Spin Interference and the Rashba Hamiltonian

M. Harmer

Czech Academy of Sciences, Řež

Graph Models of Mesoscopic Systems, Newton Institute

Abstract

Scattering for a simple graph with Rashba hamiltonian and the possibility of spin filtering is analysed. We discuss calculation of truth tables for multi-particle spin devices.

Outline

Quantum graph models

- Offers a simple model of quantum waveguides.
- Finite quantum graphs solvable.
- Can be use to make qualitative statements/identify general features of the system.
- Having explicit expression for scattering properties allows optimisation of design, at least to first order.

Nanoelectronics with spin

- Use of spin in nanoelectronic devices: Datta and Das [5] spin field effect transistor.
- Rashba effect and spin filtering: Kiselev and Kim [13, 14], Nitta et al [20], Splettstoesser et al [23], Středa and Šeba [24].
- We consider quantum graphs with spin-$\frac{1}{2}$ particles.
- Study simple(st) family of graphs with spin, where interaction is generated by Rashba effect.
- Application of this family to spin filtering.
- We consider multi-particle spin quantum graphs as models for (classical) gates and discuss the derivation of simplified truth tables.

Family of graphs with Rashba hamiltonian

- We have free hamiltonian on the semi infinite edges, with local coordinates $x_1, x_2, \ldots, x_n \in \mathbb{R}_+$.
- Assume structural inversion asymmetry on the ring so that the Rashba effect is present. Local coordinate on the ring is $\theta \in [0, 2\pi)$.

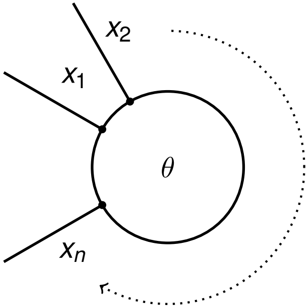

Figure 1: Rashba ring.

Hamiltonian on the ring

- The hamiltonian on the ring is

$$H_0 f = D_0^2 f - \left(\frac{\alpha}{2}\right)^2 f\,.$$

- Here

$$\begin{aligned} D_0 &= -\frac{1}{i}\frac{d}{d\theta} + \frac{\alpha}{2}\sigma_r\,, \\ \sigma_r &= \sigma_x \cos(\theta) + \sigma_y \sin(\theta)\,, \end{aligned}$$

  σ_x, σ_y, σ_z, id denote the Pauli spin matrices and α describes the strength of the Rashba spin-orbit coupling.
- Relative to spin axes the ring is in the x-y plane.

Hamiltonian on the ring

- Solutions of the eigenequation $H_0 f = k^2 f$, are

$$f_{\pm,0}(\theta, k) = e^{-i\sigma_z\theta/2}\, e^{-i\sigma_y\varphi/2}\, e^{\pm i\sigma_z\kappa_\pm(\theta-\pi)} \qquad (1)$$

where $\kappa_\pm = \sqrt{k^2 + \frac{\alpha^2}{4}} \pm \sqrt{\frac{1}{4} + \frac{\alpha^2}{4}}$ and $\tan(\varphi) = \alpha$, $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

- Using Pauli spin matrices we write the solution with both spin states together.

Spectrum on the ring

- Eigenvalues

$$\lambda_{\pm,n} = n^2 - \left(\tfrac{1}{2} \pm n\right)\left(\sqrt{1+\alpha^2} - 1\right) \qquad (2)$$

are zeroes of $\cos(\kappa_{\pm}\pi)$.

- Since $\lambda_{+,n} = \lambda_{-,-n}$ we take as the spectrum

$$\sigma\left(H_0\right) = \{\lambda_{+,n} \, : n \in \{1,2,\ldots\}\} \bigcup \{\lambda_{-,n} \, : n \in \{0,1,2,\ldots\}\} \, .$$

- Each $\lambda_{\pm,n}$ has multiplicity two; the eigenspace is spanned by $\left\{e^{\pm in\theta}\chi_{\uparrow},\, e^{\mp in\theta}\chi_{\downarrow}\right\}$ where

$$\chi_{\uparrow} = \begin{pmatrix} \cos(\varphi/2) \\ e^{i\theta}\sin(\varphi/2) \end{pmatrix}, \quad \chi_{\downarrow} = \begin{pmatrix} -e^{-i\theta}\sin(\varphi/2) \\ \cos(\varphi/2) \end{pmatrix}.$$

Spectrum on the ring

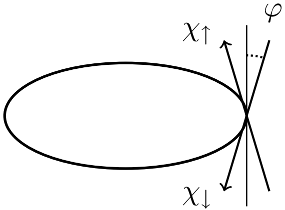

Figure 2: Spin eigen-states on the ring.

