Contractive semigroups on metric graphs

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based on a joint work with
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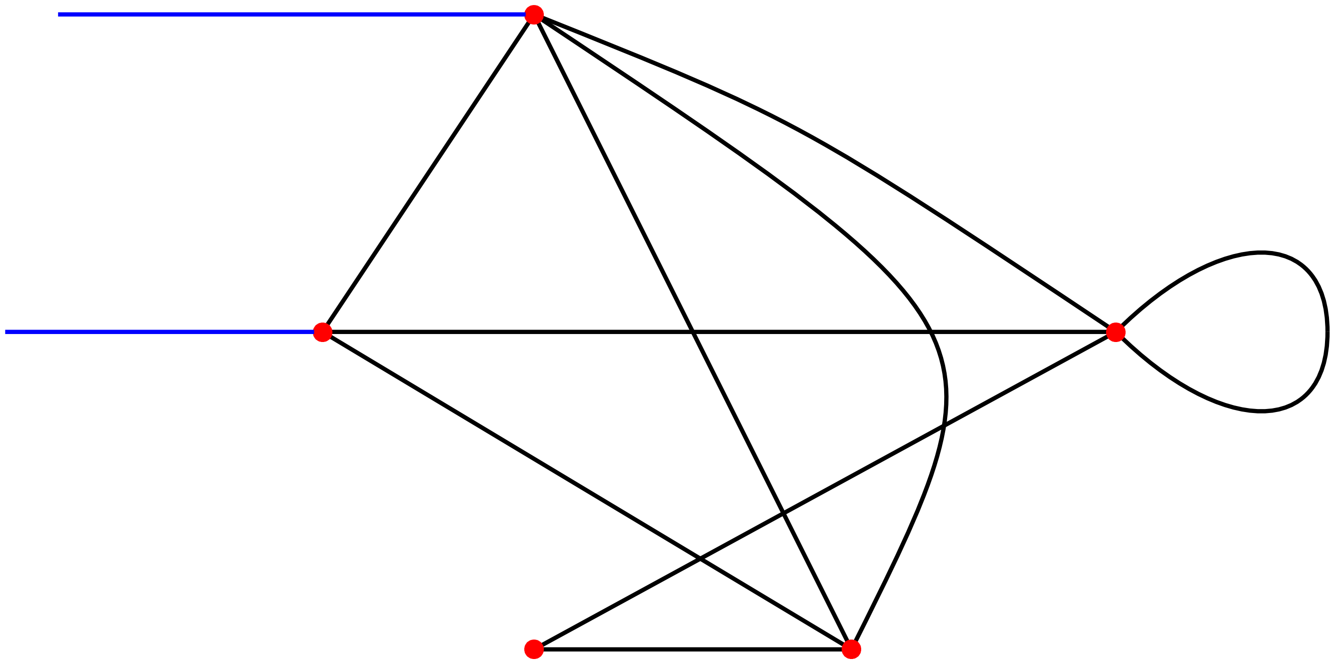
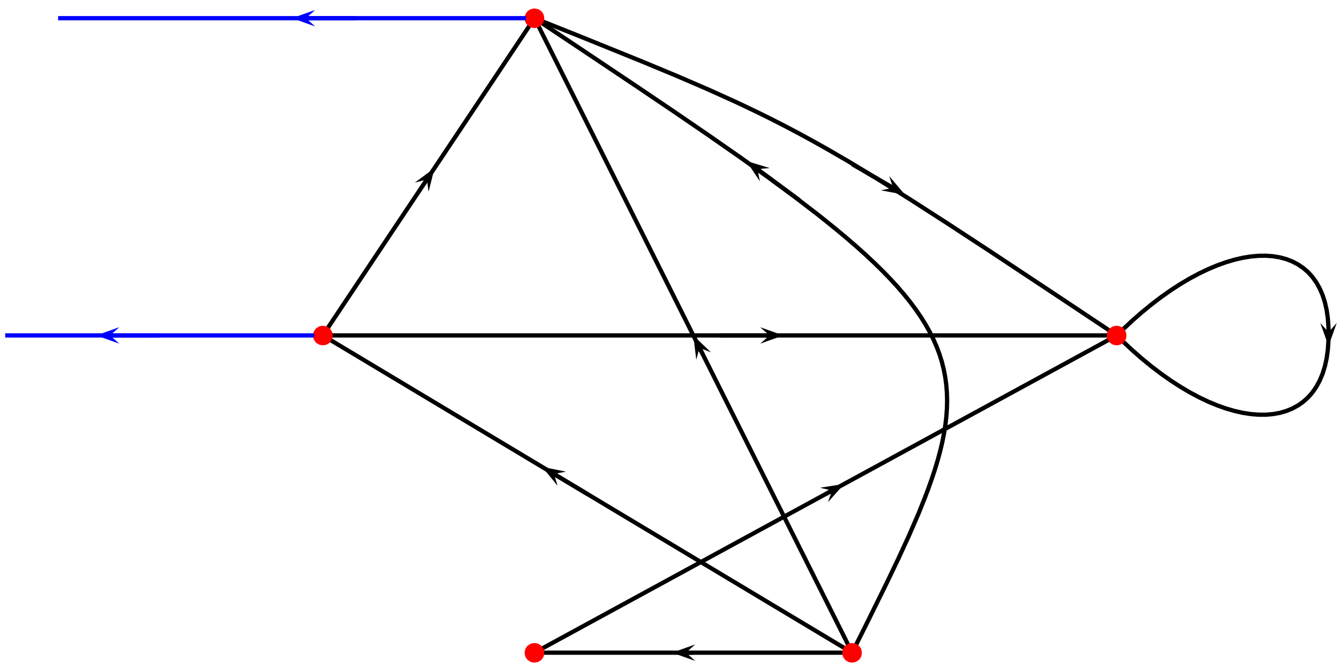
Outline

