SCHUR COMPLEMENT, DRICHLET-TO-NEUMANN MAP, AND EIGENFUNCTIONS ON SELF-SIMILAR GRAPHS

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joint work with

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VIBRATION MODES OF $3N$-GASKETS AND OTHER FRACTALS

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Abstract. We study eigenvalues and eigenfunctions on the class of self-similar symmetric finitely ramified graphs. We consider such examples as the graphs modeled on the Sierpinski gasket, a non-p.c.f. analog of the Sierpinski gasket, the Level-3 Sierpinski gasket, a fractal 3-tree, the Hexagasket, and one dimensional fractal graphs. We develop a matrix analysis, including analysis of singularities, which allows us to compute eigenvalues, eigenfunctions and their multiplicities exactly.

Contents

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

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Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunik´, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

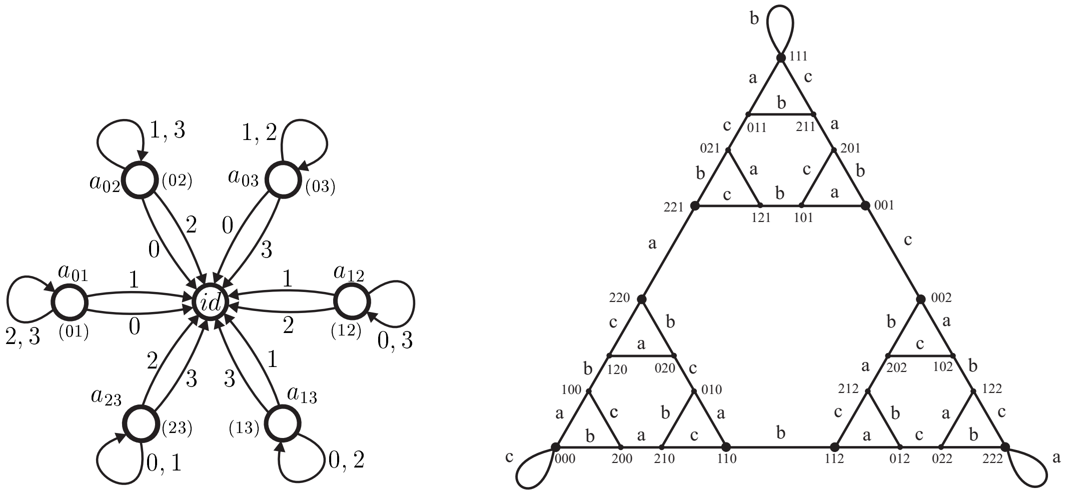

Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3

A **finitely ramified self-similar set** is a compact connected metric space F with injective contractions

$$\psi_1, ..., \psi_m : F \to F$$

$$F = \bigcup_{i=1}^{m} \psi_i(F)$$

$\forall w, w' \in W_n = \{1, ..., m\}^n$

$$F_w \cap F_{w'} = V_w \cap V_{w'},$$

$$F_w = \psi_w(F), \quad V_w = \psi_w(V_0), \quad V_0 = \partial F \subset F, \quad 2 \leqslant |V_0| < \infty$$

Notation: if $w = w_1...w_n \in W_n$ then $\psi_w = \psi_{w_1} \circ ... \circ \psi_{w_n}$

A **finitely ramified self-similar set** is a compact connected metric space F with injective contractions

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Our main geometric assumption:
for any permutation $\sigma : V_0 \to V_0$ there is an isometry $g_\sigma : F \to F$ that maps any $x \in V_0$ into $\sigma(x)$ and preserves the self-similar structure:
$\forall \sigma \; \exists \widetilde{g}_\sigma : W_1 \to W_1$ such that

$$\psi_i \circ g_\sigma = g_\sigma \circ \psi_{\widetilde{g}_\sigma(i)}$$

$\forall i \in W_1$.

$$V_n = \bigcup_{i=1}^{m} \psi_i(V_{n-1}) = \bigcup_{w \in W_n} V_w$$

$$G_n = \bigcup_{w \in W_n} G_w$$

where G_w is a complete graph with vertices V_w

Probabilistic graph Laplacians Δ_n on G_n are

$$\Delta_n f(x) = f(x) - \frac{1}{\deg_n(x)} \sum_{(x,y) \in E(G_n)} f(y)$$

where $E(G_n)$ is the set of edges of the graph G_n.