- The twofold degeneracy of the eigenvalues rises to a fourfold degeneracy when

$$\sqrt{1+\alpha^2}-1 = m \in \{0,1,\ldots\}\,.$$

In this case

$$\lambda_{-,n} = \lambda_{+,n+m}\,.$$

Greens function on the ring

The Greens function on the ring is the continuous solution of

$$
\begin{aligned}
H_0 \, G\left(\theta, \eta; k^2\right) &= k^2 \, G(\theta, \eta; k^2) \\
\frac{\partial}{\partial \theta} G(\theta, \eta; k^2)\Big|_{\theta=\eta^-}^{\theta=\eta^+} &= \mathrm{id} \\
G(\theta, \eta; k^2) &= G^\star(\eta, \theta; k^2)\,,
\end{aligned}
$$

for $k \in \mathbb{R} \setminus \sigma\left(H_0\right)$.

Greens function on the ring

Pauli matrices allow us to write both spin states together in one solution, equation (1), giving a 'compact' expression

$$G\left(\theta,\eta;k^2\right) =$$
$$= \left[f_{+,0}(\theta)\frac{e^{-i\sigma_z\kappa_+\eta}}{\cos(\kappa_+\pi)} - f_{-,0}(\theta)\frac{e^{i\sigma_z\kappa_-\eta}}{\cos(\kappa_-\pi)}\right]\frac{e^{-i\sigma_y\varphi/2}}{2i(\kappa_+ + \kappa_-)}e^{i\sigma_z\eta/2}\,\sigma_z$$
$$= \frac{e^{-i\sigma_z\theta/2}\,e^{-i\sigma_y\varphi/2}}{2i(\kappa_+ + \kappa_-)}\left[\frac{e^{i\sigma_z\kappa_+(\theta-\eta-\pi)}}{\cos(\kappa_+\pi)} - \frac{e^{-i\sigma_z\kappa_-(\theta-\eta-\pi)}}{\cos(\kappa_-\pi)}\right]\times$$
$$e^{-i\sigma_y\varphi/2}e^{i\sigma_z\eta/2}\,\sigma_z\,; \qquad \theta-\eta\in[0,2\pi)\,. \tag{3}$$

Hamiltonian on the wires

- On each wire we have the 'free' hamiltonian

$$H_j f_j = D_j^2 f_j \, , \qquad D_j = -\frac{1}{i}\frac{d}{dx_j} \, .$$

- We write both spin states of the generalised eigenfunctions as

$$f_{\pm,j} = e^{\pm i \, \mathrm{id} k x_j} \, ,$$

in the z basis.
- Interpretation of spin measurements along x or y axes in the wires.

Boundary conditions

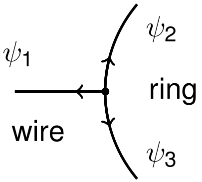

Figure 3: Boundary conditions at a junction.

- Assume no two wires are attached at the same point on the ring.
- Prescribe boundary conditions at each vertex

$$\beta^{-1}\psi_1 = \psi_2 = \psi_3 \,, \quad (4)$$
$$\beta\psi_1' + \psi_2' + \psi_3' = 0 \,.$$

Boundary conditions

- Continuity on the ring implies $H = H_0 \oplus \sum_{j=1}^{n} H_j$ with these boundary conditions is self-adjoint [17].
- β describes the strength of the coupling between the wire and the ring.
- Boundary conditions are assumed to hold with the same β for each component of the spinor, ie. the coupling is spin independent. However, β may be different at each vertex.
- Derivation with arbitrary β. For explicit calculation of polarisation $\beta = 1$.
- Scattering matrix of ‘free’ junction the same as scattering matrix of T-junction in the physics literature [1].

Scattered waves

- Scattered waves ψ_i are eigenfunctions $H\psi_i = k^2\psi_i$ satisfying the boundary conditions at the vertices and having the following form on the wires:

$$\psi_i = \left(f_{+,i} + f_{-,i}S_{ii}\right) \oplus_{j\neq i} f_{-,j}S_{ji} \,. \tag{5}$$

- Components of ψ_i and the scattering matrix S_{ji} , $i,j \in \{1,\ldots,n\}$, are 2×2 matrix valued.

Scattered waves

- Suppose $\{\theta_i\}_{i=1}^{n}$ are the points where the wires are attached to the ring we define $G_k = G(\theta, \theta_k)$.
- Due to form of boundary conditions we can write the scattered wave on the ring

$$\psi_i = \sum_k G_k A_{ki} \tag{6}$$

for some matrix A.

Scattered waves

- We see from (5) that on the wires

$$\psi_i\big|_j = \delta_{ji} + S_{ji}\,, \quad \psi_i'\big|_j = ik\left(\delta_{ji} - S_{ji}\right)\,.$$

- Likewise, defining

$$\mathcal{G}_{jk} = G\left(\theta_j, \theta_k; k^2\right)$$

we have from (6)

$$\psi_i\big|_j = \sum_k \mathcal{G}_{jk} A_{ki}\,, \quad \psi_i'\Big|_{\theta=\theta_j^-}^{\theta=\theta_j^+} = A_{ji}$$

on the ring.