Let $G = (V, E)$ be (not necessarily plane) graph,
multiple edges and loops are allowed:



V	vertex set
$E = E_{\text{int}} \cup E_{\text{ext}}$	edge set

Consider G as a directed graph:



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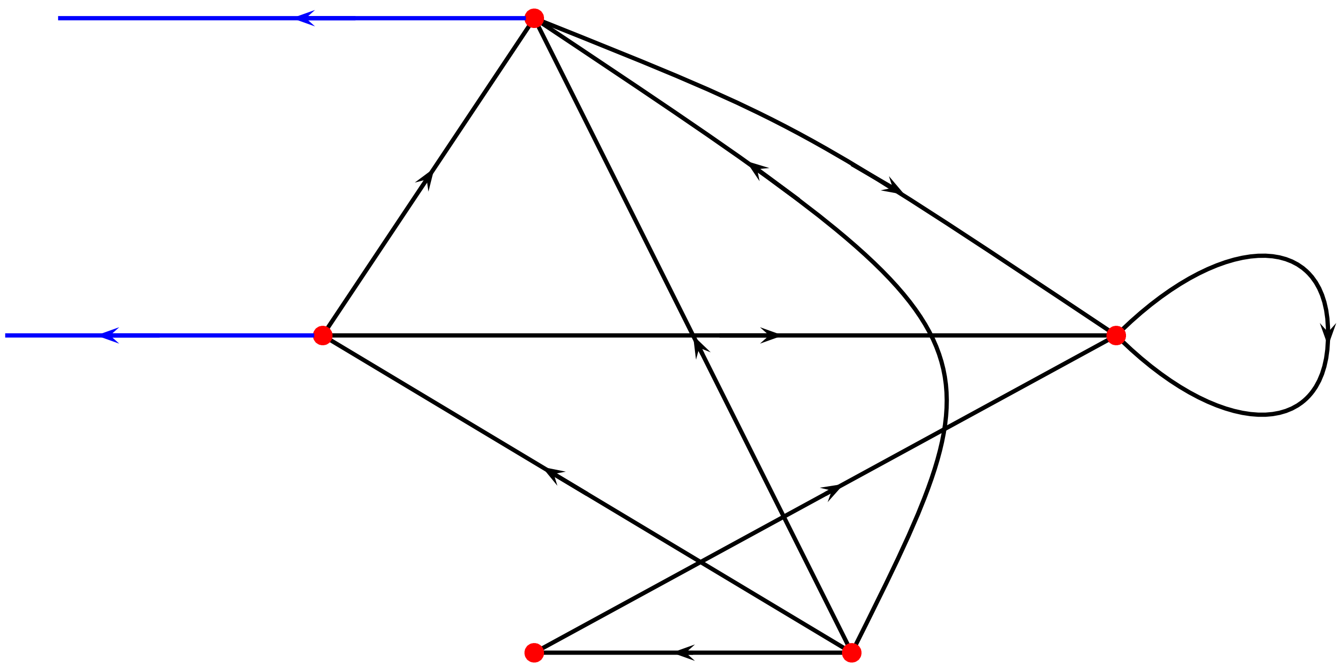

