INVERSE SPECTRAL PROBLEMS ON COMPACT (AND NONCOMPACT) GRAPHS

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Basic definitions

- Consider a compact tree T in $\mathbf{R}^{\mathbf{m}}$ with root v_0, the set of vertices $V = \{v_0, \dots, v_r\}$ and the set of edges $\mathcal{E} = \{e_1, \dots, e_r\}$ of length 1. $\Gamma = \{v_0, \dots, v_p\}$ is the set of boundary vertices.

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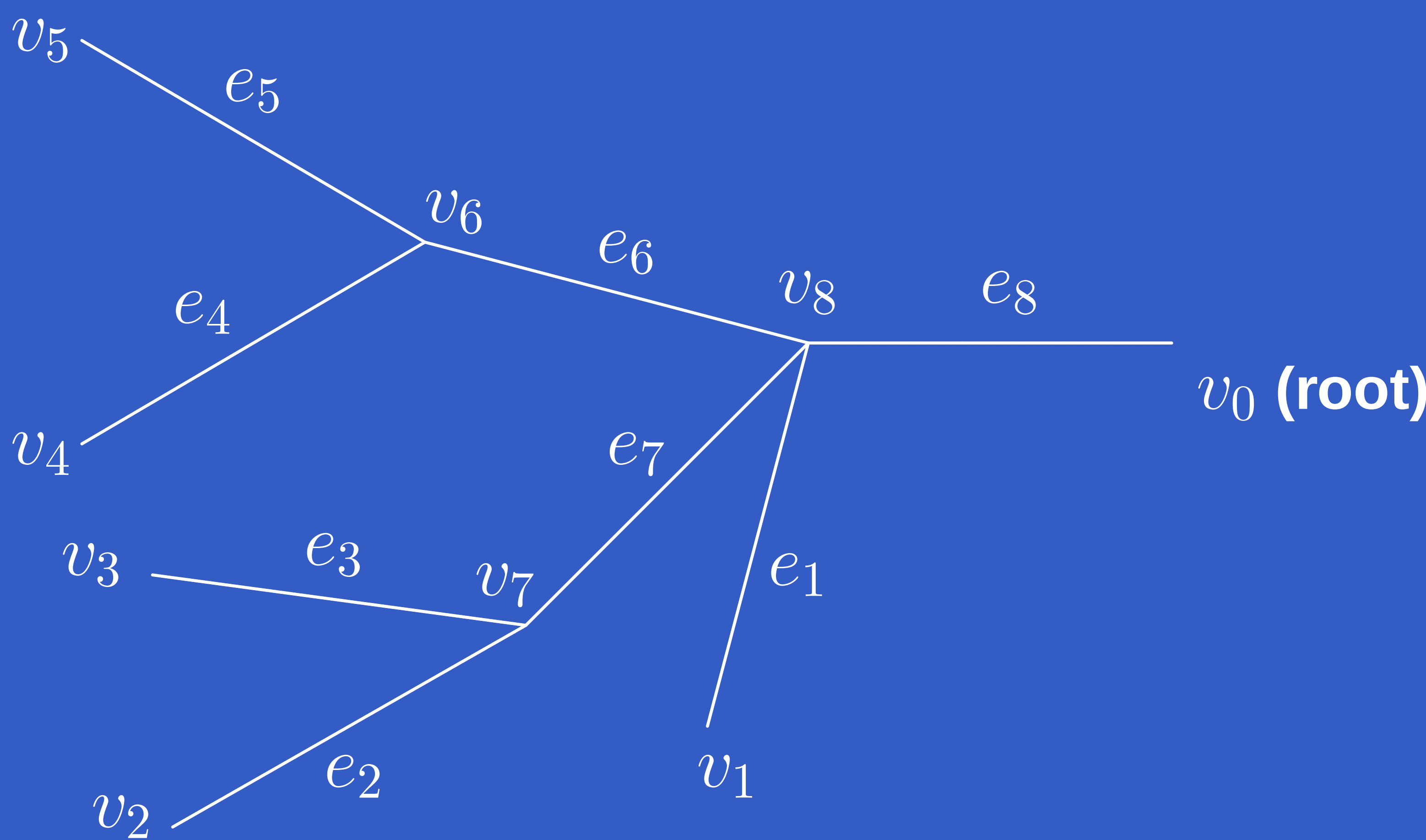
- $[a, b] := \{z \in T \mid \ a \leq z \leq b\}$.

- $R(v) := \{e \in \mathcal{E} \mid \ e = [v, w], \ w \in V\}$ set of edges emanating from v.