Scattering matrix

Eliminating A we can write the scattering matrix

$$S = (ik\beta\mathcal{G}\beta + \mathbb{I})\,(ik\beta\mathcal{G}\beta - \mathbb{I})^{-1} \tag{7}$$

in terms of the Greens function on the ring $\mathcal{G}$ and a diagonal matrix containing the coupling strengths for the n vertices β.

The wronskian

- The wronskian

$$W(f,g) = \sum_i (\langle D_i f, g\rangle + \langle f, D_i g\rangle)|_{x_i}$$

  is well defined on eigensolutions, $Hf = k^2 f$, $Hg = k^2 g$. Here $\langle\cdot,\cdot\rangle$ is the inner product on spinors.
- If f, g satisfy the boundary conditions (4) then

$$W(f,g) = 0$$

  the wronskian is zero.

Symmetries

- Using this and symmetries of the hamiltonian we can find symmetries of the scattering matrix.
- A trivial example, the wronskian of two scattered waves gives

$$\begin{aligned} 0 &= W\left(\psi_i, \psi_j\right) = \sum_k \left(\langle D\psi_i, \psi_j\rangle + \langle \psi_i, D\psi_j\rangle\right)\Big|_{x_k=0} \\ &= \sum_k -k\left(\left(\delta_{ik} - S^\dagger_{ik}\right)\left(\delta_{kj} + S_{kj}\right) + \left(\delta_{ik} + S^\dagger_{ik}\right)\left(\delta_{kj} - S_{kj}\right)\right) , \end{aligned}$$

ie. $S^\dagger S = \mathbb{I}$, the unitarity of the scattering matrix.

- Here $W\left(\psi_i, \psi_j\right)$ is 2×2 matrix valued.

Symmetries

- Suppose the graph, and boundary conditions, are invariant with respect to a reflection in one of the axes on the plane

$$R : y \leftrightarrow -y \, .$$

- Then the hamiltonian will commute with $\mathcal{R} = \sigma_y R$ and

$$\begin{aligned} 0 &= W\left(\mathcal{R}\psi_i, \psi_j\right) = \sum_k \left(\langle D\mathcal{R}\psi_i, \psi_j\rangle + \langle \mathcal{R}\psi_i, D\psi_j\rangle\right)\Big|_{x_k=0} \\ &= \sum_k k\left(\left(\delta_{ik}\sigma_y - R(S^\dagger_{ik})\sigma_y\right)\left(\delta_{kj} + S_{kj}\right) - \right. \\ &\left. \left(\delta_{ik}\sigma_y + R(S^\dagger_{ik})\sigma_y\right)\left(\delta_{kj} - S_{kj}\right)\right) \, . \end{aligned}$$

Symmetries

- Consequently

$$\begin{aligned} \hat{\sigma}_y R(S^\dagger)\hat{\sigma}_y S &= \mathbb{I} \\ \Rightarrow \quad S &= \hat{\sigma}_y R\left(S\right)\hat{\sigma}_y \end{aligned} \quad , \tag{8}$$

  where $\hat{\sigma}_y$ is block diagonal with σ_y on the diagonal.
- Further symmetries of S can be found from $\mathcal{K} = \sigma_y K$, where K is complex conjugation and $\mathcal{L} = \sigma_z L$, where L reverses the sign of α.

Three terminal device

- Consider a three terminal device with $y \leftrightarrow -y$ symmetry. The angle between the first and second and first and third wires is $\xi \in (0, \pi)$.
- Also need the coupling constants at vertices two and three the same—for simplicity we set $\beta_i = \beta$.

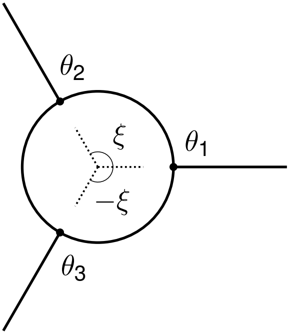

Figure 4: The three terminal Rashba ring.

Flux and polarisation

- Decompose the components of the scattering matrix in terms of spin matrices

$$s_{ij} = s_{ij,1} + i \sum_{\alpha} \sigma_{\alpha} s_{ij,\alpha} \, .$$

- Then the flux T_{ij} and polarisation in the α-axis $P_{ij,\alpha}$ for waves going from wire j to wire i are

$$T_{ij} = 2\left(\left| s_{ij,1} \right|^2 + \left| s_{ij,x} \right|^2 + \left| s_{ij,y} \right|^2 + \left| s_{ij,z} \right|^2 \right) \quad (9)$$

$$P_{ij,\alpha} = 4\Im \left(s_{ij,1} \bar{s}_{ij,\alpha} + s_{ij,\alpha-1} \bar{s}_{ij,\alpha+1} \right) \, . \quad (10)$$

