Localized shelf waves on a curved coast – existence of eigenvalues of a linear operator pencil in a curved waveguide

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with

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- Eigenvalues

1 Physical background

Measurements of velocity fields along the coasts of oceans throughout the world show that much of the fluid energy is contained in motions with periods of a few days or longer. The comparison of measurements at different places along the same coast show that in general these low-frequency disturbances propagate along coasts with shallow water to the right in the northern hemisphere and to the left in the southern hemisphere. These waves have come to be known as continental shelf waves (CSWs). Can there be non-propagating, trapped CSWs?

1 Physical background

Measurements of velocity fields along the coasts of oceans throughout the world show that much of the fluid energy is contained in motions with periods of a few days or longer. The comparison of measurements at different places along the same coast show that in general these low-frequency disturbances propagate along coasts with shallow water to the right in the northern hemisphere and to the left in the southern hemisphere. These waves have come to be known as continental shelf waves (CSWs). Can there be non-propagating, trapped CSWs?

Assumptions for deducing equations for CSWs:

- inviscid flow
- constant density
- shallow flows (ratio $\frac{\text{depth}}{\text{typical horizontal scale}}$ is small)

Rotating incompressible shallow water equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \mathbf{grad}\ \mathbf{u} - 2\Omega \mathbf{k} \times \mathbf{u} = -g\ \mathbf{grad}\ \widetilde{H}, \quad (1)$$

$$\frac{\partial \widetilde{H}}{\partial t} + \mathrm{div}[(\widetilde{H} + H)\mathbf{u}] = 0. \quad (2)$$

Here div and $\mathbf{grad}$ are taken with respect to horizontal coordinates (x, y) in a frame fixed to the rotating Earth, $\mathbf{k}$ is a vertical unit vector, $\mathbf{u}(x, y, t)$ is the horizontal velocity (with components $\mathbf{u} = (u, v)$), Ω is the (locally constant) vertical component of the Earth's rotation, g is gravitational acceleration, $\widetilde{H}(x, y, t)$ is the vertical displacement of the free surface, and $H(x, y)$ is the local undisturbed fluid depth.

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There are two types of waves — denoted Class 1 and 2 by `Lamb`. Class 2 waves are slower, low-frequency waves that vanish in the absence of depth change or in the absence of rotation. It is the Class 2 waves that give CSWs.

Linearizing, neglecting vertical displacement terms, and introducing the volume flux streamfunction $\psi(x, y)$ via

$$Hu = -\frac{\partial \psi}{\partial y}, \qquad Hv = \frac{\partial \psi}{\partial x},$$

allows (1), (2) to be written as the single equation

$$\mathrm{div}\left(\frac{1}{H}\,\mathbf{grad}\,\frac{\partial \psi}{\partial t}\right) + 2\Omega\mathbf{k}\cdot\mathbf{grad}\,\psi \times \mathbf{grad}\,(1/H) = 0. \tag{3}$$

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Equation (3) is generally described as the topographic Rossby-wave equation or the equation for barotropic CSWs. Huge amount on physical literature on it — but mostly for straight coastline (`LeBlond, Mysak, Stocker, Hutter, Johnson etc.`)

Further simplifications are by considering flow of the form

$$\psi(x,y,t) = \mathrm{Re}\{\Phi(x,y)\exp(-2\mathrm{i}\omega\Omega \mathrm{t})\}, \tag{4}$$

so $\Phi(x,y)$ gives the spatial structure of the flow and ω its non-dimensional frequency (the spectral parameter). Then Φ satisfies

$$\boxed{\frac{1}{H}\Delta\Phi + \mathbf{grad}\left(\frac{1}{H}\right)\cdot\mathbf{grad}\,\Phi + \frac{i}{\omega}\mathbf{k}\cdot\left(\mathbf{grad}\,\Phi\times\mathbf{grad}\left(\frac{1}{H}\right)\right) = 0}, \tag{5}$$

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Note an artificial Neumann condition imposed at the shelf-ocean boundary — well motivated in oceanography!

2 Mathematical statement and background

Assume the shelf is of constant width.