Graph energy is

$$\mathcal{E}_n(f,f) = \sum_{(x,y) \in E(G_n)} \big(f(x) - f(y)\big)^2$$

There is a unique $\mathcal{G}$-invariant resistance form $\mathcal{E}$ on F which is self-similar with energy renormalization factor ρ

$$\mathcal{E}(f,f) = \rho \sum_{i=1}^{m} \mathcal{E}(f_i, f_i)$$

for any $f \in Dom(\mathcal{E})$;

$$\mathcal{E}(f,f) = \int_F f \Delta_\mu f \, d\mu$$

for any $f \in Dom(\Delta_\mu)$.

$$\rho^{-n}\mathcal{E}_n \xrightarrow[n\to\infty]{} \mathcal{E}$$
$$(m\rho)^{-n}\Delta_n \xrightarrow[n\to\infty]{} \Delta_\mu$$

$$\rho m = \frac{d}{dz} R(0) > 1$$

$R(z)$ is the rational function that appears in the **spectral decimation**.

Example. $F=[0,1]$ which is self-similar with $m=2$, $\psi_1(x)=\frac{1}{2}x$, $\psi_2(x)=\frac{1}{2}x+\frac{1}{2}$,

$$\mathcal{E}(f,f)=\int_0^1 (f'(x))^2 dx$$

$$\mathcal{E}(f,f)=\int_F f\Delta_\mu f\, d\mu=\int_0^1 -ff''dx$$

for any $f\in Dom(\Delta_\mu)$.

$$\rho=2$$

$$4^{-n}\Delta_n \xrightarrow[n\to\infty]{} \Delta_\mu$$

$$\frac{2f(x)-f(x-\frac{1}{2^n})-f(x+\frac{1}{2^n})}{4^n} \xrightarrow[n\to\infty]{} -f''(x)$$

1. Spectral self-similarity, Schur complement and Drichlet-to-Neumann map

If we have a matrix M given in a block form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

then its Schur complement is

$$A - BD^{-1}C.$$

The Schur complement of the matrix $\Delta_1 - z$ is

$$S(z) = A - z - B(D - z)^{-1}C. \tag{1.2}$$

The blocks A and D in (1.1) correspond to outer (boundary) and interior vertices respectively.

Suppose v is an eigenvector of Δ_1 which is partitioned into its boundary part v_0 and interior part v_1'. Then

$$\Delta_1 v = zv$$

is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_0 \\ v_1' \end{bmatrix} = z \begin{bmatrix} v_0 \\ v_1' \end{bmatrix}$$

or as two equations

$$\begin{aligned} Av_0 + Bv_1' &= zv_0 \\ Cv_0 + Dv_1' &= zv_1' \end{aligned}$$

From the second equation we obtain $v_1' = -(D-z)^{-1}Cv_0$, provided $z \notin \sigma(D)$, which implies

$$S(z)v_0 = 0. \tag{1.3}$$

If v_0 is also an eigenvector of Δ_0 with an eigenvalue z_0, then we would like to relate (1.3) with

$$(\Delta_0 - z_0)v_0 = 0.$$

$$S(z) = A - z - B(D-z)^{-1}C$$

Main equation:

$$S(z) = \phi(z)\Big(\Delta_0 - R(z)\Big),$$

where $\phi(z)$ and $R(z)$ are scalar (meaning not matrix-valued) rational functions.

Remark 1.1. *From the calculations above one can see that $S(\lambda)$ is the so called Drichlet-to-Neumann map for the Laplacian Δ_1.*

In our examples Δ_0 is a matrix that has 1 on the diagonal and $-\frac{1}{|V_0|-1}$ off the diagonal. Therefore

$$\phi(z) = -(|V_0|-1)S_{1,2}(z)$$

$$R(z) = 1 - \frac{S_{1,1}(z)}{\phi(z)}.$$

Exceptional set

$$E(\Delta_0, \Delta_1) = \sigma(D) \cup \{z : \phi(z) = 0\}.$$

Theorem 1. *Suppose that z is not an eigenvalue of D, and not a zero of ϕ. Then z is an eigenvalue of Δ_1 with an eigenvector v if and only if $R(z)$ is an eigenvalue of Δ_0 with an eigenvector v_0, and $v = \begin{bmatrix} v_0 \\ v' \end{bmatrix}$ where*

$$v' = -(D - z)^{-1} C v_0.$$

This implies, in particular, that there is an one-to-one map from the eigenspace of Δ_0 corresponding to $R(z)$ onto the eigenspace of Δ_1 corresponding to z

$$v_0 \mapsto v = T(z) v_0$$

where

$$T(z) = I_0 - (D - z)^{-1} C.$$

The map $v_0 \mapsto v$ is called the eigenfunction extension map, and $T(z)$ is called the eigenfunction extension matrix.

$$\Delta_n = \left[\begin{array}{cc} A_n & B_n \\ C_n & D_n \end{array} \right]$$

for the block decomposition of Δ_n corresponding to the representation

$$V_n = V_{n-1} \bigcup \Big(V_n \backslash V_{n-1} \Big)$$

Theorem 2. *For all $n > 0$ we have a relation*

$$P_{n-1}(\Delta_n - z)^{-1} P_{n-1}^* = \frac{1}{\phi(z)} (\Delta_{n-1} - R(z))^{-1},$$

where P_{n-1} is defined as the restriction operator from V_n to V_{n-1}.

