Finding eigenvalues and resonances of the Laplacian on domains with regular ends

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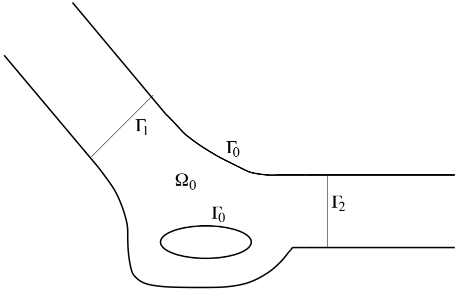
Model geometry



$\Omega \subset \mathbb{R}^d$, $d \geq 2$ — an unbounded connected domain with a finite number of cylindrical ends:

$$\Omega = \operatorname{Int}\left(\overline{\Omega_0 \cup \bigsqcup_{n=1}^{N} \mathcal{C}_n^0}\right).$$

Model geometry (contd.)

$\Omega_0 \subset \mathbb{R}^d$ is a connected bounded open set, and in some local coordinates $(x, \mathbf{y})$ the cylinder $\mathcal{C}_n^0$ is given by

$$\mathcal{C}_n^0 = \{(x, \mathbf{y}) \ : \ \mathbf{y} \in \Gamma_n^0, \ \ x \geq 0\},$$

where Γ_n^0 is the open bounded connected (but not necessarily simply connected) $(d-1)$-dimensional cross-section of $\mathcal{C}_n^0$.

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To finalize our notation: the sets $\mathcal{C}_n := \{(x, \mathbf{y}) \ : \ \mathbf{y} \in \Gamma_n^0, \ \ x \geq 1\}$ and their union $\mathcal{C} := \bigsqcup_{n=1}^N \mathcal{C}_n$ are called *cylindrical ends* of Ω and the sets $\Gamma_n := \{(1, \mathbf{y}) \in \mathcal{C}_n\} = \{(1, \mathbf{y}) \ : \ \mathbf{y} \in \Gamma_n^0\}$ and their union $\Gamma := \bigsqcup_{n=1}^N \Gamma_n$ are called *interfaces*.

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Model spectral problem

Referred later on as “Problem (P)”:

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Conditions on boundary operators B:

(B_1) a and b are piecewise smooth functions on the boundary $\partial\Omega$;

(B_2) $a(\cdot)^2 + b(\cdot)^2 \equiv 1$;

(B_3) on each connected component of the part $(\frac{1}{2}, \infty) \times \partial\Gamma_n$ of the infinite boundary of each cylindrical end $\mathcal{C}_n^0$ the functions a and b are constant (but the constants are allowed to differ between the cylinders and even between the connected components of the boundary of each cylinder).

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Neumann conditions $=$ an *acoustic* waveguide.

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Theorem (Evans, Levitin, Vassiliev (1994))

For an acoustic waveguide, if $\mathcal{O}$ is symmetric with respect to $y = 0$, then there is an embedded eigenvalue.

- procedures for finding embedded eigenvalues for obstacles of simple shapes (rectangles, circles, etc.) using special function expansions;
- asymptotic results in relation to various additional parameters, or asymptotic distribution of the counting function of the discrete and continuous spectrum;

Side track

a bottle of wine challenge a la SHARGORODSKY

find sufficient conditions for absolute continuity of the spectrum. Known in two cases and their variations: a straight strip (Dirichlet or Neumann) and Rellich's semi-strip bounded by a graph of a function (Dirichlet).

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Remark

All our methods apply in a more general case

$$-\operatorname{div}(p(\cdot)\,\mathbf{grad}\,u) + q(\cdot)u = \lambda u$$

$$(a(\cdot)u + b(\cdot)\mathbf{n}\cdot p(\cdot)\,\mathbf{grad}\,u(\cdot))|_{\partial\Omega} = 0\,.$$

with p being a sufficiently smooth positive definite $(d \times d)$-matrix valued function, and q being an $L_\infty(\mathbb{R}^d)$ scalar potential, both constant at infinity

Continuous spectrum

Transversal problem on the joint interface Γ:

$$-\Delta_{d-1} w = \kappa w\,, \qquad \left(a(\cdot) w + b(\cdot) \frac{\partial w}{\partial \nu} \right) \Bigg|_{\partial\Gamma} = 0\,; \tag{3}$$

here Δ_{d-1} is the $(d-1)$-dimensional Laplacian and $\frac{\partial}{\partial\nu}$ is the derivative with respect to the external normal ν to $\partial\Gamma$ (which coincides with the normal n to $\partial\Omega$). We recall that a and b are constant on each connected component of $\partial\Gamma$.

