On the localisation of Laplace eigenfunctions: recent progress and open problems

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(X, g) is a closed connected smooth Riemannian manifold

$-\Delta$ the Laplace operator

$\lambda_1 = 0 < \cdots \leq \lambda_j \leq \cdots$ the eigenvalues

$\phi_1, \cdots, \phi_j, \cdots$ an orthonormal L^2 basis of eigenfunctions

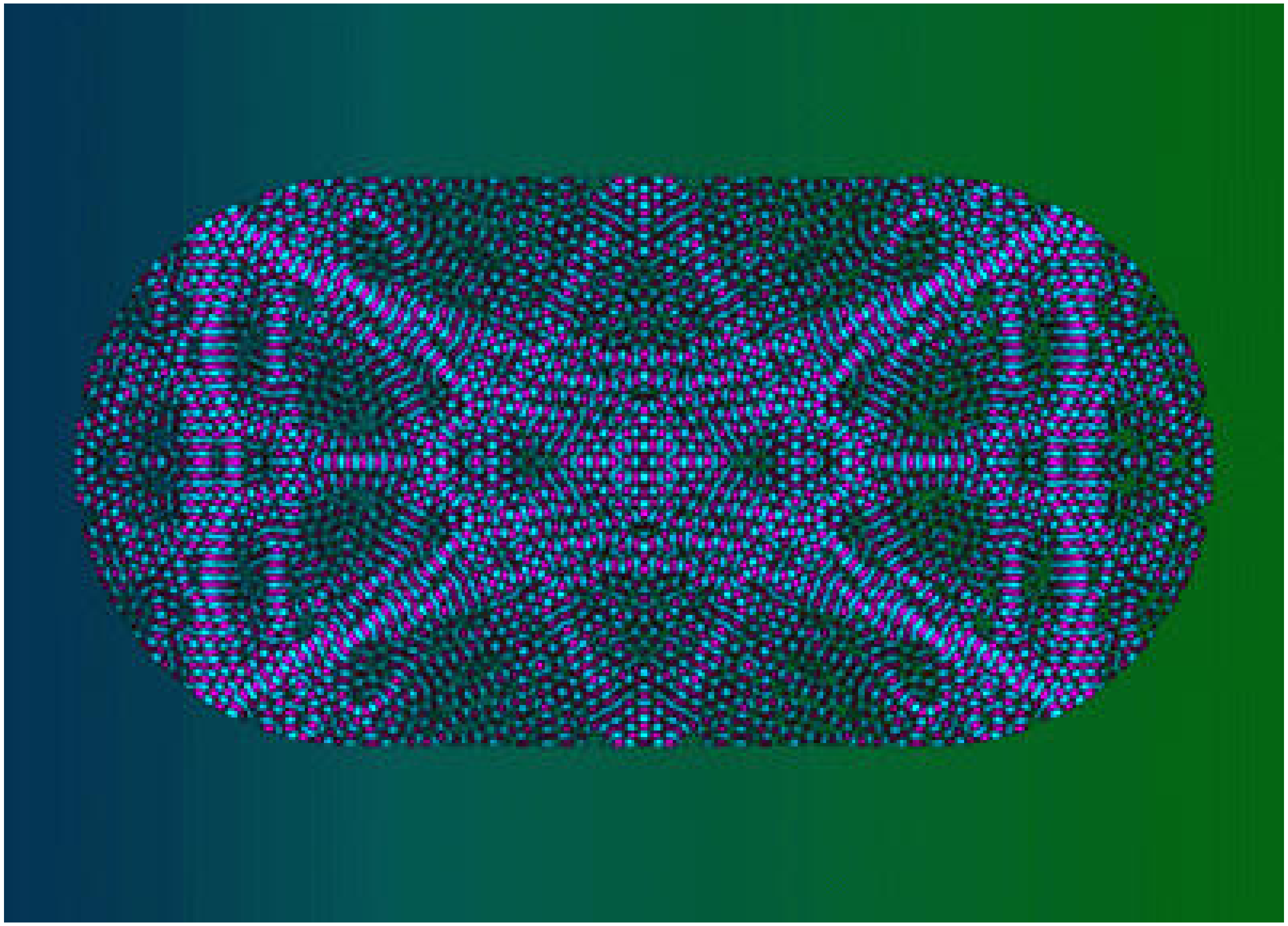
Main question: what is the asymptotic behaviour of the ϕ_j's as $j \to \infty$?