Suppose that $z_n \notin E(\Delta_0, \Delta_1)$. Then z_n is an eigenvalue of Δ_n with an eigenvector v_n if and only if

$$z_{n-1} = R(z_n)$$

is an eigenvalue of Δ_{n-1} with an eigenvector v_{n-1}, and $v_n = \left[\begin{array}{c} v_{n-1} \\ v_n' \end{array} \right]$ *where*

$$v_n' = -(D_n - z_n)^{-1} C_n v_{n-1}.$$

Theorem 3. *Let P_{n,z_n} be the eigenprojector of Δ_n corresponding to an eigenvalue $z_n \notin E(\Delta_0, \Delta_1)$, and $P_{n-1,z_{n-1}}$ be the eigenprojector of Δ_{n-1} corresponding to eigenvalue $z_{n-1} = R(z_n)$. Then*

$$P_{n,z_n} = \frac{1}{\phi(z_n)\frac{d}{dz}R(z_n)} T_n(z_n)\, P_{n-1,z_{n-1}}\, (P_{n-1} - B_n(D_n - z_n)^{-1} P'_n)$$

where

$$T_n(z) = (P_{n-1} - (D_n - z)^{-1} C_n)$$

and P'_n is defined as the restriction operator from V_n to $V_n \backslash V_{n-1}$. We often identify P'_n with the orthogonal projection from $\ell^2(V_n)$ onto the subspace of functions that vanish on V_{n-1}. In this case $P'_n = I_n - P_{n-1}$.

Proof.

$$\Delta_1^{-1} = D^{-1} + (I_0 - D^{-1}C)(A - BD^{-1}C)^{-1}(I_0 - BD^{-1})$$

provided that D and $A - BD^{-1}C$ are invertible.

∎

2. Analysis of the exceptional values.

$\mathrm{mult}_n(z)$ is the multiplicity of z as an eigenvalues of Δ_n
$\dim_n = \dim \ell^2(V_n) = |V_n|$.

Proposition 2.1. (1) *If $z \notin E(\Delta_0, \Delta_1)$, then*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-1}(R(z)),$$

and every corresponding eigenfunction at depth n is an extension of an eigenfunction at depth $n-1$.

(2) *If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z, then*

$$\mathrm{mult}_n(z) = \dim_{n-1},$$

and every corresponding eigenfunction at depth n is localized.

(3) *If $z \in \sigma(D)$, both $\phi(z)$ and $\phi_1(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{d}{dz}R(z) \neq 0$, then*

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) - \dim_{n-1} + \mathrm{mult}_{n-1}(R(z)),$$

and every corresponding eigenfunction at depth n vanishes on V_{n-1}.

(4) *If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, and $\phi(z) \neq 0$, then*

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) + \mathrm{mult}_{n-1}(R(z)).$$

In this case there are $m^{n-1}\mathrm{mult}_D(z)$ localized and $\mathrm{mult}_{n-1}(R(z))$ non-localized corresponding eigenfunctions at depth n.

(5) *If $z \in \sigma(D)$, but $\phi(z)$ and $\phi_1(z)$ do not have poles at z, and $\phi(z) = 0$, then*

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) + \mathrm{mult}_{n-1}(R(z)) + \dim_{n-1}$$

provided $R(z)$ has a removable singularity at z. In this case there are $m^{n-1}\mathrm{mult}_D(z) + \dim_{n-1}$ localized and $\mathrm{mult}_{n-1}(R(z))$ non-localized corresponding eigenfunctions at depth n.

