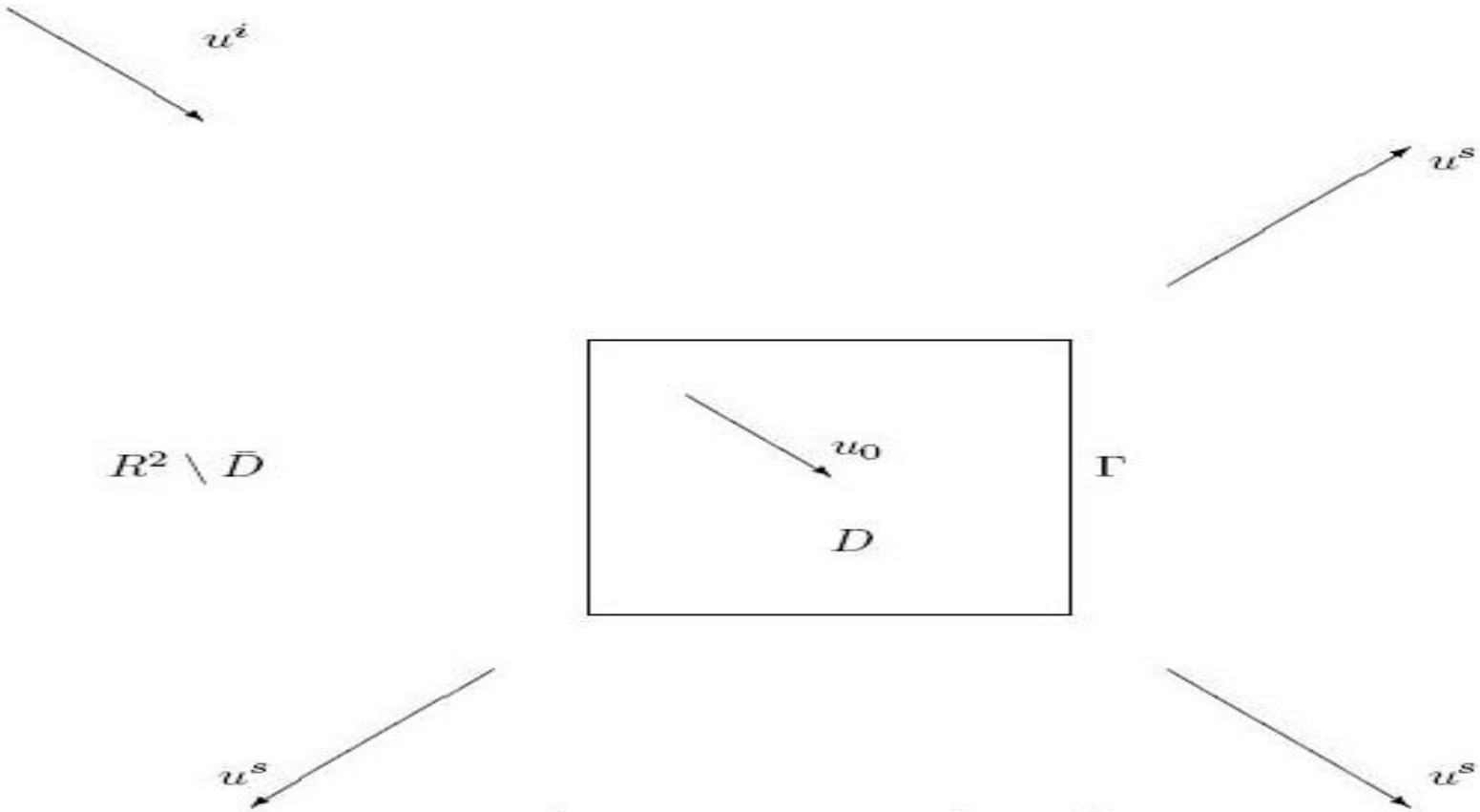
Difficulty with choice of collocation points: mass matrix $M_c := [\rho_j(s_m)]$ may be ill conditioned.

Possible extensions - curvilinear polygons



Although we don't know exact reflected wave any more, we do know **leading order**, and it seems that a similar grid will be appropriate for modelling the diffracted rays as that used for the polygon.