Flux and polarisation

- Now $S = \hat{\sigma}_y R(S)\, \hat{\sigma}_y$ (8) implies

$$s_{ij,1} = s_{i'j',1}\,, \quad s_{ij,y} = s_{i'j',y}$$
$$s_{ij,x} = -s_{i'j',x}\,, \quad s_{ij,z} = -s_{i'j',z}$$

  where $R(s_{ij}) = s_{i'j'}$.
- In particular, the polarisation satisfies

$$P_{21,x} = 4\Im\left(s_{21,1}\bar{s}_{21,x} + s_{21,z}\bar{s}_{21,y}\right) = -P_{31,x}$$
$$P_{21,y} = 4\Im\left(s_{21,1}\bar{s}_{21,y} + s_{21,x}\bar{s}_{21,z}\right) = P_{31,y}$$
$$P_{21,z} = 4\Im\left(s_{21,1}\bar{s}_{21,z} + s_{21,y}\bar{s}_{21,x}\right) = -P_{31,z} \quad .$$

- For waves incident on wire 1, expectation of spin on z-axis on wires 2 and 3 opposite.

Flux and polarisation

- There are computational difficulties in treating a 6×6 matrix as a 3×3 matrix with Pauli spin matrix valued coefficients.
- In this case, from (3, 7), we have

$$U\,S\,U^{\star} = \begin{pmatrix} \bar{b} & \bar{z}_1 & z_1 \\ z_1 & \bar{b} & z_2 \\ \bar{z}_1 & \bar{z}_2 & \bar{b} \end{pmatrix} \begin{pmatrix} b & \bar{z}_1 & z_1 \\ z_1 & b & z_2 \\ \bar{z}_1 & \bar{z}_2 & b \end{pmatrix}^{-1}$$

- Here b is a complex scalar and

$$z_l = i\sigma_z\kappa \left(\frac{e^{i\sigma_z\kappa_+(l\xi-\pi)}}{\cos(\kappa_+\pi)} - \frac{e^{-i\sigma_z\kappa_-(l\xi-\pi)}}{\cos(\kappa_-\pi)} \right),$$

has a 'nice form' in the algebra generated by id, σ_z (not so with U).

Flux and polarisation

- This means that the matrices on the right hand side can be treated 'almost as scalar' to give expressions for the flux

$$T_{21} = \frac{8}{|D|^2}\left(\left(|b|^2 + |z_2|^2\right)|z_1|^2 - \tfrac{1}{2}\left(b + \bar{b}\right)\left(\bar{z}_1^2 z_2 + z_1^2 \bar{z}_2\right)\right)$$

  and polarisation

$$P_{21,z} = \frac{8\cos(\varphi)}{|D|^2} j\left(\bar{z}_1^2 z_2 - z_1^2 \bar{z}_2\right) .$$

- Here the determinant D is a complex scalar and the contribution from conjugation by U just gives a multiplicative factor to the polarisation.

Flux and polarisation

- Using Maple these expressions give

$$T_{21}(k,\xi,\alpha) = \frac{8R}{X^2+Y^2}, \quad P_{21,z}(k,\xi,\alpha) = \frac{8\cos(\varphi)\,Q}{X^2+Y^2}.$$

- Here we need to manually cancel common factors $\cos^{-2}(\kappa_+\pi)\cos^{-2}(\kappa_-\pi)$ which appear in the numerator and denominator to avoid numerical instability...

Flux and polarisation

and

$$\begin{aligned}
X &= -\left(4\kappa^3+3\kappa\right)\sin\left(\kappa_+ + \kappa_-\right)\pi + 4\kappa^3\sin\left(\kappa_+ + \kappa_-\right)\left(2\xi-\pi\right) \\
&\quad - 8\kappa^3\sin\left(\kappa_+ + \kappa_-\right)\left(\xi-\pi\right) \\
Y &= -\left(6\kappa^2+\frac{1}{2}\right)\cos\left(\kappa_+ + \kappa_-\right)\pi - \frac{1}{2}\cos\left(\kappa_+ - \kappa_-\right)\pi \\
&\quad + 2\kappa^2\cos\left(\kappa_+ + \kappa_-\right)\left(2\xi-\pi\right) + 4\kappa^2\cos\left(\kappa_+ + \kappa_-\right)\left(\xi-\pi\right) \\
Q &= \kappa^3\left[\cos\left(2\kappa_-\pi\right) - \cos\left(2\kappa_+\pi\right)\right. \\
&\quad + \cos\left(\kappa_+ 2\xi + \kappa_- 2\left(\xi-\pi\right)\right) - \left.\cos\left(\kappa_+ 2\left(\xi-\pi\right) + \kappa_- 2\xi\right)\right] \\
&\quad + 2\kappa^3\left[\cos\left(\kappa_+\left(\xi-2\pi\right) + \kappa_-\xi\right) - \cos\left(\kappa_+\xi + \kappa_-\left(\xi-2\pi\right)\right)\right] ,
\end{aligned}$$