(6) *If $z \in \sigma(D)$, both $\phi(z)$ and $\phi_1(z)$ have poles at z, $R(z)$ has a removable singularity at z, and $\frac{d}{dz}R(z) = 0$, then*

$$\mathrm{mult}_n(z) = \mathrm{mult}_{n-1}(R(z)),$$

provided there are no corresponding eigenfunctions at depth n that vanish on V_{n-1}. In general we have

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z) - \dim_{n-1} + 2\mathrm{mult}_{n-1}(R(z))$$

(7) *If* $z \notin \sigma(D)$, $\phi(z) = 0$ *and* $R(z)$ *has a pole* z, *then* $\mathrm{mult}_n(z) = 0$ *and* z *is not an eigenvalue.*

(8) *If* $z \in \sigma(D)$, *but* $\phi(z)$ *and* $\phi_1(z)$ *do not have poles at* z, $\phi(z) = 0$, *and* $R(z)$ *has a pole* z, *then*

$$\mathrm{mult}_n(z) = m^{n-1}\mathrm{mult}_D(z)$$

and every corresponding eigenfunction at depth n *vanishes on* V_{n-1}.

We state this theorem for $n = 1$, and the analogous relation holds for any $n \geqslant 1$. We define $\phi_1(z) = \phi(z)r(z)$.

Theorem 4. (1) *In the case of Proposition 2.1(1),*

$$P_z = \frac{1}{\phi(z)\frac{d}{dz}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}).$$

(2) *In the case of Proposition 2.1(2),*

$$P_z = (P_0 - (D-z)^{-1}C)(\psi_0(z)\Delta_0 - \psi_1(z))^{-1}(P_0 - B(D-z)^{-1}) \quad (2.1)$$

where $\psi_0(x) = \phi(x)/(z-x)$ *and* $\psi_1(x) = \phi_1(x)/(z-x)$. *This implies, in particular, that there is an one-to-one map* $v_0 \mapsto v = v_0 - (D-z)^{-1}Cv_0$ *from* $\ell^2(V_0)$ *onto the eigenspace of* Δ_1 *corresponding to* z.

(3) *In the case of Proposition 2.1(3), the poles of* $\phi(z)$ *and* ϕ_1 *are simple and so* $R(z)$ *has a removable singularity at* z, $P_z P_{D,z} = P_z$ *and* $P_0 P_z = 0$, *which means that the corresponding eigenfunctions of* Δ_1 *vanish on* V_0.

Moreover,

$$rank P_{D,z} - rank P_z = rank(\psi_0(z)\Delta_0 - \psi_1(z)I_0) = corank P_{0,R(z)}$$

where $\psi_0(x) = \phi(x)(z-x)$ *and* $\psi_1(x) = \phi_1(x)(z-x)$.

In addition, the following relations hold

$$P_z = P_{D,z} + \frac{1}{\psi_0(z)} P_{D,z} C(\Delta_0 - R(z))^{-1}(I_0 - P_{0,R(z)}) B P_{D,z}$$

and $P_{D,z} C P_{0,R(z)} = 0$. *Note that* $I_0 - P_{0,R(z)}$ *is the projector from* $\ell^2(V_0)$ *onto the space, where* $(D-z)^{-1}$ *is a well defined bounded operator.*

(4) *In the case of Proposition 2.1(4),*

$$P_z = P_{D,z} + \frac{1}{\phi(z)\frac{d}{dz}R(z)} (P_0 - (D-z)^{-1}C) P_{0,R(z)} (P_0 - B(D-z)^{-1}) \quad (2.2)$$

and the projector $P_{D,z}$ *is orthogonal to the second addend in the right hand side of this formula. In particular,* $P_z P_{D,z} = P_{D,z}$.

(5) *In the case of Proposition 2.1(5),* P_z *is the sum of the right hand sides in (2.1) and (2.2).*

(6) *In the case of Proposition 2.1(6), provided there are no corresponding eigenfunction at depth* n *that vanish on* V_{n-1}, *we have*

$$P_z = \frac{2}{\psi(z)\frac{d^2}{dz^2}R(z)}(P_0 - (D-z)^{-1}C)P_{0,R(z)}(P_0 - B(D-z)^{-1}).$$

In general, this formula is combined with 3.

(7) *In the case of Proposition 2.1(7) we formally have* $P_z = 0$.

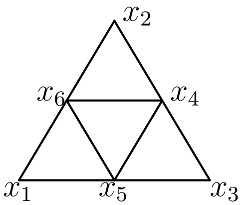
(8) *In the case of Proposition 2.1(8) we have* $P_z = P_{D,z}$.