The limit $\lambda_j \to \infty$ is the high frequency limit for the wave equation and has to do with the geodesic flow on X.

Old results (Babich-Lazutkin (67), Ralston (76)): approximate eigenfunctions (quasi-modes) concentrating on **stable** periodic geodesics ("Gaussian beams")

In this lecture, we will be interesting in the case where the dynamics is **chaotic**, it is the case if the sectionnal curvature is < 0. All closed geodesics are then UNSTABLE!

Scars: Mc Donald & Kaufman (79), Heller (84)



Quantum limits

$$\hbar^2 \Delta \phi_j + \phi_j = 0$$

with $\hbar = \lambda_j^{-\frac{1}{2}}$ (an "effective" semi-classical parameter).

Let μ_j be the Wigner (Husimi) measure on the phase space $T^\star X$. The projection of μ_j on X is $\phi_j^2 |dx|$. The μ_j's are sometimes called the "microlocal lifts" of $\phi_j^2 |dx|$.

Technical definition:

$$\int_{T^\star X} a d\mu_j = \langle \mathrm{Op}_\hbar(a) \phi_j | \phi_j \rangle$$

where $a \to \mathrm{Op}_\hbar(a)$ *is a positive semi-classical quantization, i.e.* $\mathrm{Op}_\hbar(a)$ *is a* $\hbar - \Psi DO$ *of principal symbol* a *.*

Definition 1 *A* **quantum limit** *is any weak limit of a sub-sequence* μ_{j_k}.

Theorem 1 *Any quantum limit is a* probability *measure* μ *with support in the* unit cotangent bundle Z *and* μ *is* invariant *by the geodesic flow.*

We will say that we have a strong scar if μ has a singular part w.r. to the Liouville measure dL.

3 recent progress:

- Lindenstrauss (2005): no scars for an Hecke eigenbasis in the arithmetic case

- Faure, Nonnenmacher & de Bièvre (2003): strong scars do exist for the quantum cat map

- Anantharaman (2005), Anantharaman & Nonnenmacher (2006): if the geodesic flow is Anosov, the Kolmogorov-Sinaï entropy of any quantum limit is > 0.

Classical dynamics I. Ergodicity

Definition 2 *The dynamical system U^t preserving the probability measure μ is* ergodic *if every measurable set which is invariant by U^t is of measure 0 or 1.*

As a consequence, we get the celebrated *Birkhoff ergodic:*

Theorem 2 *For every $f \in L^1(Z, d\mu)$ and almost every $z \in Z$:*

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T f(U^t z) dt = \int_Z f d\mu \ .$$

The cat map is ergodic and the geodesic flow of every closed Riemannian manifold with < 0 sectional curvature is ergodic too.

II. Liapounov exponent

Definition 3 *The global Liapounov exponent* Λ_+ *of the dynamical system is defined as the lower bounds of the* Λ*'s which satisfy*

$$\|dU^t(z)\| = O(e^{\Lambda t}) \ ,$$

uniformly w.r. to z*.*

For cat maps given by $A \in SL_2(\mathbb{Z})$, $\Lambda_+ = |\lambda_+|$. If X is a Riemannian manifold of sectional curvature -1, $\Lambda_+ = 1$.

III. K-S entropy

Let (Z, U^t, μ) be a smooth dynamical system. If $\mathcal{P} = \{\Omega_j | j = 1, \cdots, N\}$ is a finite measurable partition of Z, we define the *entropy* of $\mathcal{P}$ by:

$$h(\mathcal{P}) := -\sum \mu(\Omega_j) \log \mu(\Omega_j) \ .$$

In terms of information theory, it is the average information you get by knowing in which of the Ω_j*'s the point* z *lies.*

Let $\mathcal{P}^N$ be the partition whose sets are

$$\Omega_{j_0,j_1,\cdots,j_N} = \{z \in \Omega_{j_0} \text{ so that } \forall l = 1, \cdots, N,\ U^l(z) \in \Omega_{j_l}\} \ .$$

We define

$$h_{\mathrm{KS}}(\mathcal{P}) := \lim_{N\to\infty} h(\mathcal{P}^N)/N \ ;$$

the limit exists because μ is invariant by the dynamics and this implies the subbadditivity of the sequence.