An example



Tree T with boundary vertices $v_0, \ldots, v_5$.
Edges $E = \{e_1, \ldots, e_8\}$, Vertices $V = \{v_0, \ldots, v_8\}$

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- Let $V^{(\mu)} := \{v \in V \mid |v| = \mu\}$, $\mu = \overline{0, \sigma}$ be the set of vertices of order μ, and let $\mathcal{E}^{(\mu)} := \{e \in \mathcal{E} \mid e = [v, w], v \in V^{(\mu-1)}, w \in V^{(\mu)}\}$, be the set of edges of order μ.

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- Each edge $e_j =: [v_{n_j}, v_j]$ has length 1 with

$$e_j = \{v_j + x(v_{n_j} - v_j) \mid 0 \leq x \leq 1\}.$$

Basic definitions

- An integrable function Y on T may be represented as a vector $Y(x) = [y_j(x)]_{j \in J}$, $x \in [0, 1]$, where $J := \{j \mid \; j = \overline{1, r}\}$, and

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- Let $q = [q_j]_{j\in J}$ be an integrable complex-valued function on T which is called the potential.

- Consider the Sturm-Liouville equation on T:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad x \in [0,1], \qquad (1)$$

where $y_j(x)$, $y_j'(x)$ are absolutely continuous.

Matching conditions

- In each internal vertex v_k, $k = \overline{p+1, r}$:

$$\left.\begin{aligned} y_k(0) &= a_{kj} y_j(1) \quad \text{for all} \quad e_j \in R(v_k), \\ y_k'(0) &= \sum_{e_j \in R(v_k)} (a_{kj}^1 y_j'(1) + a_{kj}^0 y_j(1)), \end{aligned}\right\} \qquad (2)$$

  where a_{kj}, a_{kj}^0, a_{kj}^1 are complex numbers, and $a_{kj} a_{kj}^1 \neq 0$.

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  where a_{kj}, a_{kj}^0, a_{kj}^1 are complex numbers, and $a_{kj} a_{kj}^1 \neq 0$.

- We note that if $a_{kj} = a_{kj}^1 = 1$, $a_{kj}^0 = 0$ for all k, j, then the conditions (2) are called the standard conditions.

Regularity condition

- Assume that for $k = \overline{p+1, r}$

$$r_k := \sum_{e_j \in R(v_k)} \frac{a^1_{kj}}{a_{kj}} \neq -1. \qquad (3)$$

Cond.(3) is the regularity condition for matching.

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- Note that in (2) we have $2r - p - 1$ conditions. In order to define a boundary value problem for (1) we introduce the following linear forms in the boundary vertices v_j, $j \in \Gamma$:

$$U_{js}(Y) := \sum_{\nu=0}^{1} h_{js}^{\nu} Y_{|v_j}^{(\nu)}, \qquad s = 0, 1;\ j = \overline{0, p},$$

The BVPs L and L_k

- L is the boundary value problem for (1) with the matching conditions (2) and with the boundary conditions

$$U_{j0}(Y) = 0, \quad j = \overline{0, p}.$$

- L_k, $k = \overline{0, p}$, L is the boundary value problem for (1) with the matching conditions (2) and with the boundary conditions

$$U_{k1}(Y) = 0,\ U_{j0}(Y) = 0, \quad j = \overline{0, p} \setminus k.$$

- Let $\{\lambda_l\}_{l \geq 0}$ and $\{\lambda_{lk}\}_{l \geq 0}$ be the eigenvalues of L and L_k, respectively.

Weyl solutions

Let $\Psi_k(x,\lambda) = [\psi_{kj}(x,\lambda)]_{j\in J}$, $k = \overline{0,p}$, be solutions of (1) satisfying (2) and

$$U_{j0}(\Psi_k) = \delta_{jk}, \quad j = \overline{0,p}. \tag{4}$$

The functions Ψ_k are called the Weyl solutions of (1) with respect to the vertex v_k and the functionals U_{j0}. The functions $M_k(\lambda) := U_{k1}(\Psi_k)$ are called the Weyl functions, and

$$M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}},$$

is called the Weyl vector for (1).

Weyl functions and Weyl vector

For definiteness, we will consider the case when

$$U_{j0}(Y) = Y'_{|v_j} + h_j Y_{|v_j},\ U_{j1}(Y) = Y_{|v_j},$$

i.e. $h^1_{j0} = h^0_{j1} = 1,\ h^1_{j1} = 0,\ h^0_{j0} = h_j$.