Proof.

$$M = \sum_{z\in\sigma(M)} zP_z.$$

and pass to the limit as $x \to z$ □

3. Sierpiński gasket.



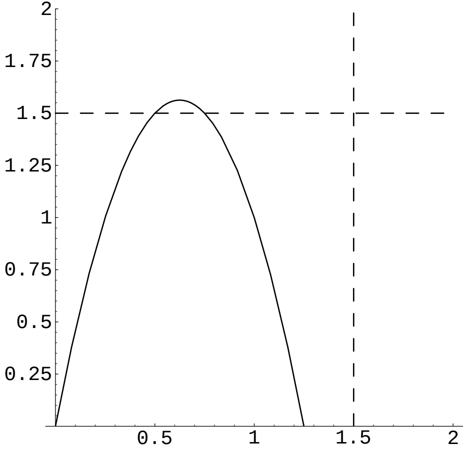
$$\Delta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

$$(D-z)^{-1}C = \begin{pmatrix} \frac{1}{-5+2(7-4z)z} & \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{2(-1+z)}{5+2z(-7+4z)} \\ \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{1}{-5+2(7-4z)z} & \frac{2(-1+z)}{5+2z(-7+4z)} \\ \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{1}{-5+2(7-4z)z} \end{pmatrix}$$

$$E(\Delta_0, \Delta_1) = \left\{ \frac{5}{4}, \frac{1}{2}, \frac{3}{2} \right\}$$

$$\phi(z) = \frac{3 - 2z}{5 - 14z + 4z^2}$$

$$R(z) = (5 - 4z)z$$



Proposition 2.1(3):

$$
\begin{aligned}
\mathrm{mult}_1(\tfrac{5}{4}) &= 2-3+1=0,\\
\mathrm{mult}_2(\tfrac{5}{4}) &= 6-6+1=1,\\
\mathrm{mult}_3(\tfrac{5}{4}) &= 18-15+1=4,\\
\mathrm{mult}_1(\tfrac{1}{2}) &= 1-3+2=0,\\
\mathrm{mult}_2(\tfrac{1}{2}) &= 3-6+3=0,\\
\mathrm{mult}_3(\tfrac{1}{2}) &= 9-15+6=0,
\end{aligned}
$$

Proposition 2.1(2):

$$
\begin{aligned}
\mathrm{mult}_1(\tfrac{3}{2}) &= 3,\\
\mathrm{mult}_2(\tfrac{3}{2}) &= 6,\\
\mathrm{mult}_3(\tfrac{3}{2}) &= 15,
\end{aligned}
$$

Table 1 shows the ancestor-offspring structure of the eigenvalues of the Sierpiński gasket. The symbol * indicates branches

$$\xi_1(z) = \frac{5 - \sqrt{25 - 16z}}{8}$$

and

$$\xi_2(z) = \frac{5 + \sqrt{25 - 16z}}{8}$$

of the inverse function $R^{-1}(z)$ computed at the ancestor value z.

<table>
<tr><td>$z \in \sigma(\Delta_0)$
$\mathrm{mult}_0(z)$</td><td colspan="4">$\mathbf{0}$
1</td><td colspan="8">$\mathbf{\frac{3}{2}}$
2</td><td colspan="6"></td></tr>
<tr><td>$z \in \sigma(\Delta_1)$
$\mathrm{mult}_1(z)$</td><td colspan="3">$\mathbf{0}$
1</td><td rowspan="3">$\mathbf{\frac{5}{4}}$</td><td colspan="4">$\mathbf{\frac{3}{4}}$
2</td><td rowspan="3">$\mathbf{\frac{1}{2}}$</td><td colspan="3">$\mathbf{\frac{3}{2}}$
3</td><td colspan="6"></td></tr>
<tr><td>$z \in \sigma(\Delta_2)$
$\mathrm{mult}_2(z)$</td><td colspan="2">$\mathbf{0}$
1</td><td rowspan="2">$\mathbf{\frac{5}{4}}$</td><td colspan="2">$\boldsymbol{\xi_1}(\mathbf{\frac{3}{4}})$
2</td><td colspan="2">$\boldsymbol{\xi_2}(\mathbf{\frac{3}{4}})$
2</td><td colspan="2">$\mathbf{\frac{3}{4}}$
3</td><td rowspan="2">$\mathbf{\frac{1}{2}}$</td><td colspan="2">$\mathbf{\frac{3}{2}}$
6</td><td colspan="2">$\mathbf{\frac{5}{4}}$
1</td><td colspan="2"></td></tr>
<tr><td>$z \in \sigma(\Delta_3)$
$\mathrm{mult}_3(z)$</td><td>$\mathbf{0}$
1</td><td>$\mathbf{\frac{5}{4}}$</td><td>$*$
2</td><td>$*$
2</td><td>$*$
2</td><td>$*$
2</td><td>$*$
3</td><td>$*$
3</td><td>$\mathbf{\frac{3}{4}}$
6</td><td>$\mathbf{\frac{1}{2}}$</td><td>$*$
1</td><td>$*$
1</td><td>$\mathbf{\frac{3}{2}}$
15</td><td>$\mathbf{\frac{5}{4}}$
4</td></tr>
</table>

TABLE 1. Ancestor-offspring structure of the eigenvalues on the Sierpiński gasket

Proposition 3.1. (i) $\sigma(\Delta_0) = \{0, \frac{3}{2}\}$.