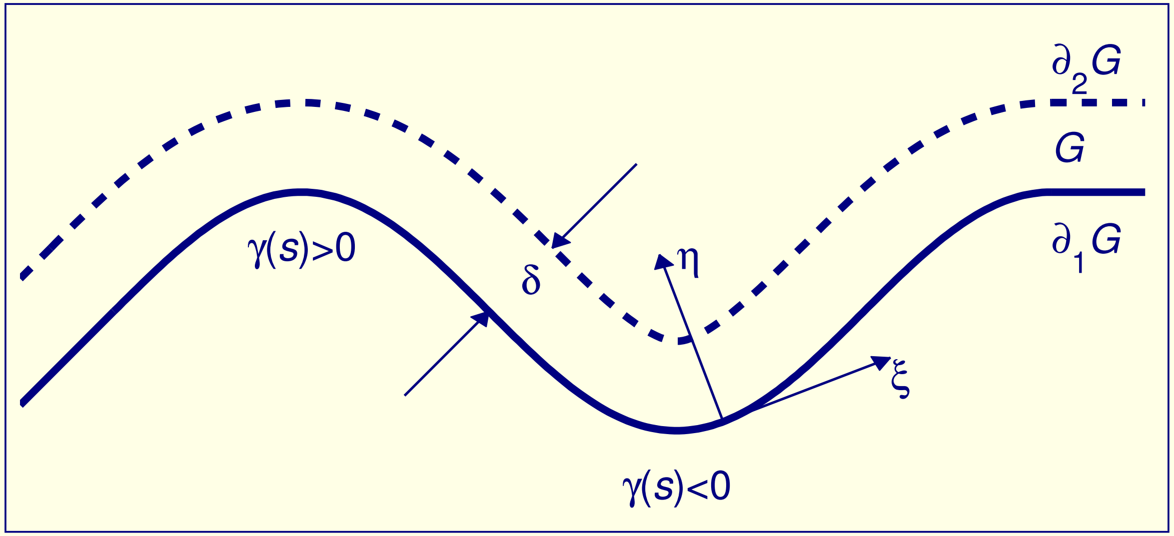

Figure 1: Domain G and curvilinear coordinates ξ, η. The solid line denotes the boundary $\partial_1 G$ with the Dirichlet boundary condition (the coast), and the dotted line — the boundary $\partial_2 G$ with the Neumann boundary condition.

Let $\gamma = \gamma(\xi)$ be the (signed) curvature of the (Dirichlet) part of the boundary.

Assumptions (not the minimal possible):

- Smoothness condition $\gamma \in C^\infty(\mathbb{R})$;
- Locally curved strip:

$$\operatorname{supp} \gamma \Subset [-R, R] \qquad \text{for some } R > 0\,, \tag{8}$$

  We set $\kappa^+ = \sup_{\xi\in[-R,R]} \gamma(\xi)$, $\kappa^- = -\inf_{\xi\in[-R,R]} \gamma(\xi)$.

- No self-intersections: $(\xi, \eta) \mapsto (x, y)$ is an injection on G. Locally gives $\kappa^\pm \le A\delta^{-1}$ with $A = \text{const} \in [0, 1)$.

Euclidean metric in curvilinear coordinates is $dx^2 + dy^2 = g d\xi^2 + d\eta^2$, where $g(\xi,\eta) := (1+\eta\gamma(\xi))^2$.

Notation:

$$p(\xi,\eta) = (g(\xi,\eta))^{1/2} = 1 + \eta\gamma(\xi)\,,$$

and in all the integrals

$$\mathrm{d}G_\gamma = p(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = (1+\eta\gamma(\xi))\,\mathrm{d}\xi\,\mathrm{d}\eta = p(\xi,\eta)\,\mathrm{d}G_0\,.$$

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Recall the problem: for a given curvature profile $\gamma(\xi)$ and a given positive depth profile $H(\xi, \eta) := \mathrm{e}^{\beta(\xi,\eta)}$, study the ω-spectrum of the problem (5)–(7).

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Re-write in new coordinates but with serios simplifications: only consider longitudinally uniform monotone depth profiles $\boxed{\beta(\xi, \eta) \equiv \beta(\eta)\,,\ \beta'(\eta) > 0}$.

We get after some simplifications

$$\omega\left(-\frac{1}{p^2}\frac{\partial^2\Phi}{\partial\xi^2}-\frac{\partial^2\Phi}{\partial\eta^2}+\frac{1}{p^3}\frac{\partial p}{\partial\xi}\frac{\partial\Phi}{\partial\xi}+\left(\beta'-\frac{1}{p}\frac{\partial p}{\partial\eta}\right)\frac{\partial\Phi}{\partial\eta}\right) = -\frac{i}{p}\beta'\frac{\partial\Phi}{\partial\xi}, \tag{9}$$

(with $\beta'=\dfrac{\mathrm{d}\beta}{\mathrm{d}\eta}$). Boundary conditions stay the same:

$$\Phi\big|_{\partial_1 G}=\left.\frac{\partial\Phi}{\partial\eta}\right|_{\partial_2 G}=0. \tag{10}$$

Important remark: When deducing equation (9), we have cancelled, on both sides, a common positive factor $h(\xi,\eta):=\dfrac{1}{H(\xi,\eta)}=\mathrm{e}^{-\beta(\xi,\eta)}$. However, we have to use this factor when considering corresponding variational equations, in order to keep the resulting forms symmetric. This leads to a special choice of weighted Hilbert spaces below.

Look back: Similar problems for the Laplace operator have been extensively studied in the literature — either in a curved strip, or in a straight strip with an obstacle, or in a strip with compactly perturbed boundary. In the case of Laplacians with Dirichlet boundary conditions these problems are usually called ‘quantum waveguides’; the Neumann case is usually referred to as ‘acoustic waveguides’. The important result concerning quantum waveguides was established in EXNER–ŠEBA (1989) and DUCLOS–EXNER (1995): in the curved waveguides there always exist a trapped mode. Later this result was extended to more general settings; in particular, in DITTRICH–KŘÍŽ (2002) (also KREJČIŘÍK—KŘÍŽ (2005)) it was shown that in the case of mixed boundary conditions (i.e., Dirichlet conditions on one side of the strip and Neumann conditions on the other side) trapped modes exist if the strip is curved ‘in the direction of the Dirichlet boundary’. The case of acoustic waveguides is more complicated because any eventual eigenvalues are embedded into the essential spectrum and are, therefore, highly unstable. Therefore, it is believed that in general the existence of trapped modes in this case is due to some sort of the symmetry of the problem (EVANS, LEVITIN, VASSILIEV; DAVIES, LP, . . .).