Let $\varphi_j(x,\lambda),\ S_j(x,\lambda)$, $j \in J,\ x \in [0,1]$ be solutions of equation (1) on the edge e_j under the initial conditions

$$\varphi_j(0,\lambda),\quad \varphi'_j(0,\lambda) = -h_j, \quad (i.e.\ \ U_{j0}(\varphi_j) = 0)$$

$$S'_j(0,\lambda) = 1\ \ S_j(0,\lambda) = 0.$$

Properties of Weyl functions

Denote

$$M^1_{kj}(\lambda) = \psi_{kj}(0, \lambda),$$

$$M^0_{kj}(\lambda) = \psi'_{kj}(0, \lambda) + h_j \psi_{kj}(0, \lambda).$$

Then

$$\psi_{kj}(x, \lambda) = M^0_{kj}(\lambda) S_j(x, \lambda) + M^1_{kj}(\lambda) \varphi_j(x, \lambda). \tag{5}$$

In particular, for $k = \overline{1, p}$, we have

$$\psi_{kk}(x, \lambda) = S_k(x, \lambda) + M_k(\lambda) \varphi_k(x, \lambda). \tag{6}$$

Weyl functions and spectral data

For compact T the Weyl functions $M_k(\lambda)$ are meromorphic in λ with the poles $\{\lambda_l\}_{l\geq 0}$:

$$M_k(\lambda) = \frac{\Delta_k(\lambda)}{\Delta(\lambda)}, \quad k = \overline{1,p}, \tag{7}$$

where $\Delta(\lambda)$ and $\Delta_k(\lambda)$ are the characteristic functions for L and L_k, respectively. If all poles are simple we introduce the data $S := \{\lambda_l, \alpha_{lk}\}_{l\geq 0,\, k=\overline{1,p}}$, where α_{lk} are residues of $M_k(\lambda)$ at λ_l; the data S are called the spectral data for L.

Three inverse problems

We study three inverse problems of recovering the potential $q = [q_j]_{j \in J}$ and the coefficients $h = [h_j]_{j \in J}$ from the following spectral characteristics:

1) from the Weyl vector $M = [M_k]_{k=\overline{1,p}}$;

2) from the system of $p+1$ spectra
$\Sigma := \{\lambda_l, \lambda_{lk}, l \geq 0, k = \overline{1,p}\}$;

3) from the spectral data S.

Asymptotics for Weyl solutions

Lemma 1. *For* $\nu = 0, 1$,

$$\rho = \sqrt{\lambda} \in \Lambda_\delta = \{\rho \mid \ \arg \rho \in [\delta, \pi - \delta]\},$$

and $|\rho| \to \infty$, *uniformly in* $x \in [0, 1]$,

$$\psi_{0j}^{(\nu)}(x, \lambda) = B_j(\rho) \exp(i\rho\mu)((-i\rho)^{\nu-1} \exp(-i\rho x)[1]$$

$$-(i\rho)^{\nu-1} d_j \exp(i\rho x)[1]),$$

where $d_j = 1$ *for* $j = \overline{1, p}$, *and* $d_j = (1 + r_j)^{-1}(1 - r_j)$ *for* $j = \overline{p+1, r}$. *Moreover, for* $\rho \in \Lambda_\delta, \ |\rho| \to \infty$,

$$B_j(\rho) = b_j[1], \quad b_j \neq 0, \ b_{p+1} = 1.$$

Similar asymptotics holds for all other Weyl solutions.

Asymptotics for Weyl solutions

Lemma 2. *For* $k = \overline{1, p}$, $\nu = 0, 1$, *one has*

$$\psi_{kk}^{(\nu)}(x, \lambda) = (i\rho)^{\nu-1} \exp(i\rho x)[1], \ x \in [0, 1),$$

$$M_k(\lambda) = (i\rho)^{-1}[1], \ \rho \in \Lambda_\delta, \ |\rho| \to \infty.$$

Let $\delta > 0$ be sufficiently small and fixed. Denote $G_\delta := \{\rho \mid \ |\rho - \rho_l| \geq \delta, \forall l \geq 0\}$, where $\lambda_l = \rho_l^2$ are eigenvalues of L. Then

$$|\psi_{kk}^{(\nu)}(x, \lambda)| \leq C|\rho^{\nu-1} \exp(i\rho x)|, \ x \in [0, 1],$$

$$|M_k(\lambda)| \leq C|\rho|^{-1}, \ \rho \in G_\delta \cap \Lambda.$$

Local inverse problem

Fix $k = \overline{1, p}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

IP(k): Given $M_k(\lambda)$, construct $q_k(x)$, $x \in [0, 1]$ and h_k.