Then $h_{\mathrm{KS}}(\mu) = \inf_{\mathcal{P}} h_{\mathrm{KS}}(\mathcal{P})$.

IV. Hyperbolicity

Cat maps as well as geodesic flows on manifolds with < 0 curvature are *hyperbolic systems* in the sense of *Anosov*. They are the smooth dynamical systems which have the strongest chaotic properties. Let us give the definitions for flows:

Definition 4 *A smooth dynamical system* (Z, U^t) *generated by the vector field* V *is* Anosov *if there is a continuous splitting*

$$TZ = E_+ \oplus E_- \oplus \mathbb{R}V$$

so that, if dU^t *is the differential of* U^t*, the splitting is preserved by* dU^t*, and, if* dU^t_+ *(resp.* dU^t_-*) is the restriction of* dU^t *to* E_+ *(resp.* E_-*), there exists* $C > 0$ *and* $k > 0$ *so that:*

$$\forall t \geq 0,\ \|dU^t_+\| \leq Ce^{-kt},$$

$$\forall t \leq 0,\ \|dU^t_-\| \leq Ce^{kt},$$

The bundle E_+ *(resp.* E_-*) are called the* stable *(resp.* unstable*) bundles.*

We define then the unstable Jacobian $J_u(z)$ as the Jacobian determinant of $dU^1_-(z)$ w.r. to some Riemannian metric on Z (the rate of expansion at time 1 in the unstable direction). We have the following nice result which is a combination of results by Ruelle, Pesin and Ledrappier-Young:

Theorem 3 *If the dynamical system (Z, U^t) is Anosov and L is an invariant absolutely continuous measure, for every invariant probability measure μ, we have*:*

$$h_{\mathsf{KS}}(\mu) \leq \int_Z \log(J_u(z))d\mu \ .$$

Moreover, with equality if and only if $\mu = L$.

*The Jacobian $J_u(z)$ depends on the choice of a metric on Z, but the previous integral does not

Time scales in semi-classics

I. Ehrenfest time

The wave packets in quantum mechanics cannot be localized into sets of "size" less than $\hbar$. The Ehrenfest time is the time it takes for a cell of size $\hbar$ to be expanded to the whole phase space, more precisely:

Definition 5 *The* Ehrenfest time T_{E} *is defined by*

$$T_{\mathsf{E}} := \frac{|\log \hbar|}{\Lambda_+} \ .$$

Many estimates in semi-classics which are well known for finite time can be extended to times which are of the order of a suitable fraction of T_{E}. For example Egorov Theorem and the semi-classical trace formula.

II. Heisenberg time

The Heisenberg time is the time needed to resolve the spectrum from the observation of a wave at some point $x_0 \in X$: we have $u(x_0, t) = \sum a_j \mathrm{exp}(-itE_j/\hbar)$ and we can get approximate values of the E_j's only by knowing $u(x_0, t)$ on a windows of time larger than the Heisenberg time.

This time is of the order of $\hbar/\delta E$ where δE is the (mean) spacing of eigenvalues. Using Weyl's law, δE is of the order $\hbar^d$ where d is the dimension of the configuration space.

Definition 6 *The Heisenberg time is*

$$T_{\mathrm{H}} := \frac{\hbar}{\delta E} \ .$$

This time is usually of the order of $\hbar^{-(d-1)}$ which is much larger than the Ehrenfest time.

Asymptotic calculation of the eigenmodes need a knowledge of the quantum dynamics until the Heisenberg time. It is possible to do that (at the moment) only for (almost)integrable systems for which is the Heisenberg time is $+\infty$.

Gutzwiller type trace formulae are valid up to Ehrenfest time and are not quantization rules except for integrable systems for which they are equivalent, via the Bohr-Sommerfeld rules, to the Poisson summation formula.

The beginning of this story is the celebrated Schnirelman (74):

Theorem 4 *Let X be a closed Riemannian manifold whose geodesic flow is* ergodic. *Let (ϕ_j, λ_j) be an eigendecomposition of the Laplace operator. There exists a density one sub-sequence (λ_{j_k}) of the eigenvalues sequence** *so that all quantum limits of the φ_j's are the Liouville measure on the unit cotangent bundle.*

Since more than twenty years, existence of atypical sub-sequences is considered as an important problem. In particular, Rudnick and Sarnak formulated the so-called *Quantum Unique Ergodicity conjecture* (QUE): there are no exceptional sub-sequences at least for manifold of < 0 curvature.