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The assumptions (B_1), (B_2) and (B_3) ensure that the problem (3) has a purely discrete spectrum of eigenvalues $(\kappa_j)_{j=1}^{\infty}$; we denote by $(w_j(\mathbf{y}))_{j=1}^{\infty}$ the corresponding eigenfunctions normalized in $L_2(\Gamma)$. We also set $\kappa_0 := -\infty$.

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It is well known that the essential spectrum of Problem (P) is then given by $[\kappa_1, +\infty)$ and moreover its multiplicity at a point $\lambda \geq \kappa_1$ is equal to $\#\{\kappa_j \leq \lambda\}$. We shall often refer to the number κ_j as the j-th *threshold*.

Reduction to the interface — domain decomposition

Let $u \in H^1(\Omega)$ be an eigenfunction of Problem (P) corresponding to an eigenvalue λ. We define

$$f := u|_\Gamma , \quad g = \left.\frac{\partial u}{\partial x}\right|_\Gamma . \tag{4}$$

Thus f and g satisfy

$$f = -\mathcal{R}_\lambda^{\mathcal{C}} g = \mathcal{R}_\lambda^0 g,$$

where $\mathcal{R}_\lambda^{\mathcal{C}}$ and $\mathcal{R}_\lambda^0$ are the *Neumann-to-Dirichlet maps*, so that a necessary condition for λ to be an eigenvalue of Problem (P) is that $\sigma = -1$ should be an eigenvalue of the operator pencil problem

$$(\mathcal{R}_\lambda^{\mathcal{C}} - \sigma \mathcal{R}_\lambda^0) g = 0 . \tag{5}$$

Neumann-to-Dirichlet operators — general properties

Let $\Omega \subset \mathbb{R}^n$ has sufficiently smooth boundary Γ, and let v be a solution of

$$-\Delta v = \lambda v \quad \text{in} \quad \Omega, \qquad \frac{\partial v}{\partial n} = g \quad \text{in} \quad \Gamma$$

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- Eigenvalues of $\mathcal{R}_\lambda$ cross from -0 to $+0$ at points of $\operatorname{spec}(-\Delta_D(\Omega))$ and from $+\infty$ to $-\infty$ at points of $\operatorname{spec}(-\Delta_N(\Omega))$;
- $\dim\{\text{negative eigenvalues of } \mathcal{R}_\lambda\} = \mathcal{N}(-\Delta_N(\Omega); \lambda) - \mathcal{N}(-\Delta_D(\Omega); \lambda)$.

[Friedlander, Safarov, Agranovich,...]

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Cylindrical ends: $\mathcal{R}_\lambda^{\mathcal{C}}$ is easy and explicit by separation of variables, more below...

Central domain (Ω_0): where the difficulty lives

Denote by μ_m the eigenvalues and by U_m the corresponding eigenfunctions of the homogeneous problem

$$-\Delta U_m = \mu_m U_m \text{ in } \Omega_0, \quad \frac{\partial U_m}{\partial n} = 0 \text{ on } \Gamma, \quad BU_m = 0 \text{ on } \Gamma_0, \tag{6}$$

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$$\begin{aligned} R_{k\ell} &= (\mathcal{R}^0_\lambda \phi_k, \phi_\ell)_\Gamma = \langle \nabla \Phi_k, \nabla \Phi_\ell \rangle - \lambda \langle \Phi_k, \Phi_\ell \rangle \\ &= \sum_{m=1}^{\infty} \langle \nabla U_m, \nabla U_m \rangle \langle \Phi_k, U_m \rangle \langle U_m, \Phi_\ell \rangle = \sum_{m=1}^{\infty} (\mu_m - \lambda) \langle \Phi_k, U_m \rangle \langle U_m, \Phi_\ell \rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{\mu_m - \lambda} (\phi_k, U_m|_\Gamma)_\Gamma \cdot (U_m|_\Gamma, \phi_\ell)_\Gamma \,. \end{aligned}$$

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Theorem

$\lambda \in [\kappa_J, \kappa_{J+1})$ is an eigenvalue of Problem (P) if and only if -1 is a σ-eigenvalue of the matrix pencil

$$\sigma \mathsf{S}_{J+1:\infty,1:\infty} \mathsf{D}(\lambda) \left(\mathsf{S}_{J+1:\infty,1:\infty}\right)^* - \mathsf{T}(\lambda), \tag{7}$$

and, if $J > 0$, the correspoding eigenvector c of (7) satisfies the orthogonality condition

$$\mathsf{S}_{1:J,1:\infty} \mathsf{D}(\lambda) \left(\mathsf{S}_{J+1:\infty,1:\infty}\right)^* \mathsf{c} = 0. \tag{8}$$

Main result (contd.)