Uniqueness of the solution of IP(k):

Consider a tree $\tilde{T}$ of the same form but with different $\tilde{q}$ and $\tilde{h}$. Everywhere below if a symbol α denotes an object related to T, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{T}$.

Uniqueness theorem

Lemma 3. *If $M_k(\lambda) = \tilde{M}_k(\lambda)$, then $q_k(x) = \tilde{q}_k(x)$ a.e. on [0,1], and $h_k = \tilde{h}_k$. Thus, the specification of the Weyl function M_k uniquely determines the potential q_k on the edge e_k and the coefficient h_k.*

Using the method of spectral mappings for the Sturm-Liouville operator on the edge e_k one can get a constructive procedure for the solution of the local inverse problem IP(k):

Main equation

Take the BVP $\tilde{L}$ with $\tilde{q} = 0$ and $\tilde{h} = 0$. Then $\tilde{\varphi}_k(x, \lambda) = \cos \rho x$. Fix $k = \overline{1, p}$. For each fixed $x \in [0, 1]$, $\varphi_k(x, \lambda)$ is the unique solution of the following linear integral equation

$$\tilde{\varphi}_k(x, \lambda) = \varphi_k(x, \lambda) + \frac{1}{2\pi i} \int_\gamma \tilde{D}_k(x, \lambda, \mu) \hat{M}_k(\mu) \varphi_k(x, \mu)\, d\mu, \tag{8}$$

where $\tilde{D}_k(x, \lambda, \mu) = \int_0^x \tilde{\varphi}_k(t, \lambda) \tilde{\varphi}_k(t, \mu)\, dt$,

and

$$\hat{M}_k(\mu) := M_k(\mu) - \tilde{M}_k(\mu).$$

Reconstruction formula

The potential q_k on the edge e_k can be constructed from the solution of the integral equation (8) via the formula

$$q_k(x) = \frac{1}{2\pi i}\int_\gamma (\varphi_k(x,\lambda)\tilde{\varphi}_k(x,\lambda))'\hat{M}_k(\lambda)\,d\lambda$$

or by the formula $q_k(x) = \lambda + \varphi_k''(x,\lambda)/\varphi_k(x,\lambda)$. Moreover $h_k = -\varphi'(0,\lambda)$.

It is also possible to construct the potential from the discrete spectral data $\{\lambda_l, \alpha_{lk}\}_{l\geq 0}$.

Discrete spectral data

For this purpose one can calculate the contour integral in (8) by the residue theorem and transform the integral equation (8) to the following linear equation in a space of bounded sequences (for each fixed x):

$$\tilde{\varphi}_{kns}(x) = \varphi_{kns}(x) + \sum_{l,j} \tilde{P}_{kns}^{lj}(x)\varphi_{klj}(x),$$

$$l, n \geq 0,\ \ s, j = 0, 1,$$

where for $\lambda_l^1 = \tilde{\lambda}_l$, $\alpha_{lk}^0 = \alpha_{lk}$ $\alpha_{lk}^1 = \tilde{\alpha}_{lk}$, $\varphi_{kns}(x) = \varphi_k(x, \lambda_n^s)$, $\tilde{\varphi}_{kns}(x) = \tilde{\varphi}_k(x, \lambda_n^s)$,

$$\tilde{P}_{kns}^{lj}(x) = (-1)^j \tilde{D}_k(x, \lambda_n^s, \lambda_l^j)\alpha_{lk}^j, \quad \lambda_l^0 = \lambda_l.$$

Problem $\mathbf{Z(T, v_0, a)}$

Let $\Psi = [\psi_j]_{j\in J}$ be the solution of equation (1) satisfying (2) and the boundary conditions