Each internal edge $j \in E_{\text{int}}$ will be associated with a bounded open interval $I_j := (0, a_j)$, $a_j > 0$, according to its orientation.
Each external edge $j \in E_{\text{ext}}$ will be associated with the semiline $I_j := (0, \infty)$.

- The Hilbert space $$\mathcal{H} = \bigoplus_{j\in E} L^2(I_j) = \bigoplus_{j\in E_{\text{int}}} L^2((0,a_j)) \oplus \bigoplus_{j\in E_{\text{ext}}} L^2((0,\infty))$$

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- $\psi \in \mathcal{H} :\Longleftrightarrow \{\psi_j\}_{j \in E}$ with $\psi_j \in L^2(I_j)$ for $j \in E$
- For an arbitrary $\psi \in \mathcal{H}$ with $\psi_j \in H^2(I_j)$ (Sobolev space) set

$$\underline{\psi} = \begin{pmatrix} \{\psi_j(0)\}_{j \in E_{\text{ext}}} \\ \{\psi_j(0)\}_{j \in E_{\text{int}}} \\ \{\psi_j(a_j)\}_{j \in E_{\text{int}}} \end{pmatrix} \in \mathcal{K}, \qquad \underline{\psi}' = \begin{pmatrix} \{\psi_j'(0)\}_{j \in E_{\text{ext}}} \\ \{\psi_j'(0)\}_{j \in E_{\text{int}}} \\ \{-\psi_j'(a_j)\}_{j \in E_{\text{int}}} \end{pmatrix} \in \mathcal{K} \cong \mathbb{C}^{2|E_{\text{int}}| + |E_{\text{ext}}|}$$

and $[\psi] := \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \in \mathcal{K}_d := \mathcal{K} \oplus \mathcal{K}$ (space of boundary values).

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and $[\psi] := \begin{pmatrix} \underline{\psi} \\ \underline{\psi}' \end{pmatrix} \in \mathcal{K}_d := \mathcal{K} \oplus \mathcal{K}$ (space of boundary values).

- Let Δ_0 be the Laplace operator in $\mathcal{H}$ defined on Sobolev functions in H^2 with the boundary conditions $[\psi] = 0$. It is a symmetric, closed, densely defined operator.

Our aim is to study extensions of Δ_0 which generate strongly continuous semigroups on $L^2(G)$

$$\mathrm{e}^{t\Delta} \qquad \text{and} \qquad \mathrm{e}^{-\mathrm{i}t\Delta} \qquad \text{for} \quad t > 0.$$

- quasi-contractive semigroups: $\|T(t)\| \leq \mathrm{e}^{wt}$ holds for some $w \geq 0$ and all $t > 0$,
- contractive semigroups: $\|T(t)\| \leq 1$ holds for all $t > 0$.