(ii) *For any* $n \geqslant 1$ *we have*

$$\sigma(\Delta_n) = \{\tfrac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\{0, \tfrac{3}{4}\} \right).$$

In particular, for $n \geqslant 2$

$$\sigma(\Delta_n) = \{0, \tfrac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\{\tfrac{3}{4}\} \right) \bigcup \left(\bigcup_{m=0}^{n-2} R_{-m}\{\tfrac{5}{4}\} \right).$$

(iii) *For any* $n \geqslant 0$, $\dim_n = \frac{3^{n+1}+3}{2}$.

(iv) *For any* $n \geqslant 0$, $\mathrm{mult}_n(0) = 1$.

(v) *For any* $n \geqslant 0$, $\mathrm{mult}_n(\frac{3}{2}) = \frac{3^n+3}{2}$.

(vi) *If* $z \in R_{-k}\{\frac{3}{4}\}$ *then* $\mathrm{mult}_n(z) = \frac{3^{n-k-1}+3}{2}$ *for* $n \geqslant 1$, $0 \leqslant k \leqslant n-1$.

(vii) *If* $z \in R_{-k}\{\frac{5}{4}\}$ *then* $\mathrm{mult}_n(z) = \frac{3^{n-k-1}-1}{2}$ *for* $n \geqslant 2$, $0 \leqslant k \leqslant n-2$.

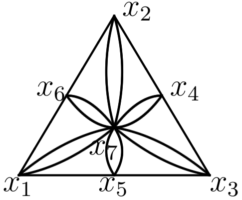
$$P_{n+1,\frac{3}{2}} = \Big(P_n - \big(D_n - \tfrac{3}{2}\big)^{-1} C_n\Big)\Big(\Delta_n + \tfrac{3}{2}\Big)^{-1}\Big(P_n - B_n\big(D_n - \tfrac{3}{2}\big)^{-1}\Big).$$

$$P_{n+1,\frac{5}{4}} = P_{D_n,\frac{5}{4}} + 12\, P_{D_n,\frac{5}{4}} C_n \Delta_n^{-1} B_n P_{D_n,\frac{5}{4}}.$$

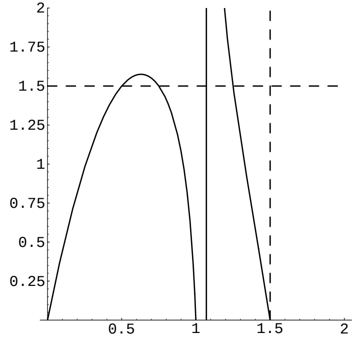
$$P_{D_n,\frac{5}{4}} \approx \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$D_n \approx \frac{1}{4}\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}.$$

4. A non-p.c.f. analog of the Sierpiński gasket.



$$R(z) = -\frac{24(z-1)z(2z-3)}{14z-15}, \qquad E(\Delta_0, \Delta_1) = \left\{\frac{3}{2}, 1, \frac{1}{2}, \frac{15}{14}\right\}$$



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<tr><td>$z \in \sigma(\Delta_0)$</td><td colspan="5">$\mathbf{0}$</td><td colspan="7">$\mathbf{\frac{3}{2}}$</td></tr>
<tr><td>$\mathrm{mult}_0(z)$</td><td colspan="5">1</td><td colspan="7">2</td></tr>
<tr><td>$z \in \sigma(\Delta_1)$</td><td colspan="3">$\mathbf{0}$</td><td rowspan="4">$\mathbf{1}$</td><td rowspan="4">$\mathbf{\frac{3}{2}}$</td><td rowspan="4">$\mathbf{\frac{1}{2}}$</td><td colspan="3">$\mathbf{\frac{3}{4}}$</td><td colspan="3">$\mathbf{\frac{5}{4}}$</td><td colspan="3">$\mathbf{\frac{3}{2}}$</td></tr>
<tr><td>$\mathrm{mult}_1(z)$</td><td colspan="3">1</td><td colspan="3">2</td><td colspan="3">2</td><td colspan="3">2</td></tr>
<tr><td>$z \in \sigma(\Delta_2)$</td><td>$\mathbf{0}$</td><td>$\mathbf{1}$</td><td>$\mathbf{\frac{3}{2}}$</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>$\mathbf{\frac{1}{2}}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{5}{4}}$</td><td>$\mathbf{\frac{1}{2}}$</td><td>$\mathbf{1}$</td><td>$\mathbf{\frac{3}{2}}$</td></tr>
<tr><td>$\mathrm{mult}_2(z)$</td><td>1</td><td></td><td></td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td>2</td><td>2</td><td>1</td><td>6</td><td>7</td></tr>
</table>

TABLE 2. Ancestor-offspring structure of the eigenvalues on the non-p.c.f. analog of the Sierpiński gasket.