$$\Psi_{|v_0} = a, \quad U_{j0}(\Psi) = 0, \; j = \overline{1,p}. \qquad (9)$$

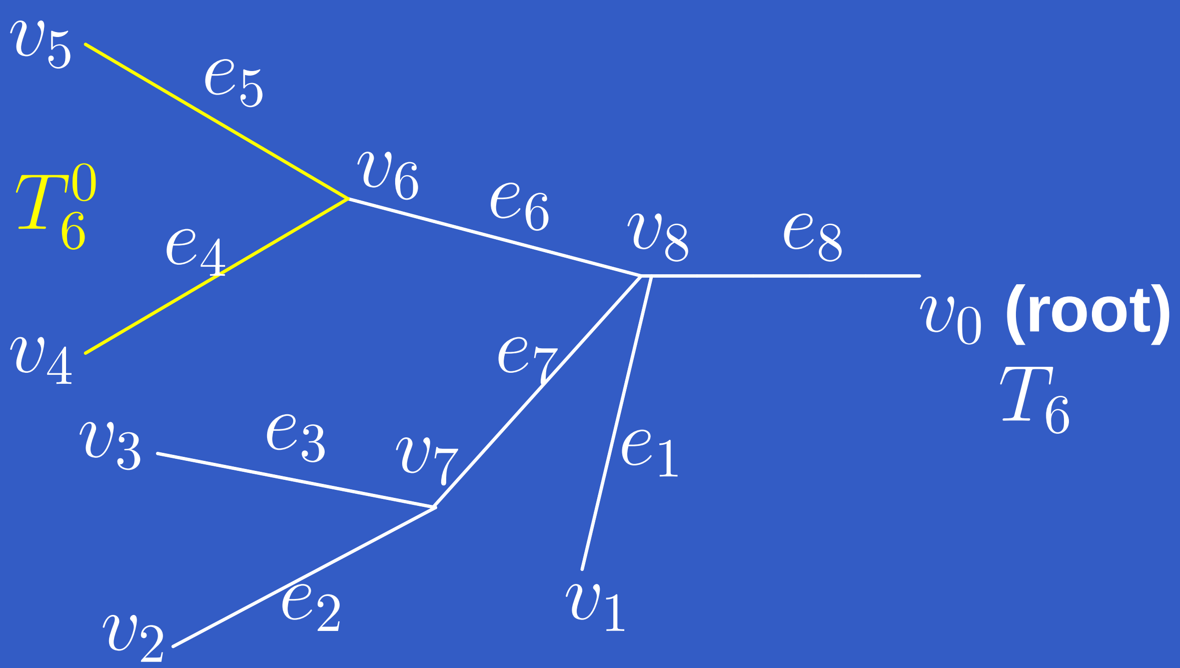
Substituting

$$\psi_j(x,\lambda) = m_j^0(\lambda) S_j(x,\lambda) + m_j^1(\lambda)\varphi_j(x,\lambda). \qquad (10)$$

into (2) and (9) we obtain a linear algebraic system for $m_j^0(\lambda) = \psi_j'(0,\lambda) + h_j\psi_j(0,\lambda)$, $m_j^1(\lambda) = \psi_j(0,\lambda), j \in J$. The determinant of this system is $\Delta_0(\lambda)$. Solving this system by Kramer's rule we find the transition matrix $[m_j^0(\lambda), m_j^1(\lambda)]_{j\in J}$ for T with respect to v_0 and a.

Weyl solutions for internal vertices

For $v_k \in V$ put $T_k^0 := \{z \in T \mid \ v_k < z\}$.
$T_k := T \setminus T_k^0$ is a tree with the root v_0.
Let Γ_k be the set of boundary vertices of T_k, and let E_k be the set of boundary edges of T_k. Denote $J_k := \{j \mid \ e_j \in T_k\}$. If $Y = [y_j]_{j \in J}$ is a function on T, then $\{Y\}_k := [y_j]_{j \in J_k}$ is a function on T_k.



Weyl functions for T_k

Fix $v_k \notin \Gamma$ (i.e. $k = \overline{p+1, r}$). Let

$$\Psi_k(x, \lambda) = [\psi_{kj}(x, \lambda)]_{j \in J_k}$$

be the solution of equation (1) on T_k satisfying (2) and the boundary conditions $U_{j0}(\Psi_k) = \delta_{kj}$, $v_j \in \Gamma_k$, where

$$U_{k0}(Y) = Y'_{|v_k} + h_k Y_{|v_k}, \quad h_k \in \mathbf{R}.$$

Ψ_k is the Weyl solution of (1) on T_k with respect to the vertex v_k.
Denote by $M_k(\lambda) := \psi_{kk}(0, \lambda)$, $k = \overline{p+1, r}$ the Weyl functions for T_k with respect to v_k.