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Motivation:

- Diffusion on metric graphs [Walsh (1978)], [Barlow, Pitman, Yor (1989)], [Freidlin, Wentzell (1993)]; C-contractive semigroups are important

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- Network models related to blood flow [Carlson (2006)]
- Spectral determinants [Friedlander (2006)]

An extension Δ of Δ_0 is self-adjoint if and only if there are A and B such that

$$A\underline{\psi} + B\underline{\psi}' = 0 \qquad \text{for any} \qquad \psi \in \mathrm{Dom}(\Delta)$$

and one of the following equivalent conditions holds:

- $(A \quad B)$ has maximal rank and $AB^* = BA^*$,
- $S(k;A,B) := -(A + \mathrm{i}kB)^{-1}(A - \mathrm{i}kB)$ is unitary for some (and hence for all) $k > 0$.

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A self-adjoint operator $\Delta(A,B)$ is non-negative if (and only if for “star graphs”) one of the following equivalent conditions holds:

- $AB^* \leq 0$,
- $S(\mathrm{i}\varkappa;A,B) := -(A-\varkappa B)^{-1}(A+\varkappa B)$ is a contraction for some (and hence for all) $\varkappa > 0$.

The resolvent $(-\Delta(\mathcal{M}) - k^2)^{-1}$ is the integral operator with the $(|E_{\text{int}}| + |E_{\text{ext}}|) \times (|E_{\text{int}}| + |E_{\text{ext}}|)$ matrix-valued integral kernel given by

$$r(x,y;k) = r^{(0)}(x,y,k) + \frac{\mathrm{i}}{2k}\Phi(x,k)R_+(k)^{-1}[I - S(k;A,B)T(k)]^{-1}S(k;A,B)R_+(k)^{-1}\Phi(y,k)^T,$$

where $\Phi(x,k) := \begin{pmatrix} \phi(x;k) & 0 & 0 \\ 0 & \phi_+(x;k) & \phi_-(x;k) \end{pmatrix}$,

$$R_+(k) := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}ka} \end{pmatrix}, \qquad T(k) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}ka} \\ 0 & \mathrm{e}^{\mathrm{i}ka} & 0 \end{pmatrix},$$

with diagonal matrices $\phi(x;k) = \operatorname{diag}\{\mathrm{e}^{\mathrm{i}kx_j}\}_{j\in E_{\text{ext}}}$, $\phi_\pm(x;k) = \operatorname{diag}\{\mathrm{e}^{\pm\mathrm{i}kx_j}\}_{j\in E_{\text{int}}}$, and

$$\mathrm{e}^{\mathrm{i}ka} = \operatorname{diag}\{\mathrm{e}^{\pm\mathrm{i}ka_j}\}_{j\in E_{\text{int}}}, \qquad [r^{(0)}(x,y,k)]_{j,j'} = \mathrm{i}\delta_{j,j'}\frac{\mathrm{e}^{\mathrm{i}k|x_j-y_j|}}{2k}, \quad x_j, y_j \in I_j.$$

$-\Delta$ is called dissipative if

$$\operatorname{Im}\langle\psi,-\Delta\psi\rangle \geq 0 \qquad \text{holds for all} \qquad \psi\in\operatorname{Dom}(-\Delta).$$

It is called maximal dissipative (m-dissipative) if it possesses no proper dissipative extension.

Maximal Dissipative Laplace Operators

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How to describe m-dissipative extension?

[Kochubei (1975)] in terms of spaces of boundary values:

$$(S-I)\underline{\psi}+\mathrm{i}(S+I)\underline{\psi}' = 0 \quad \text{for any contraction} \quad S, \quad \|S\| \leq 1.$$

- S is unitary $\quad\Rightarrow\quad$ Δ is self-adjoint

Maximal Accretive Laplace Operators

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The boundary conditions

$$(S-I)\underline{\psi} + (S+I)\underline{\psi}' = 0$$

for any contraction S, $\|S\| \leq 1$, define an m-accretive Laplace operator.

Related results: [Tsekanovskii and coworkers (1980), (1992)].

Maximal Accretive Laplace Operators

Theorem. The Laplace operator $-\Delta(A,B)$ is m-accretive if (and only if for "star graphs") one of the following equivalent conditions holds:

- $(A \quad B)$ has maximal rank and $\operatorname{Re} AB^* \leq 0$,
- $S(\mathrm{i}\varkappa; A, B) := -(A - \varkappa B)^{-1}(A + \varkappa B)$ is contractive for some (and hence for all) $\varkappa > 0$.