Proposition 2.1(3):

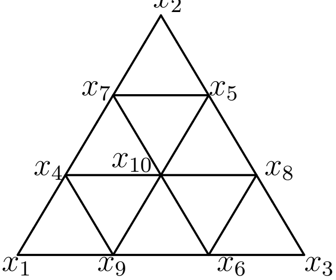
$$\begin{aligned} \mathrm{mult}_1(1) &= \; 2-3+1=0, \\ \mathrm{mult}_1(\tfrac{1}{2}) &= \; 1-3+2=0, \\ \mathrm{mult}_2(1) &= \; 12-7+1=6, \\ \mathrm{mult}_2(\tfrac{1}{2}) &= 6-7+2=1 \end{aligned}$$

Proposition 2.1(4):

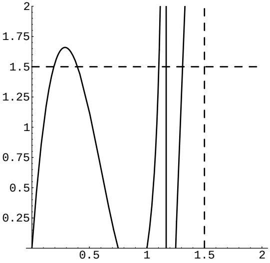
$$\mathrm{mult}_1(\tfrac{3}{2}) = 1+1=2$$

$$\mathrm{mult}_2(\tfrac{3}{2}) = 6+1=7.$$

5. Level-3 Sierpiński gasket.



$$R(z) = \frac{6z(z-1)(4z-5)(4z-3)}{6z-7}, \quad E(\Delta_0, \Delta_1) = \left\{ \frac{3}{2}, \frac{1}{4}(3 \pm \sqrt{5}), \frac{5}{4}, \frac{3}{4}, \frac{7}{6} \right\}$$



<table>
<tr><td>$z \in \sigma(\Delta_0)$</td><td colspan="10">$\mathbf{0}$</td><td colspan="10">$\mathbf{\frac{3}{2}}$</td><td colspan="7"></td></tr>
<tr><td>$\mathrm{mult}_0(z)$</td><td colspan="10">*1*</td><td colspan="10">*2*</td><td colspan="7"></td></tr>
<tr><td>$z \in \sigma(\Delta_1)$</td><td colspan="4">$\mathbf{0}$</td><td colspan="4">$\mathbf{1}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{5}{4}}$</td><td colspan="8">$\mathbf{\frac{3\pm\sqrt{2}}{4}}$</td><td colspan="2">$\mathbf{\frac{3\pm\sqrt{5}}{4}}$</td><td colspan="4">$\mathbf{\frac{3}{2}}$</td><td colspan="3"></td></tr>
<tr><td>$\mathrm{mult}_1(z)$</td><td colspan="4">*1*</td><td colspan="4">*1*</td><td></td><td></td><td colspan="4">*2*</td><td colspan="4">*2*</td><td></td><td></td><td colspan="4">*4*</td><td colspan="3"></td></tr>
<tr><td>$z \in \sigma(\Delta_2)$</td><td>$\mathbf{0}$</td><td>$\mathbf{1}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{5}{4}}$</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td></td><td colspan="2">$\mathbf{\frac{3\pm\sqrt{2}}{4}}$</td><td colspan="2">$\mathbf{\frac{3\pm\sqrt{5}}{4}}$</td><td>$\mathbf{\frac{3}{2}}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\mathbf{\frac{4}{5}}$</td></tr>
<tr><td>$\mathrm{mult}_2(z)$</td><td>*1*</td><td>*1*</td><td></td><td></td><td>*1*</td><td>*1*</td><td>*1*</td><td>*1*</td><td></td><td></td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td>*2*</td><td></td><td></td><td>*4*</td><td>*4*</td><td></td><td></td><td>*16*</td><td>*3*</td><td>*3*</td></tr>
</table>

TABLE 3. Ancestor-offspring structure of the eigenvalues on the Level-3 Sierpiński gasket.