Flux and polarisation

with

$$\begin{aligned} R &= \kappa^2 + 4\kappa^4 + \tfrac{1}{2}\kappa^2 \left[\cos\left(2\kappa_+\pi\right) + \cos\left(2\kappa_-\pi\right) - \cos\left(\kappa_+ + \kappa_-\right)\xi \right. \\ &\quad - \cos\left(\kappa_+ + \kappa_-\right)\left(\xi - 2\pi\right) - \cos\left(\kappa_+\left(\xi - 2\pi\right) + \kappa_-\xi\right) \\ &\quad \left. - \cos\left(\kappa_+\xi + \kappa_-\left(\xi - 2\pi\right)\right)\right] \\ &\quad + 2\kappa^4\left[\cos\left(\kappa_+ + \kappa_-\right)\left(\xi - 2\pi\right) + \cos\left(\kappa_+ + \kappa_-\right)\left(3\xi - 2\pi\right)\right] \\ &\quad + 4\kappa^4\left[\cos\left(\kappa_+ + \kappa_-\right)\xi + \cos\left(\kappa_+ + \kappa_-\right)2\left(\xi - \pi\right)\right] . \end{aligned}$$

In these we have put $\beta = 1$.

Vanishing of polarisation

- We observe that for

$$\kappa_+ = \kappa_- + m + 1$$

  the expression $Q \equiv 0$ and the polarisation is identically zero.
- This happens for values of α such that $\sqrt{1+\alpha^2} - 1 = m \in \{0, 1, \dots\}$, ie. when the eigenvalues on the ring become fourfold degenerate.
- Physical explanation in terms of ‘integral number of effective flux quanta through the ring’.

Vanishing of polarisation

- Alternatively, we apply a gauge transformation on the ring $VH_0V^\star$,

$$V(\theta) = e^{-i\sigma_z(m+1)\theta/2}\, e^{i\sigma_y\varphi/2}\, e^{i\sigma_z\theta/2}\,.$$

- The hamiltonian on the ring is then $-\frac{d^2}{d\theta^2} - \left(\frac{\alpha}{2}\right)^2$ and all spin interaction is concentrated at the boundary conditions.
- However, for $\sqrt{1+\alpha^2} - 1 = m \in \{0, 1, \dots\}$ the boundary conditions remain spin independent so that we have vanishing polarisation.
- A similar argument shows that the *flux* is almost periodic in α.

Plots of flux and polarisation

- In figures 5–9 we plot the conductance T_{21} (upper curve) and $P_{21,z}$ (lower curve) against the energy $\lambda = k^2$.
- We put $\alpha = 0.8$—only significant effect observed with respect to α is vanishing at points of degeneracy.
- Eigenvalues on the ring are indicated by

$$\Box = \lambda_{+,n}\,, \quad \Diamond = \lambda_{-,n}$$

Leftmost $\Box$ and $\Diamond$ are at $\lambda_{+,1}$ and $\lambda_{-,1}$ respectively with $\lambda_{-,0} \leq 0$ off the left of the plot.

Plots of flux and polarisation

- For $\xi = \pi/2$, figure 5, the polarisation is maximum at odd resonant eigenvalues and damped at even eigenvalues.
- A similar device is analysed, and an explanation in terms of interference effects is given, in [14].
- For $\xi = p\pi/q$ flux and polarisation are periodic, repeating after $2q$ resonances $\lambda_{\pm,n}$, with zeroes at $\lambda_{\pm,lq}$.
- Perhaps this behaviour can also be explained by spin interference and topological phase effects [18].
- $\xi = 2\pi/3$ best candidate for spin filter.

Flux and polarisation for $\xi = \pi/2$

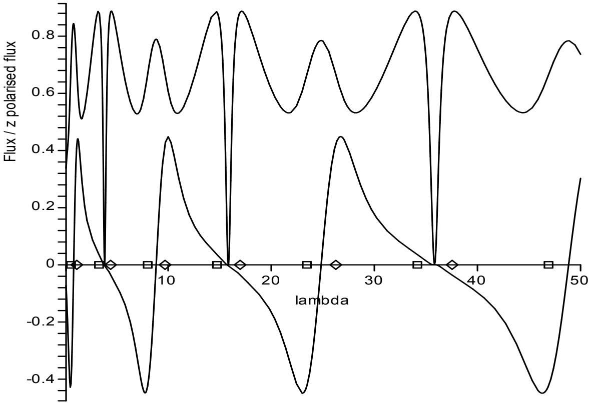

Figure 5: T_{21} and $P_{21,z}$ for $\alpha = 0.8$ and $\xi = \frac{\pi}{2}$.