Remark: If $-\Delta$ is m-accretive, then $\Delta = \Delta(A,B)$ with some $(A \quad B)$ having maximal rank (since $-\Delta^*$ is m-accretive, too).

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The following statements are equivalent:

- *The Laplace operator* $-\Delta(A,B)$ *is m-dissipative,*
- $(A \quad B)$ *has maximal rank and* $\operatorname{Im} AB^* \leq 0$*,*
- $S(-k; A, B) := -(A - \mathrm{i}kB)^{-1}(A + \mathrm{i}kB)$ *is contractive for some (and hence for all)* $k > 0$*.*

Assume: à la [Kuchment (2004)]

P an orthogonal projection in $\mathcal{K}$ (possibly zero),
$-L$ an accretive operator in $\mathcal{K}$ such that $LP^\perp = 0$.

Set $A = P^\perp + L$, $B = P$. Then $AB^* = L$ and $(A \quad B)$ has maximal rank. Hence, $-\Delta(A, B)$ is m-accretive.

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Do we get all m-accretive extensions? Yes! Take an arbitrary contraction S_0. If -1 is an eigenvalue of S_0, then the corresponding eigenprojection Q is orthogonal, that is, $Q^* = Q$. Split off the subspace corresponding to Q. Denote by $\widehat{S}_0$ the restriction of S onto the complimentary subspace. Take its Cayley transform $\widehat{L} = \mathrm{i}(I + \widehat{S}_0)^{-1}(I - \widehat{S}_0)$. Its negative is accretive. Let L be the extension of $\widehat{L}$ to the whole of $\mathcal{K}$. Then we obtain

$$S(1; P^{\perp} + L, P) = -(P^{\perp} + L + \mathrm{i}P)^{-1}(P^{\perp} + L - \mathrm{i}P) = S_0$$

where $P = Q^{\perp}$.

Let

$$\mathcal{M} = \left\{ [\psi] = \begin{pmatrix} \psi \\ \underline{\psi}' \end{pmatrix} \middle| (P^{\perp} + L)\underline{\psi} + P\underline{\psi}' = 0 \right\}$$

with $LP^{\perp} = P^{\perp}L = 0$.

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Integration by parts yields that $\mathcal{M}^*$ is the orthogonal complement of

$$\left\{ \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} [\varphi] \,\middle|\, \varphi \in \mathrm{Dom}(\Delta(\mathcal{M})) \right\} = \mathrm{Ker}(P^\perp + L, P) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$= \mathrm{Ker}(-P, P^\perp + L) \Rightarrow \mathcal{M}^* = \mathrm{Ran} \begin{pmatrix} -P \\ P^\perp + L^* \end{pmatrix}.$$

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Now compute: $(P^\perp + L^*, P) \begin{pmatrix} -P \\ P^\perp + L^* \end{pmatrix} = -L^*P + PL^* = L^*P^\perp - P^\perp L^* = 0.$

Denote

$$\mathcal{M} := \left\{ [\psi] = \begin{pmatrix} \psi \\ \underline{\psi'} \end{pmatrix} \middle| A\underline{\psi} + B\underline{\psi}' = 0 \right\}.$$

Calculate the quadratic form of $-\Delta(\mathcal{M})$:

$$\begin{aligned} \mathrm{Re}\,\langle \psi, -\Delta(\mathcal{M})\psi \rangle_{\mathcal{H}} &= \sum_{j \in E} \|\psi_j\|_{\mathcal{H}_j} + \mathrm{Re}\,\langle [\psi], Q[\psi] \rangle_{\mathcal{K}_d} \qquad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \sum_{j \in E} \|\psi_j\|_{\mathcal{H}_j} + \mathrm{Re}\,\langle [\psi], P_{\mathcal{M}} Q P_{\mathcal{M}} [\psi] \rangle_{\mathcal{K}_d}, \end{aligned}$$

where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M} \subset \mathcal{K}_d$.
Hence, $-\Delta(\mathcal{M})$ is accretive if $\mathrm{Re}\,P_{\mathcal{M}} Q P_{\mathcal{M}} \leq 0$.