*The sub-sequence λ_{j_k} of the sequence λ_j is of density 1 if

$$\lim_{\lambda \to +\infty} \frac{\#\{k | \lambda_{j_k} \leq \lambda\}}{\#\{j | \lambda_j \leq \lambda\}} = 1 \ .$$

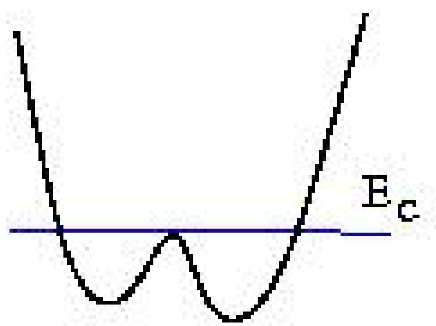
Arithmetic case

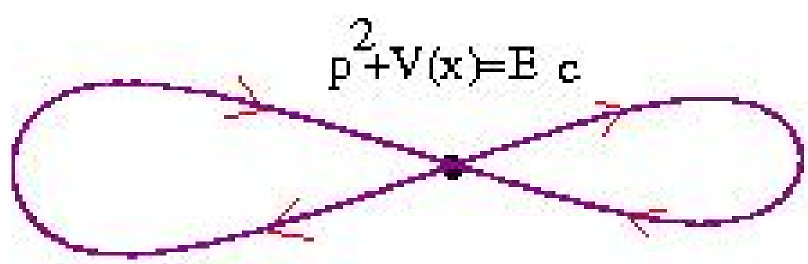
E. Lindenstrauss (2005) proved the QUE for an Hecke eigenbasis of *arithmetic* Riemann surfaces with constant curvature. His proof uses sophisticated results of M. Ratner.

Localized states for the cat map.

The only counter-example to QUE is for linear cat maps (de Bièvre-Faure-Nonnenmacher, 2003).

The baby-scars example: a 2 wells potential near the critical level: $\hbar^2\phi''_\hbar + V(x)\phi_\hbar = (E_c + O(\hbar))\phi_\hbar$, with E_c the local maximum of V.





The classical cat map (or Arnold cat map) is a symplectic map of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ induced from the action on $\mathbb{R}^2$ of a map $U \in SL_2(\mathbb{Z})$ which is hyperbolic. Such a map can be quantized using finite dimensional Hilbert spaces $\mathcal{H}_N$ of dimension N and the semi-classical limit corresponds to $N \to \infty$.

The basic fact is that the quantum cat map $\widehat{U}_N$ is a unitary periodic operator (i.e. $\widehat{U}_N^{T(N)} = e^{iT(N)\alpha_n}\mathrm{Id}$) in sharp contrast with the classical cat map which is chaotic! The period is the period of the permutation induced by U on $(\mathbb{Z}/N\mathbb{Z})^2$. The period $T(N)$ satisfies

$$2T_{\mathsf{E}} = \frac{2|\log \hbar|}{\lambda} \leq T(N) \leq 3N \ .$$

We will choose a sequence N_k so that the periods are close to $2T_{\mathsf{E}}$. Let us denote $T_k := T(N_k)$. For such sequences, we have $T_H \sim T_{\mathsf{E}}$.

Lower bounds on the entropy: the A-N Theorems

Theorem 5 *Let (X, g) be a smooth closed Riemannian manifold of dimension d with < 0 sectional curvature. Let μ be any quantum limit of the Laplace operator. We have the following lower bound for the entropy of μ:*

$$h_{\mathsf{KS}}(\mu) \geq \int_Z \log J_u(z) d\mu - \frac{1}{2}(d-1)\Lambda_+ \ .$$

If the curvature is $\equiv -1$, it gives

$$h_{\mathsf{KS}}(\mu) \geq \frac{1}{2} h_{\mathsf{KS}}(dL) \ .$$

Before that N. Anantharaman proved:

Theorem 6 *If X is a closed Riemannian manifold with < 0 curvature, then, for any quantum limit μ, $h_{\mathrm{KS}}(\mu) > 0$.*

It implies that no convex combinations of averages on closed geodesics is a quantum limit.