Flux and polarisation for $\xi = \pi/3$

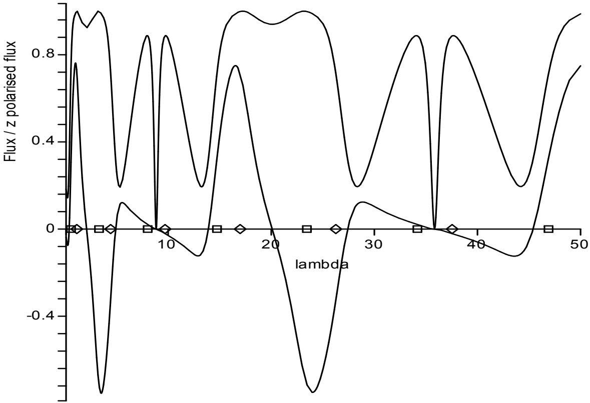

Figure 6: T_{21} and $P_{21,z}$ for $\alpha = 0.8$ and $\xi = \frac{\pi}{3}$.

Flux and polarisation for $\xi = 2\pi/3$

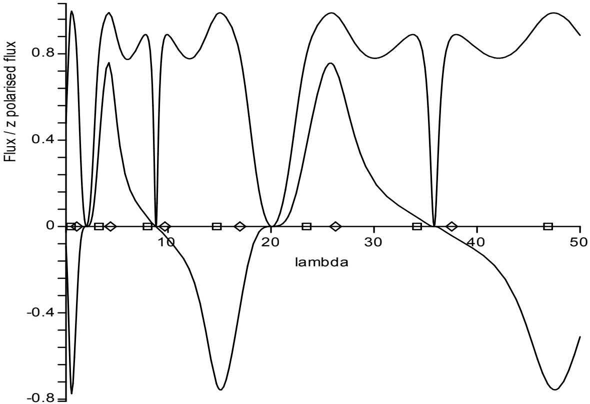

Figure 7: T_{21} and $P_{21,z}$ for $\alpha = 0.8$ and $\xi = \frac{2\pi}{3}$.

Flux and polarisation for $\xi = \pi/4$

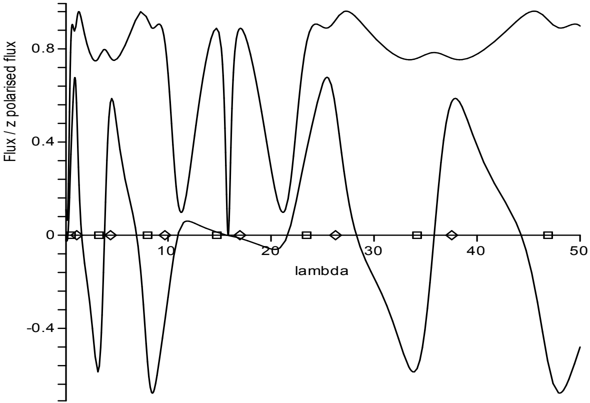

Figure 8: T_{21} and $P_{21,z}$ for $\alpha = 0.8$ and $\xi = \frac{\pi}{4}$.

Flux and polarisation for $\xi = 3\pi/4$

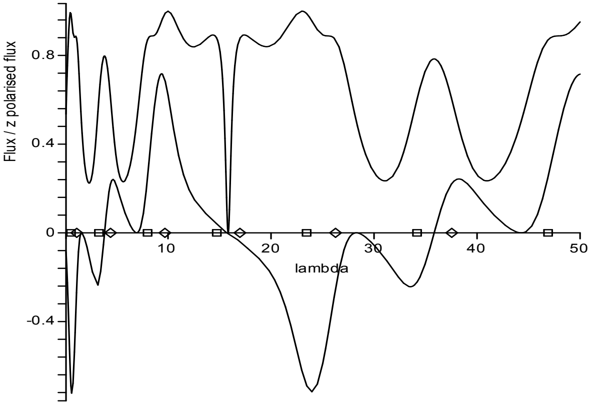

Figure 9: T_{21} and $P_{21,z}$ for $\alpha = 0.8$ and $\xi = \frac{3\pi}{4}$.

Simple model of spin gate

- We consider a (classical) n terminal device with p input terminals.
- Input is the spin state (on the z-axis) of incoming electrons on p terminals.
- Output is the spin expectation measured on the $n - p$ output terminals.

Simple model of spin gate

- In the spirit of quantum graphs we propose the simplest possible, first order, model of such a device.
- The multi-particle interaction is the ‘exchange interaction’, ie. multi-particle wavefunctions on the gate given by alternating products of one particle wave functions. We need some multi-particle interaction; the sum of single particle scattering cannot produce an interesting gate.
- Input states modeled as scattered waves incident on the corresponding terminal/wire. All incoming waves assumed to have the same energy, temperature zero.
- Output calculated as sum of square norms of outward radiating coefficients of multi-particle scattered wave.