Back to oceanography:

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Formally introduce the operators $\mathcal{L}_\gamma : \Phi \mapsto -\frac{1}{p^2}\frac{\partial^2\Phi}{\partial\xi^2} - \frac{\partial^2\Phi}{\partial\eta^2} + \frac{1}{p^3}\frac{\partial p}{\partial\xi}\frac{\partial\Phi}{\partial\xi} + \left(\beta' - \frac{1}{p}\frac{\partial p}{\partial\eta}\right)\frac{\partial\Phi}{\partial\eta}$ and $\mathcal{M}_\gamma : \Phi \mapsto -\frac{i}{p}\beta'\frac{\partial\Phi}{\partial\xi}$. Then (9) can be formally re-written as

$$\omega\mathcal{L}_\gamma\Phi = \mathcal{M}_\gamma\Phi\,, \tag{11}$$

or via an *operator pencil*

$$\mathcal{A}_\gamma \equiv \mathcal{A}_\gamma(\omega) = \omega\mathcal{L}_\gamma - \mathcal{M}_\gamma \tag{12}$$

as

$$\mathcal{A}_\gamma(\omega)\Phi = 0 \tag{13}$$

From now on everything is understood variationally.

3 Essential spectrum

Lemma 1. *Assume that condition* $\beta' > 0$ *holds. Then*

$$\operatorname{spec}_{\mathrm{ess}}(\mathcal{A}_\gamma) = \operatorname{spec}_{\mathrm{ess}}(\mathcal{A}_0) = [-\Omega_*, \Omega_*]\,, \tag{14}$$

where

$$\Omega_* = \sup_{\phi \in \widetilde{H}^1_0(0,\delta)} \frac{\frac{1}{2}\int_0^\delta \beta'(\eta) \mathrm{e}^{-\beta(\eta)} |\phi(\eta)|^2 \, \mathrm{d}\eta}{\sqrt{\int_0^\delta \mathrm{e}^{-\beta(\eta)} |\phi'(\eta)|^2 \, \mathrm{d}\eta \cdot \int_0^\delta \mathrm{e}^{-\beta(\eta)} |\phi(\eta)|^2 \, \mathrm{d}\eta}} > 0\,. \tag{15}$$

4 Eigenvalues — main result

Our main result consists in stating some sufficient conditions on the depth profile $\beta(\eta)$ and the curvature profile $\gamma(\xi)$ which guarantee the existence of an eigenvalue of the pencil $\mathcal{A}_\gamma$ lying outside the essential spectrum.

Theorem 2. *Assume, as before, that the condition $\beta' > 0$ holds. Assume additionally that*

$$\boxed{\beta''(\eta) < 0} \qquad \text{for } \eta \in (0, \delta)\,. \tag{16}$$

Then there exists a constant $C_\beta > 0$, which depends only on the depth profile β, such that $\mathrm{spec}_{\mathrm{dis}}(\mathcal{A}_\gamma) \neq \emptyset$ whenever γ satisfies conditions stated above and

$$\int \gamma(\xi)\mathrm{d}\xi > C_\beta \int \gamma(\xi)^2 \mathrm{d}\xi \tag{17}$$

An integral sufficient condition (17) may be replaced by a pointwise, although more restrictive, condition:

Corollary 3. *Assume that the conditions on* β *hold hold. Then there exists a constant* $c_{\beta,R} = \dfrac{C_\beta}{2R}$ *which depends only on the depth profile* β *and a given* $R > 0$ *such that* $\mathrm{spec}_{\mathrm{dis}}(\mathcal{A}_\gamma) \neq \emptyset$ *whenever* $\gamma \not\equiv 0$ *satisfies conditions stated at the beginning and*

$$\boxed{0 \le \gamma(\xi) < c_{\beta,R} \qquad \text{for } |\xi| \le R}\,. \tag{18}$$

Lemma 4. *Let* $(a, b) \subset (0, +\infty)$ *be a finite interval, and let a function* $g : (a, b) \to \mathbb{R}$ *be non-increasing. Then*

$$\left(\int_a^b xg(x)f(x)\,\mathrm{d}x\right)\cdot\left(\int_a^b f(x)\,\mathrm{d}x\right)-\left(\int_a^b g(x)f(x)\,\mathrm{d}x\right)\cdot\left(\int_a^b xf(x)\,\mathrm{d}x\right)\le 0$$

for any non-negative function $f : (a, b) \to \mathbb{R}$*.*

Conclusions:

Under relatively mild restriction, we

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Under relatively mild restriction, we

- state the problem rigorously
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