Transmission problems



$$\Delta u^s(\mathbf{x}) + k^2 u^s(\mathbf{x}) = 0, \quad k^2 = \omega^2 \mu_0 \epsilon_0$$

$$\Delta u_0(\mathbf{x}) + k_0^2 u_0(\mathbf{x}) = 0, \quad k_0^2 = \omega^2 \mu_0 \epsilon + i\mu_0 \sigma \omega$$

$$(2 + K_0 - K)u^t + (S - S_0)\frac{\partial u^t}{\partial n} = 2u^i$$

$$(2 + K' - K_0')\frac{\partial u^t}{\partial n} + (T_0 - T)u^t = 2\frac{\partial u^i}{\partial n}$$

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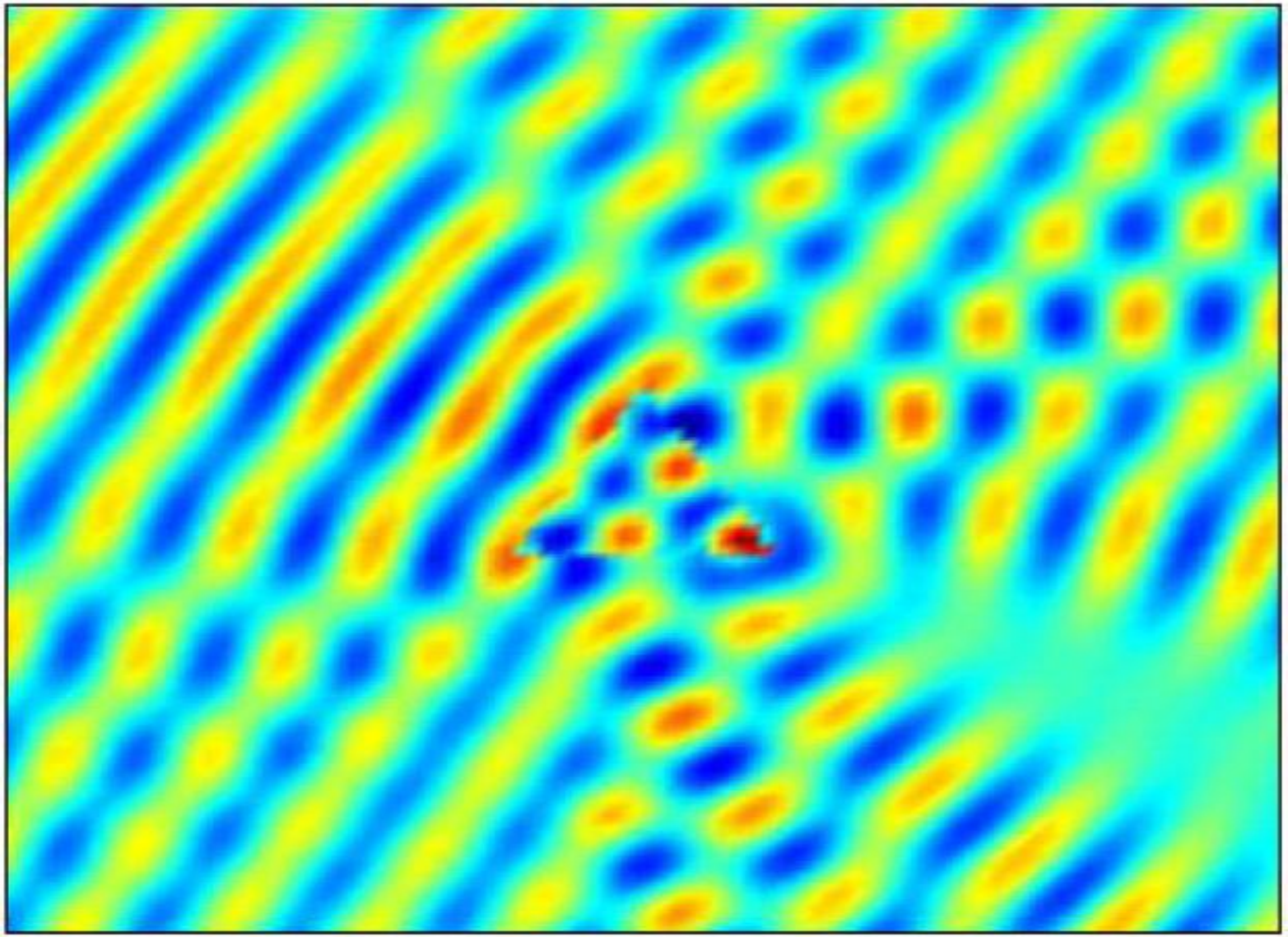

Summary and conclusions

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- Using Green's representation theorem in a half-plane we can understand behaviour of the field on the boundary and its derivatives for scattering by a convex polygon.
- For a convex polygon, design of an optimal graded mesh for piecewise polynomial approximation is then straightforward.
- The number of degrees of freedom need only grow logarithmically with the wavenumber to maintain a fixed accuracy.
- Ongoing considerations
  - Galerkin vs. Collocation - stability and convergence analysis.
  - Better schemes for evaluating oscillatory integrals.
  - *hp* ideas.
  - Extensions to related scattering problems.