Multi-particle wave functions

- For simplicity ignore spin. We consider the matrix of single particle wave functions

$$\psi_{ji}(x) = \psi_i\big|_j = e^{ikx_j} + e^{-ikx_j} S_{ji}\,.$$

- This can be thought of as a linear transformation acting on $E = \mathbb{C}^n$

$$\psi(x) \cdot e_j = \sum_{i=1}^{n} \psi_{ji} e_i = \sum_{i=1}^{n} \psi_i(x_j) e_i\,.$$

Multi-particle wave functions

- We fix a multi-index $\underline{i}_p = \{i_1 < i_2 < \cdots < i_p\}$. This corresponds to a choice of the p input terminals/wires.
- The corresponding projection

$$P = \sum_{l=1}^{p} |e_{i_l}\rangle\langle e_{i_l}|$$

gives us a sum of incoming waves on the input terminals

$$P\,\psi(x)\cdot e_j = \sum_{l=1}^{p} \psi_{i_l}(x_j) e_{i_l}\,,$$

ie. those from which an input is constructed.

Multi-particle wave functions

- The multi-particle state corresponding to an input on the p input terminals is then

$$\begin{aligned} & \frac{1}{\nu_p} P\,\psi(x^1)\cdot e_{j_1} \wedge P\,\psi(x^2)\cdot e_{j_2} \wedge \cdots \wedge P\,\psi(x^p)\cdot e_{j_p} \\ = \; & \frac{1}{\nu_p} \sum_{l_1,l_2,\ldots}^{p} \psi_{i_{l_1}}(x^1_{j_1}) e_{i_{l_1}} \wedge \psi_{i_{l_2}}(x^2_{j_2}) e_{i_{l_2}} \wedge \cdots \wedge \psi_{i_{l_p}}(x^p_{j_p}) e_{i_{l_p}} \\ = \; & \Psi\left(x^1_{j_1}, \ldots, x^p_{j_p}\right)\, e_{\underline{i}_p}\,. \end{aligned}$$

- This allows us to ‘easily’ express the desired coefficients as determinants of minors of the scattering matrix.

Spin inputs

- The multi-particle *spin* states are indexed by the spin state of the incoming waves

$$\underline{\sigma}_p = \{\sigma_1, \ldots, \sigma_p\, ;\, \sigma_i \in \{\uparrow, \downarrow\}\}$$

- Now $E = \mathbb{C}^{2n}$ is an n-dimensional space of spinors and the projection

$$P_{\underline{\sigma}_p} = \sum_{l=1}^{p} |e_{\sigma_l, i_l}\rangle\langle e_{\sigma_l, i_l}|$$

specifies the choice of p input terminals/wires as well as the spin states on these terminals.

Spin inputs

- The multi-particle state corresponding to the input $\underline{\sigma}_p$ is

$$\Psi_{\underline{\sigma}_p}\left(\tau_1, x^1_{j_1}; \ldots; \tau_p, x^p_{j_p}\right) \; e_{\underline{\sigma}_p, \underline{i}_p}$$
$$= \frac{1}{\nu_p} P_{\underline{\sigma}_p} \psi(x^1) \cdot e_{\tau_1, j_1} \wedge \cdots \wedge P_{\underline{\sigma}_p} \psi(x^p) \cdot e_{\tau_p, j_p} \, .$$

- Here the indices τ_1, j_1 denote the spinor component and the terminal/wire on which the i-th particle is being measured.

Truth tables

Expanding the multi-particle scattered waves (and summing over all but one particle) we get as coefficients of outgoing terms

$$\frac{p}{\nu_p}\begin{pmatrix} p-1 \\ q-1 \end{pmatrix}\sum_{\tau_2, j_2 \dots} P_{\underline{\sigma}_p} S \cdot e_{\tau_1, j_1} \wedge \cdots$$
$$\wedge P_{\underline{\sigma}_p} S \cdot e_{\tau_q, j_q} \wedge P_{\underline{\sigma}_p} \cdot e_{\tau_{q+1}, j_{q+1}} \wedge \cdots \wedge P_{\underline{\sigma}_p} \cdot e_{\tau_p, j_p}$$

at order S^q, $q \leq p$.

Truth tables

Define simplified indices

$$\underline{\alpha}_p = \{(\sigma_1, i_1), (\sigma_p, i_p) \dots\}$$

and denote subsets as $\underline{\alpha}_q \leq \underline{\alpha}_p$. Then the polarisation measured on terminal j_1 is the sum over orders S^q where

$$\frac{p!}{\nu_p^2 (q-1)!} \sum_{\underline{\alpha}_q \leq \underline{\alpha}_p} \left[\sum_{\underline{\beta}_q, \beta_1 = (\uparrow, j_1)} \left| \det S_{\underline{\beta}_q \times \underline{\alpha}_q} \right|^2 - \sum_{\underline{\beta}_q, \beta_1 = (\downarrow, j_1)} \left| \det S_{\underline{\beta}_q \times \underline{\alpha}_q} \right|^2 \right] .$$

Summary

- Scattering on quantum graph with spin-$\frac{1}{2}$ particles. Treating both spin states simultaneously.
- Demonstration of spin filtering using Rashba hamiltonian on such graphs.
- Future research: scattering on spin-$\frac{1}{2}$ multi-particle graphs. Simplified derivation of truth tables.