Recover $\Psi_m(x,\lambda)$ and $M_m(\lambda)$

Lemma 4. *Fix $v_m \notin \Gamma$. Let $e_k = [v_m, v_k] \in R(v_m)$. Then*

$$\Psi_m(x,\lambda) = \{\frac{1}{A_{mk}(\lambda)}\Psi_k(x,\lambda)\}_m, \qquad (11)$$

$$M_m(\lambda) = \frac{a_{mk}}{A_{mk}(\lambda)}\psi_{kk}(1,\lambda), \qquad (12)$$

where

$$A_{mk}(\lambda) = \sum_{e_j \in R(v_m)} a^1_{mj}\psi'_{kj}(1,\lambda)$$

$$+a_{mk}(h_m + \sum_{e_j \in R(v_m)} \frac{a^0_{mj}}{a_{mj}})\psi_{kk}(1,\lambda), \qquad (13)$$

and $\Psi_m, \ M_m$ does not depend on k.

Connecting relations

Denote $M^1_{kj}(\lambda) = \psi_{kj}(0,\lambda)$, $M^0_{kj}(\lambda) = \psi'_{kj}(0,\lambda) + h_j\psi_{kj}(0,\lambda)$ for $k = \overline{p+1,r}$, $j \in J_k$.
Then (5) and (6) are valid for $k = \overline{1,r}$, $j \in J_k$, where $J_k = J$ for $k = \overline{1,p}$. In particular, this yields

$$\psi^{(\nu)}_{kj}(1,\lambda) = M^0_{kj}(\lambda)S^{(\nu)}_j(1,\lambda) + M^1_{kj}(\lambda)\varphi^{(\nu)}_j(1,\lambda), \quad (14)$$

$$\psi^{(\nu)}_{kk}(1,\lambda) = S^{(\nu)}_k(1,\lambda) + M_k(\lambda)\varphi^{(\nu)}_k(1,\lambda). \quad (15)$$

Solution of Inverse Problem 1

Let the Weyl vector $M(\lambda) = [M_k(\lambda)]_{k=\overline{1,p}}$ for the tree T be given. The procedure for the solution of Inverse Problem 1 consists in the realization of the so-called A_μ- procedures successively for $\mu = \sigma, \sigma - 1, \ldots, 1$, where σ is the height of the tree T. Let us describe A_μ- procedures.

A_σ- procedure

1) For each edge $e_k \in \mathcal{E}^{(\sigma)}$, we solve the local inverse problem IP(k) and find $q_k(x)$, $x \in [0,1]$ on the edge e_k and h_k.

2) For each $e_k \in \mathcal{E}^{(\sigma)}$, we construct

$$\varphi_k(x,\lambda) \text{ and } S_k(x,\lambda),\ x \in [0,1],$$

and calculate

$$\psi_{kk}^{(\nu)}(1,\lambda) = S_k^{(\nu)}(1,\lambda) + M_k(\lambda)\varphi_k^{(\nu)}(1,\lambda). \tag{15}$$

Returning procedure

3) Returning procedure. For each fixed $v_m \in V^{(\sigma-1)} \setminus \Gamma$ and for all $e_j, e_k \in R(v_m),\ j \neq k$, we construct $M^s_{kj}(\lambda),\ s = 0, 1$, by

$$M^0_{kj}(\lambda) = 0,\ \ M^1_{kj}(\lambda) = \frac{a_{mk}}{a_{mj}} \frac{\psi_{kk}(1,\lambda)}{\varphi_j(1,\lambda)},\ \ j \neq k.$$

4) For each fixed $v_m \in V^{(\sigma-1)} \setminus \Gamma$ we calculate the Weyl function $M_m(\lambda)$ by

$$M_m(\lambda) = \frac{a_{mk}}{A_{mk}(\lambda)} \psi_{kk}(1,\lambda),$$

where $A_{mk}(\lambda)$ and $\psi'_{kj}(1,\lambda)$ are constructed via (13) and (14).

Induction

Now we carry out A_μ- procedures for $\mu = \overline{1, \sigma - 1}$ by induction. Fix $\mu = \overline{1, \sigma - 1}$, and suppose that $A_\sigma, \ldots, A_{\mu+1}$- procedures have been already carried out. Let us carry out A_μ- procedure.

A_μ- procedure

For each $v_k \in V^{(\mu)}$, the Weyl functions $M_k(\lambda)$ are given. Indeed, if $v_k \in V^{(\mu)} \cap \Gamma$, then $M_k(\lambda)$ are given a priori, and if $v_k \in V^{(\mu)} \setminus \Gamma$, then $M_k(\lambda)$ were calculated on the previous steps according to $A_\sigma, \ldots, A_{\mu+1}$-procedures.

1) For each edge $e_k \in \mathcal{E}^{(\mu)}$, we solve the local inverse problem IP(k) and find $q_k(x)$, $x \in [0,1]$ on the edge e_k and h_k. If $\mu = 1$, then Inverse Problem 1 is solved, and we stop our calculations. If $\mu > 1$, we go on to the next step.