Here $\mathsf{S} = (S_{km})_{1\le k,m<\infty}$ is an infinite matrix (independent of λ) with entries defined by

$$S_{km} = (w_k, U_m|_\Gamma)_\Gamma\,, \tag{9}$$

and $\mathsf{D}(\lambda)$ and $\mathsf{T}(\lambda)$ are infinite diagonal matrices with diagonal entries

$$D_{kk}(\lambda) = (\mu_k - \lambda)^{-1}, \quad T_{kk}(\lambda) = -\frac{1}{\sqrt{\kappa_k - \lambda}}, \quad 1 \le k < \infty. \tag{10}$$

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Procedure: compute $\sigma_n(\lambda)$ and solve for $\sigma = -1$. But how many do we need? Two results help.

Estimates of the counting function

Proposition (Monotonicity in λ)

Pencil eigenvalues $\sigma_n(\lambda)$ are monotone increasing functions of λ in intervals not containing thresholds κ_j and Neumann eigenvalues μ_k.

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Proposition ((easy) Estimate of the counting function)

The number of eigenvalues of Problem (P) in any interval $I = [\Lambda_1, \Lambda_2]$ does not exceed

$$K := \mathcal{N}(-\Delta_{N,B}(\Omega_0); \Lambda_2) - \mathcal{N}(-\Delta_{D,B}(\Omega); \Lambda_1) + \mathcal{N}(-\Delta_{d-1}(\Gamma); I)$$

Improving convergence

Compute $R_{k\ell}(\lambda_0)$ at some point numerically (rather than via eigenfunction expansion) and then use

$$R_{k\ell}(\lambda) - R_{k\ell}(\lambda_0) = \sum_{m=1}^{\infty} \frac{\lambda - \lambda_0}{(\mu_m - \lambda)(\mu_m - \lambda_0)} (\phi_k, U_m|_\Gamma)_\Gamma \cdot (U_m|_\Gamma, \phi_\ell)_\Gamma \,. \tag{11}$$

The series in (11) converges much more rapidly than the previous one.

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- finding eigenvalues and eigenfunctions for problems on bounded domains with mixed boundary conditions (e.g. MATLAB PDE Toolbox, FEMLAB);
- performing quadratures on boundaries and cross-sections of domains;
- finding the first few eigenvalues of a matrix pencil;
- finding the zeros of a monotone real-valued function of a real variable.

Resonances

In general, embedded eigenvalues are very unstable — e.g. they may become *resonances* under small perturbations of geometry destroying symmetry (e.g. [Aslanyan–Parnovsky–Vassiliev]).

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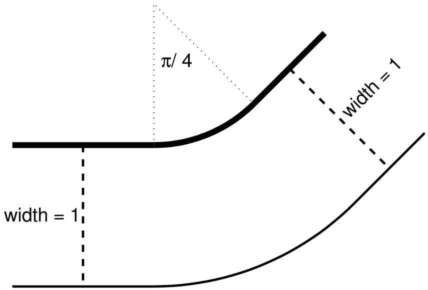
Trivial modifications to the algorithm:

- set $J = 0$ and choose the signs of square roots: $\operatorname{Re}(\sqrt{\kappa_j - \lambda}) > 0$
- forget the orthogonality conditions;
- look for complex λ such that $\sigma(\lambda) = 1$ (in practice, look for the local maxima of the condition number of some matrix pencil).

Example 1: bent waveguide with mixed conditions

(after Exner, Šeba, Duclos, Krejčiřík, Kříž et al.)

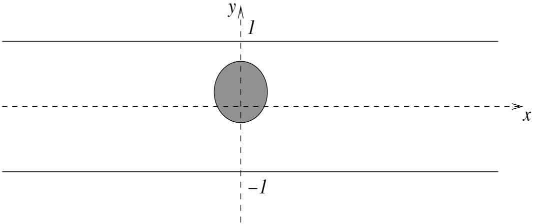


There is always an eigenvalue below the essential spectrum.

Example 2: obstructed waveguide

(after Evans/L./Vassiliev and Aslanyan/Parnovski/Vassiliev)

$$\Omega = \{(x, y) \in \mathbb{R}^2 \; : \; x > 0, \; |y| < 1\} \setminus \overline{\mathcal{O}}.$$



Eigenvalues for symmetric obstacle become resonances as obstacle moves off-centre.