This cannot be obtained by local considerations around the closed geodesic.

What does it says about scars is perhaps more clear from the following result by Ledrappier and Lindenstrauss: for a surface X with < 0 curvature $h_{\mathsf{KS}}(\mu) > \frac{1}{2}\int_Z \log J_u(z) d\mu$ implies that the projection of μ on X is a.c..

Tool-box for the A-N Theorem

T_I. Entropic uncertainty principle

The way to get a lower bound for the entropy is by an adaptation of the entropic uncertainty principle conjectured by Kraus and proved by Maassen and Uffink. This principle states that, if a unitary matrix has "small" entries, then any of its eigenvectors must have a "large" Shannon entropy. The simplest statement is as follows:

Theorem 7 *Let* $\Omega = (\omega_{ij})$ *an* $N \times N$ *unitary matrix and for* $\phi = (\phi_j) \in \mathbb{C}^N$, $h(\phi) := -\sum_{j=1}^{N} |\phi_j|^2 \log |\phi_j|^2$. *Then, for any* $\psi \in \mathbb{C}^N$,

$$h(\psi) + h(\Omega\psi) \geq -2 \log \max |\omega_{ij}| \ .$$

T_{II}. Large time Egorov Theorem

Let us denote by $\widehat{U}(t) := \exp(-it\sqrt{-\Delta})$ the quantization of the geodesic flow ("wave group"). The usual Egorov theorem (which is enough to prove Schnirelman's Theorem) says that

$$\widehat{U}(-t)\mathrm{Op}_\hbar(a)\widehat{U}(t) = \mathrm{Op}_\hbar(a \circ U^t) + O(\hbar) \ .$$

It is impossible to extend this estimate to times larger than $T_E/2$ because then the symbol $a \circ U^t$ does not more belong to a nice class of symbols. But is is possible to do it until that time (Bouzouina-Robert, 2002).

T_{III}. The main estimate

We start with a pseudo-differential partition of unity π_k (suporeted in Ω_k) which satisfies $\sum \pi_k^\star \pi_k = \mathrm{Id}$ which is refined similarly to the classical definition of the K-S entropy: for any sequence $\epsilon = (\varepsilon_0, \cdots, \varepsilon_n) \in \{1, \cdots, M\}^{n+1}$, we define:

- $\Pi_\epsilon := \left(\pi_{\varepsilon_n} \widehat{U}(1) \pi_{\varepsilon_{n-1}} \cdots \widehat{U}(1) \pi_{\varepsilon_0}\right)^\star$

- $J_u^\epsilon := \Pi_{l=0}^n \sup_{z \in \Omega_{\varepsilon_l}} J_u(z)$.

Theorem 8 • *If X is a closed $d-$manifold with < 0 sectional curvature and $\epsilon = (\varepsilon_0, \cdots, \varepsilon_n)$, we have, for all $\delta' > 0$ and $n \leq T_{\mathsf{E}}$:*

$$\|\Pi_{\epsilon'} \widehat{U}(n) \Pi_{\epsilon}^{\star}\|_{L^2 \to L^2} = O\left(\hbar^{-(d-1)/2-\delta'} (J_u^{\epsilon})^{-\frac{1}{2}} (J_u^{\epsilon'})^{-\frac{1}{2}}\right) .$$

• *For an hyperbolic quantum map on a $2d-$torus, the same estimate holds with $(d-1)/2$ replaced by $d/2$.*

Sketch of the proof of A-N Theorem

We want to apply (an extension of) the entropic inequality of T_I using a decomposotion of the eigenfunction ϕ_j of the followwing form: we start with a pseudo-differential partition of unity π_k which satisfies $\sum \pi_k^\star \pi_k = \mathrm{Id}$ which is refined similarly to the classical definition of the K-S entropy: $\Pi_\varepsilon = \pi_{\varepsilon_n} \circ \widehat{U} \circ \cdots \circ \widehat{U} \circ \pi_{\varepsilon_1}$. We use the estimates in T_{III} with $n \sim T_E$ and in order to get a nice estimate we need sub-additivity which is provide by Egorov for large time as in T_{II}.

Thank you very much for your attention!