References I

S. Datta.
Electronic Transport in Mesoscopic systems.
Cambridge University Press, Cambridge, 1995.

S. Albeverio and P. Kurasov.
Singular Perturbations of Differential Operators.
London Mathematical Society Lecture Note Series 271.
Cambridge University Press, Cambridge, 2000.

N. Bagraev, A. Mikhailova, B. S. Pavlov, L. V. Prokhorov, and A. Yafyasov.
Parameter regime of a resonance quantum switch.
Phys. Rev. B, 71:165308, 2005.

References II

T. Cheon and P. Exner.
An approximation to δ' couplings on graphs.
J. Phys. A, 37:L329–L335, 2004.

S. Datta and B. Das.
Electronic analog of the electro-optic modulator.
Appl. Phys. Lett., 56:665, 1990.

P. Exner, H. Neidhardt, and V. A. Zagrebnov.
Potential approximations to δ': an inverse Klauder phenomenon with norm-resolvent convergence.
Commun. Math. Phys., 224:593–612, 2001.

References III

P. Exner and K. Němcová.
Leaky quantum graphs: approximation by point interaction hamiltonians.
J. Phys. A, 36:10173–10193, 2003.

M. Governale, D. Boese, U. Zülicke, and C. Schroll.
Filtering spin with tunnel-coupled electron wave guides.
Phys. Rev. B, 65:140403, 2002.

M. Harmer.
Hermitian symplectic geometry and extension theory.
Journal of Physics A: Mathematical and General, 33:9193–9203, 2000.

References IV

M. Harmer.
Hermitian symplectic geometry and the factorisation of the scattering matrix on graphs.
Journal of Physics A: Mathematical and General, 33:9015–9032, 2000.

M. Harmer, B. Pavlov, and A. Yafyasov.
Boundary condition at the junction.
Journal of Computational Electronics (IWCE 11).

Pavel Hejčík and Taksu Cheon.
Irregular dynamics in a solvable one-dimensional quantum graph.
Phys. Lett. A, 356:290–293, 2006.

References V

A. A. Kiselev and K. W. Kim.
T-shaped ballistic spin filter.
Appl. Phys. Lett., 78:775, 2001.

A. A. Kiselev and K. W. Kim.
T-shaped spin filter with a ring resonator.
J. Appl. Phys., 94:4001–4005, 2003.

A. A. Kiselev and K. W. Kim.
Prohibition of equilibrium spin currents in multiterminal ballistic devices.
Phys. Rev. B, 71:153315, 2005.

References VI

V. Kostrykin and R. Schrader.
The generalized star product and the factorization of scattering matrices on graphs.
J. Math. Phys, 42:1563–1598, 2001.

V. Kostrykin and R. Schrader.
Quantum wires with magnetic fluxes.
Commun. Math. Phys., 237:161–179, 2003.

Yuli Lyanda-Geller.
Topological transitions in Berry phase interference effects.
Phys. Rev. Lett., 71:657, 1993.

References VII

F. Meijer, A. Morpurgo, and T. Klapwijk.
One-dimensional ring in the presence of Rashba spin-orbit interaction: derivation of the correct Hamiltonian.
Phys. Rev. B, 66:033107, 2002.

J. Nitta, F. E. Meijer, and H. Takayanagi.
Spin-interference device.
Appl. Phys. Lett., 75:695–697, 1999.

B. S. Pavlov.
The theory of extensions and explicitly solvable models.
Uspekhi Math. Nauk-Russian. Math. Surveys, 42(6):127–168, 1987.

References VIII

I. A. Shelykh, N. G. Galkin, and N. T. Bagraev.
Rashba spin-controlled quantum splitter.
preprint.

J. Splettstoesser, M. Governale, and U. Zülicke.
Persistent current in ballistic mesoscopic rings with Rashba spin-orbit coupling.
Phys. Rev. B, 68:165341, 2003.

P. Středa and P. Šeba.
Antisymmetric spin filtering in one-dimensional electron systems with uniform spin-orbit coupling.
Phys. Rev. Lett., 90:256601, 2003.

References IX

T. Taniguchi and M. Büttiker.
Friedel phases and phases of transmission amplitudes in quantum scattering systems.
Phys. Rev. B, 60:13814, 1999.

R. Winkler.
Rashba spin splitting in two-dimensional electron and hole systems.
Phys. Rev. B, 62:4245, 2000.