- Compare: $-\Delta(\mathcal{M})$ is dissipative if and only if $\mathrm{Im}\,P_{\mathcal{M}} Q P_{\mathcal{M}} \leq 0$.

What we have is

$$P_{\mathcal{M}^\perp} = \begin{pmatrix} A^* \\ B^* \end{pmatrix} (AA^* + BB^*)^{-1} (A, B).$$

Lemma. Let P_1 and P_2 be orthogonal projections of equal dimension. Then the following conditions are equivalent:

(i) $P_1(P_2 - P_2^\perp)P_1 \geq 0$,

(ii) $P_1^\perp(P_2 - P_2^\perp)P_1^\perp \leq 0$.

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How to apply this lemma? Set $P_1 = P_{\mathcal{M}}$. Observe that

$$2\mathrm{Re}\, Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = P_+ - P_- \qquad \text{with} \qquad P_\pm := \frac{1}{2}\begin{pmatrix} I & \mp I \\ \mp I & I \end{pmatrix}$$

Since $\mathcal{M}$ and $\mathrm{Ran} P_\pm$ have equal dimension, the zero index condition is satisfied. Therefore, $\mathrm{Re}\, P_{\mathcal{M}} Q P_{\mathcal{M}} \geq 0$ holds if and only if $\mathrm{Re}\, P_{\mathcal{M}^\perp} Q P_{\mathcal{M}^\perp} \leq 0$. But

$$\mathrm{Re}\, P_{\mathcal{M}^\perp} Q P_{\mathcal{M}^\perp} = \begin{pmatrix} A^* \\ B^* \end{pmatrix} (AA^* + BB^*)^{-1} \mathrm{Re}\,(AB^*)(AA^* + BB^*)^{-1}(A, B).$$

Maximal Quasi-Accretive Laplace Operators

- If $(A \quad B)$ has maximal rank, then $-\Delta(A,B)$ is maximal quasi-accretive. The corresponding semigroup is quasi-contractive, that is,

$$\|\mathrm{e}^{t\Delta(A,B)}\| \leq \mathrm{e}^{wt} \quad \text{for some} \quad w \geq 0 \quad \text{and all} \quad t > 0.$$

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- Is $\Delta(A,B)$ normal? **Claim:** $\Delta(A,B)$ is not normal unless $\Delta(A,B)$ is self-adjoint. **Proof:** Assume $\Delta(A,B)$ is normal. Then $\mathrm{Dom}(\Delta(A,B)) = \mathrm{Dom}(\Delta(A,B)^*)$. Hence $L_{A,B}^* = L_{A,B}$. Therefore, $\Delta(A,B)$ is self-adjoint.

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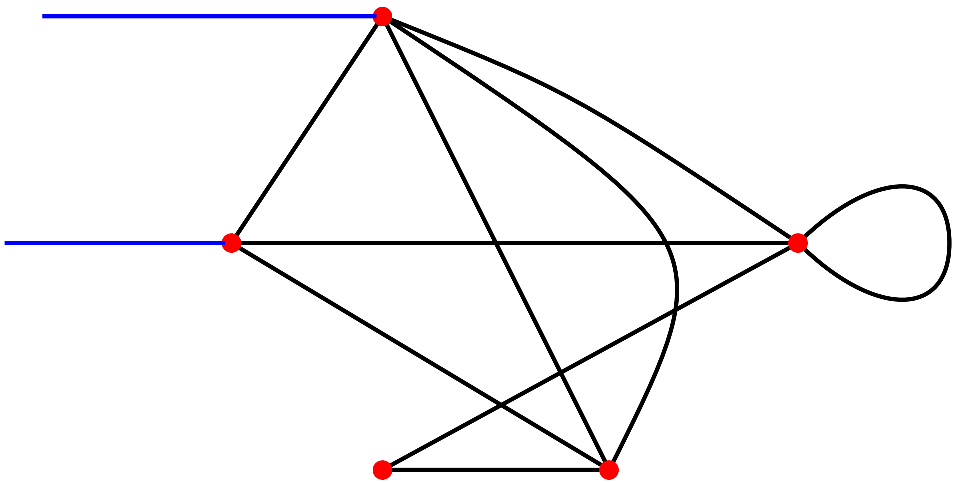
- The spectrum