Example 3: Resonances of a scatterer

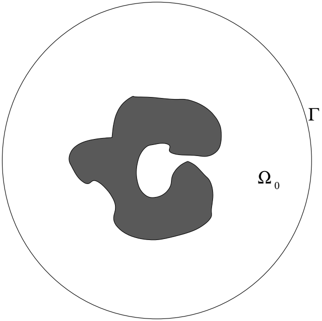


Figure: Scattering by an obstacle. The transverse boundary Γ is replaced by a circle of radius R, the cylindrical ends by the region $x^2 + y^2 > R^2$.

Use Hankels to construct Neumann-to-Dirichlet map from the outside, act as before on the inside

Example 3: Resonances of a cavity resonator

(after Brown/Hislop/Martinez)

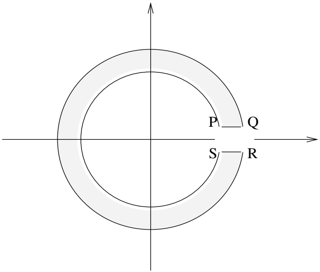
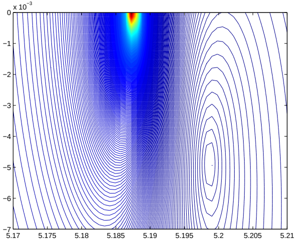


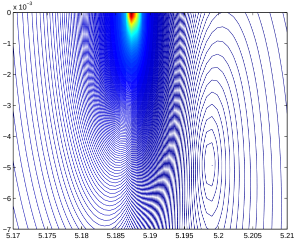
Figure: Cavity Resonator. The points P, Q, R and S have coordinates $(1, \epsilon)$, $(1.1, \epsilon)$, $(1.1, -\epsilon)$ and $(1, -\epsilon)$ respectively, and are joined by two circular arcs

Example 3: Resonances of a cavity resonator (contd.)



Method works well near the real axis if one is careful to distinguish between zeros and poles.

Example 3: Resonances of a cavity resonator (contd.)



Method works well near the real axis if one is careful to distinguish between zeros and poles. Resonance very close to a Neumann eigenvalue for the interior domain; $R = 1.5$, $\epsilon = 0.3$. At lower resolutions these are virtually indistinguishable. This is a problem, as not all Neumann eigenvalues are close to resonances! The contours are the level sets of a determinant which is zero at the resonance $5.199 - 5 \times 10^{-3} i$. Unfortunately the same determinant also has a pole at the nearby Neumann eigenvalue 5.187.

Example 3: Resonances of a cavity resonator (contd.)

Method gives only qualitative picture far from the real axis as resonances are very unstable:

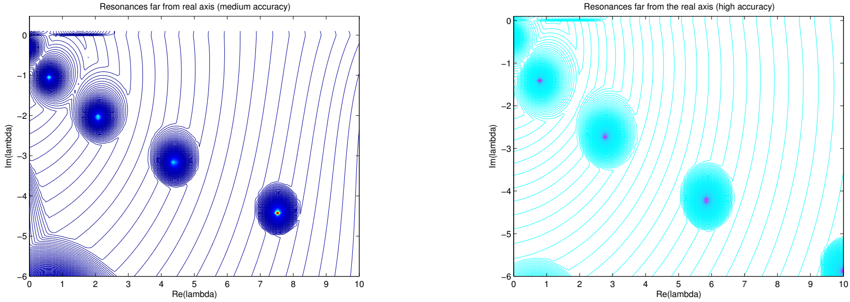


Figure: Resonances far from the real axis for $\epsilon = 0.2$: the medium and high accuracy results are only qualitatively similar

Example 4: Scattering by a Gaussian potential

(after Lin)
We consider the resonances of the Schrödinger operator in $L_2(\mathbb{R}^2)$ given by $-\Delta + q(x,y)$ where q is a superposition of three Gaussians:

$$q(x,y) = C \sum_{j=1}^{3} \exp(-\nu(x-x_j)^2 - \nu(y-y_j)^2),$$

with $(x_1,y_1) = (0,-1)$, $(x_2,y_2) = (\sin(\pi/3), \cos(\pi/3))$, $(x_3,y_3) = (-x_2,y_2)$, $C = 40$ and $\nu = 2$.

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Resonances near the real axis are calculated with moderate accuracy and seem not to produce spurious solutions characteristic for the complex scaling method.

References

- math.SP/0611237
  http://front.math.ucdavis.edu/math.SP/0611237
  Michael Levitin, Marco Marletta: A simple method of calculating eigenvalues and resonances in domains with infinite regular ends

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  Michael Levitin, Marco Marletta: A simple method of calculating eigenvalues and resonances in domains with infinite regular ends
- Scripts (Matlab + PDE Toolbox) at
  http://www.ma.hw.ac.uk/~levitin/LevitinMarletta/ or
  http://www.cf.ac.uk/maths/marlettam/