2) For each $e_k \in \mathcal{E}^{(\mu)}$, we construct by (15) $\varphi_k(x,\lambda)$ and $S_k(x,\lambda)$, $x \in [0,1]$, and calculate $\psi_{kk}^{(\nu)}(1,\lambda)$, $\nu = 0, 1,$.

Returning procedure with $Z(T_i^1, v_m, \psi_{kk}(1,\lambda))$

3) Returning procedure. For each fixed $v_m \in V^{(\mu-1)} \setminus \Gamma$ and for any fixed $e_k, e_i \in R(v_m)$, $i \neq k$, we consider the tree $T_i^1 := T_i^0 \cup \{e_i\}$ with the root v_m. Solving the problem $Z(T_i^1, v_m, \psi_{kk}(1,\lambda))$, we calculate the transition matrix $[M_{kj}^0(\lambda), M_{kj}^1(\lambda)]$ for $e_j \in T_i^1$.

4) For each fixed $v_m \in V^{(\mu-1)} \setminus \Gamma$ we calculate the Weyl function $M_m(\lambda)$ by

$$M_m(\lambda) = \frac{a_{mk}}{A_{mk}(\lambda)} \psi_{kk}(1,\lambda), \tag{12}$$

where $A_{mk}(\lambda)$ and $\psi'_{kj}(1,\lambda)$ are constructed via (13) and (14).

Main theorem

Thus, we have obtained the solution of Inverse Problem 1 and proved its uniqueness, i.e. the following assertion holds.

Theorem 1. *The specification of the Weyl vector* M *uniquely determines the potential* q *on* T *and* h*. The solution of Inverse Problem 1 can be obtained by executing successively* $A_\sigma, A_{\sigma-1}, \ldots, A_1$*-procedures.*

Solution of Inverse Problem 2

Let the system of spectra $\Sigma := \{\lambda_l,\ \lambda_{lk};\ l \geq 0;\ k = \overline{1,p}\}$ be given. The numbers $\{\lambda_l\}_{l\geq 0}$ and $\{\lambda_{lk}\}_{l\geq 0}$ coincide with the zeros of the characteristic functions $\Delta(\lambda)$ and $\Delta_k(\lambda)$, respectively. Using (7) and Hadamard's factorization theorem, we get

$$M_k(\lambda) = m_k \prod_{l=0}^{\infty} (1 - \frac{\lambda}{\lambda_{lk}})(1 - \frac{\lambda}{\lambda_l})^{-1}, \tag{16}$$

$$m_k = \lim_{|\rho|\to\infty} (i\rho)^{-1} \prod_{l=0}^{\infty} (1 - \frac{\lambda}{\lambda_l})(1 - \frac{\lambda}{\lambda_{lk}})^{-1},\ \rho \in \Lambda_\delta.$$

Thus, using the given spectra Σ, one can construct uniquely the Weyl vector $M(\lambda)$ by (16).

Theorem 2. *The specification of the system of spectra* Σ *uniquely determines the potential* q *on* T *and* h. *For constructing the solution of Inverse Problem 2 we calculate the Weyl vector* M *by (16), and then construct* q *and* h *by solving Inverse Problem 1.*

Solution of Inverse Problem 3

Let all poles of $M(\lambda)$ be simple, and let the spectral data S be given. Take positive numbers $R_N \to \infty$ such that for sufficiently small $\delta > 0$, the circles $|\rho| = R_N$ lie in G_δ for all N. Then