$$\sigma_r(\Delta(A,B)) = \emptyset, \qquad \sigma(\Delta(A,B)) = \sigma_c(\Delta(A,B)) \cup \sigma_p(\Delta(A,B)).$$

Assume that the boundary conditions are local, that is, $S(k;A,B) = \bigoplus_{v \in V} S(k;A_v,B_v)$ such that each block corresponds to a particular vertex.



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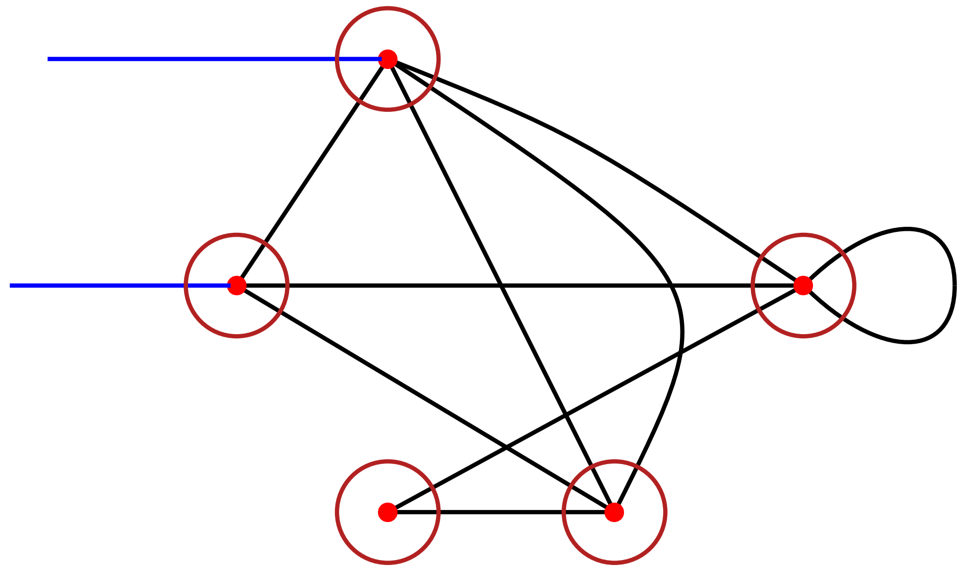
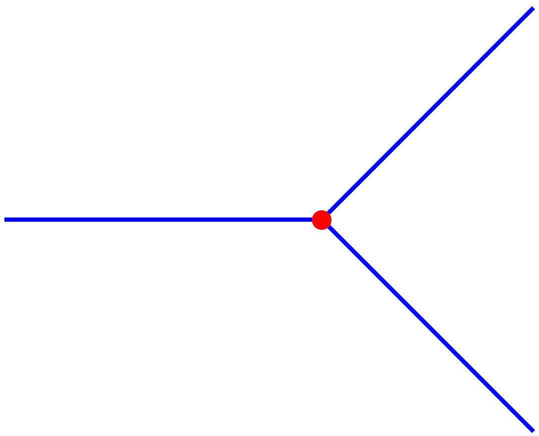


Assume, in addition,

$$I + S(\mathrm{i}\varkappa, A_v, B_v) > 0 \qquad \text{(componentwise)}$$

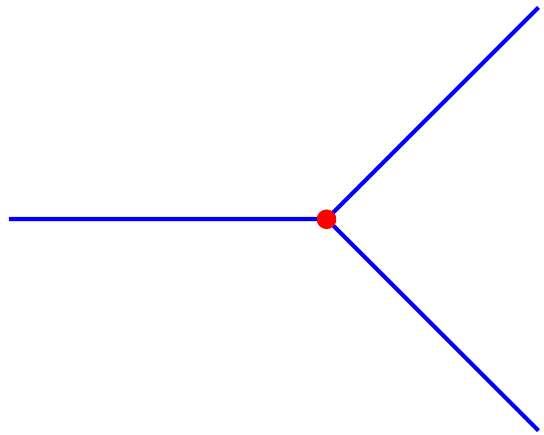
for all vertices $v \in V$ and all sufficiently large $\varkappa > 0$. Then the semigroup $\mathrm{e}^{t\Delta(A_v,B_v)}$ is positivity preserving.

Let G be a “star graph”, that is a graph with a single vertex and no internal edges:



Assume that $\mathrm{e}^{t\Delta(A,B)}$ is positivity preserving. Then $\mathrm{e}^{t\Delta(A,B)}$ is L^∞-contractive whenever $S(\mathrm{i}\varkappa, A, B)$ is substochastic for all $\varkappa > 0$.

Let G be a "star graph", that is a graph with a single vertex and no internal edges:

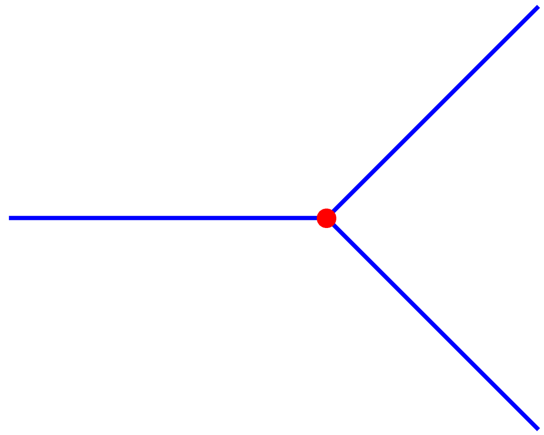


The semigroup $e^{t\Delta(A,B)}$ is a Feller semigroup if and only if

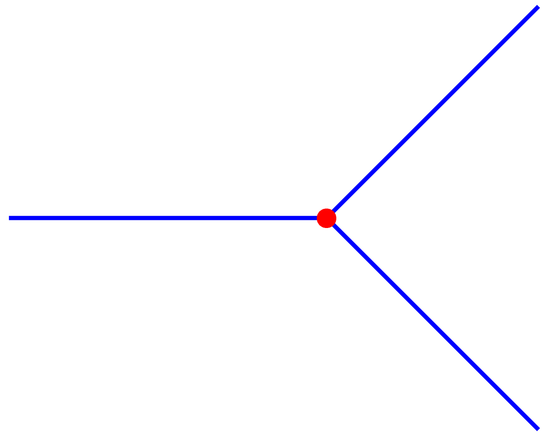
$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & -\ell \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ p_1 & p_2 & p_3 & \dots & p_{n-1} & p_n \end{pmatrix}$$

with $\sum_{k=1}^{n} p_k = 1$, $p_k \in (0,1)$.

Walsh-Type Brownian Motion



Walsh-type Brownian motion with $\ell = 0$: Choose an edge $j \in E$ and a starting point $x \in \mathbb{R}_+$. Condider a Brownian motion X on $\mathbb{R}$ starting at x. Until this Brownian motion hits zero identify X with a Brownian motion on the edge. When X hits zero, take the first excursion from zero, and with probability p_i, identify it with an excursion of X on the edge $i \in E$. And so on.



Walsh-type Brownian motion with $\ell > 0$: Let τ be a random variable with exponential distribution $\ell \mathrm{e}^{-\ell r}$, $r > 0$.
Local time:

$$L_t(v) := \lim_{\varepsilon \to +0} \frac{1}{4\varepsilon} \mathrm{meas}\{s \in [0,t] \,|\, \mathrm{dist}(X_s, v) \leq \varepsilon\}.$$

If $L_t(v) > \tau$ kill the process.

Thank You For Your Attention!

A Very Incomplete List of References

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