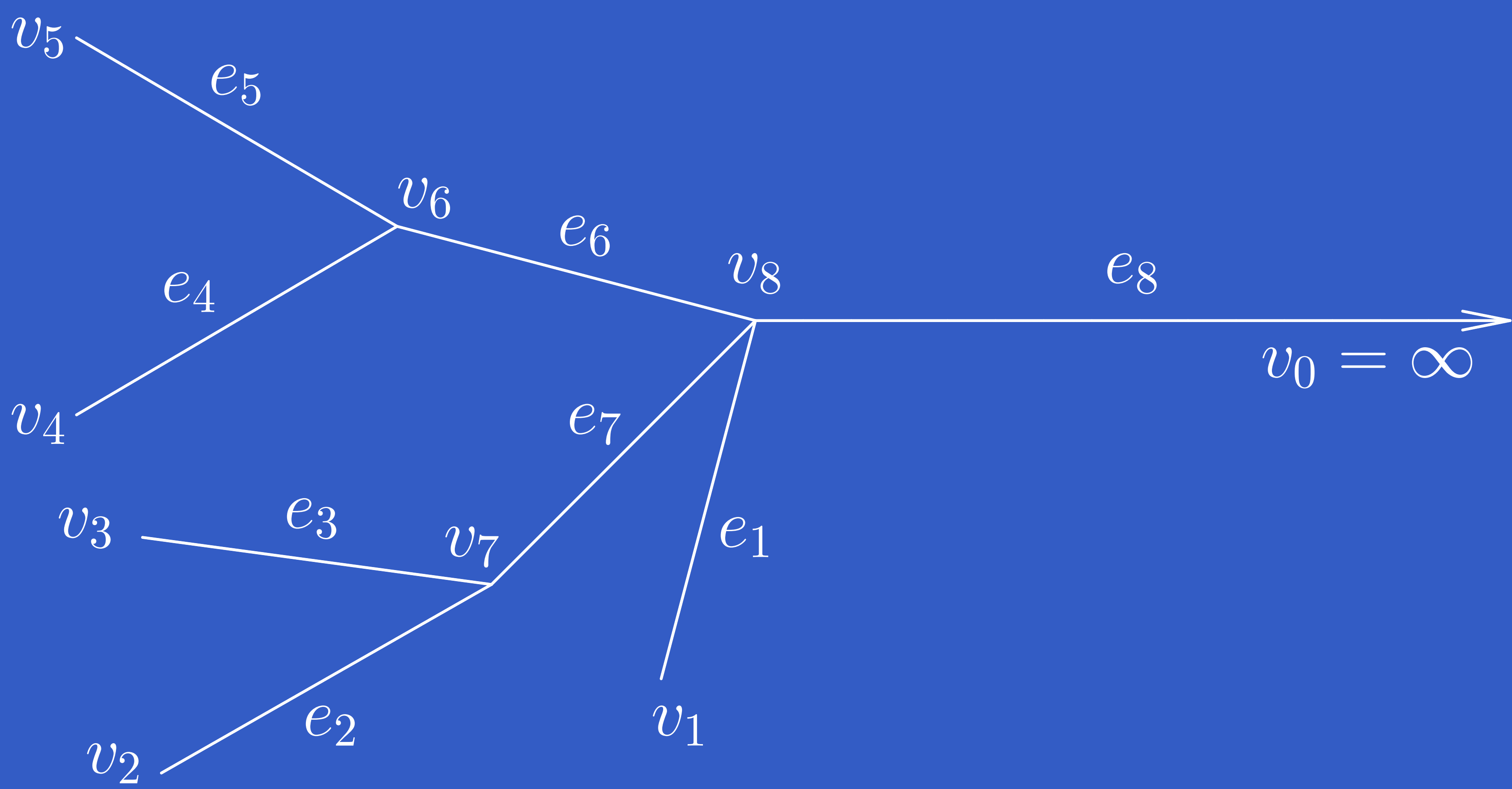
$$M_k(\lambda) = \sum_l \frac{\alpha_{lk}}{\lambda - \lambda_l}, \qquad (17)$$

where the series in (34) converge "with brackets".

Thus, using the given spectral data S, one can construct uniquely the Weyl vector $M(\lambda)$ by (17). In other words, the solution of Inverse Problem 3 is reduced to the solution of Inverse Problem 1, and the following assertion holds.

Theorem 3. *The specification of the spectral data* S *uniquely determines the potential* q *on* T *and* h*. For constructing the solution of Inverse Problem 3 we calculate the Weyl vector* M *by (17), and then construct* q *and* h *by solving Inverse Problem 1.*

A noncompact tree



References and generalizations

Similar results have been obtained by Yurko and (partially) Yurko/Freiling for
(i) trees with one edge of infinite length
(ii) second order pencils
(iii) n-th order equations.
See Inverse Problems 21 (2005), 1075-1086;
Applicable Analysis 2007;
Results in Mathematics 2007;
Preprints.

General references on Inverse Problems

Beals R., Deift P. and Tomei C., Direct and Inverse Scattering on the Line, Math. Surveys and Monographs, Vol. 28. Amer. Math. Soc. Providence: RI, 1988.

Freiling G. and Yurko V.A., Inverse Sturm-Liouville Problems and their Applications, NOVA Science Publishers, New York, 2001.

Yurko V.A., Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002, 303pp.

Inverse problems of order $n > 2$ were studied by L.A. Sakhnovich (1958...), I.G. Khachatryan (1976...).