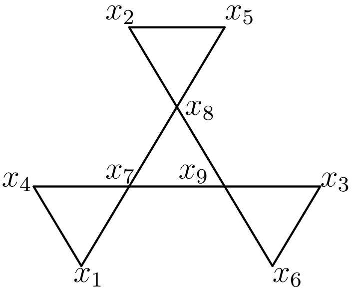
Proposition 2.1(3):

$$\begin{aligned}
\mathrm{mult}_1(\tfrac{3}{4}) &= \ 2-3+1=0,\\
\mathrm{mult}_2(\tfrac{3}{4}) &= \ 12-10+1=3,\\
\mathrm{mult}_1(\tfrac{5}{4}) &= \ 2-3+1=0,\\
\mathrm{mult}_2(\tfrac{5}{4}) &= \ 12-10+1=3,\\
\mathrm{mult}_1(\tfrac{3\pm\sqrt{5}}{4}) &= \ 1-3+2=0,\\
\mathrm{mult}_2(\tfrac{3\pm\sqrt{5}}{4}) &= \ 6-10+4=0.
\end{aligned}$$

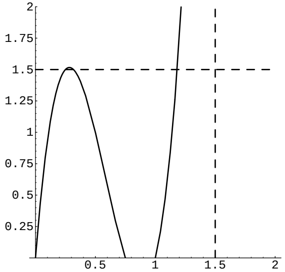
Proposition 2.1(5):

$$\begin{aligned}
\mathrm{mult}_1(\tfrac{3}{2}) &= \ 1+0+3=4,\\
\mathrm{mult}_2(\tfrac{3}{2}) &= \ 6+0+10=16.
\end{aligned}$$

6. A fractal 3-tree.



$$R(z) = 4z(z-1)(4z-3), \quad E(\Delta_0, \Delta_1) = \left\{ \tfrac{3}{2}, \tfrac{3}{4}, \tfrac{1}{4}(3 \pm \sqrt{3}) \right\}$$



Proposition 2.1(5):

$$\mathrm{mult}_1(\tfrac{3}{2}) = 4^0(2) + 0 + 3 = 5,$$

$$\mathrm{mult}_2(\tfrac{3}{2}) = 4^1(2) + 0 + 9 = 17.$$

Proposition 2.1(3):

$$\mathrm{mult}_1(\tfrac{3}{4}) = 4^0(2) - 3 + 1 = 0,$$

$$\mathrm{mult}_2(\tfrac{3}{4}) = 4^1(2) - 9 + 1 = 0.$$

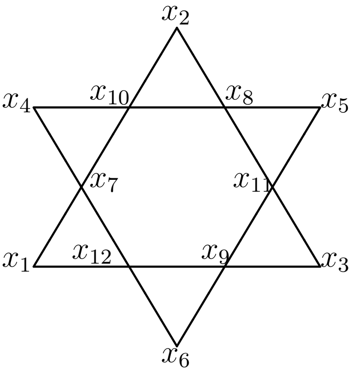
$$\mathrm{mult}_1(\tfrac{1}{4}(3 \pm \sqrt{3})) = 4^0(1) - 3 + 2 = 0,$$

$$\mathrm{mult}_2(\tfrac{1}{4}(3 \pm \sqrt{3})) = 4^1(1) - 9 + 5 = 0.$$

$z \in \sigma(\Delta_0)$	$\mathbf{0}$							$\mathbf{\frac{3}{2}}$						
$\text{mult}_0(z)$	*1*							*2*						
$z \in \sigma(\Delta_1)$	$\mathbf{0}$			$\mathbf{\frac{3}{4}}$	$\mathbf{1}$			$\mathbf{\frac{1}{4}}$			$\mathbf{\frac{3\pm\sqrt{3}}{4}}$	$\mathbf{\frac{3}{2}}$		
$\text{mult}_1(z)$	*1*				*1*			*2*				*5*		
$z \in \sigma(\Delta_2)$	$\mathbf{0}$	$\mathbf{\frac{3}{4}}$	$\mathbf{1}$		$*$	$*$	$*$	$*$	$*$	$*$		$\mathbf{\frac{1}{4}}$	$\mathbf{\frac{3\pm\sqrt{3}}{4}}$	$\mathbf{\frac{3}{2}}$
$\text{mult}_2(z)$	*1*		*1*		*1*	*1*	*1*	*2*	*2*	*2*		*5*		*17*

Table 4. Ancestor-offspring structure of the eigenvalues of the fractal tree.

7. Hexagasket.



$$R(z) = \frac{2z(z-1)(7+8z(2z-3))}{2z-1}, \quad E(\Delta_0, \Delta_1) = \left\{ \frac{3}{2}, \frac{3\pm\sqrt{5}}{4}, \frac{3\pm\sqrt{2}}{4}, \frac{1}{2} \right\}$$

<table>
<tr><td>$z \in \sigma(\Delta_0)$</td><td colspan="8">$\mathbf{0}$</td><td colspan="9">$\mathbf{\frac{3}{2}}$</td><td colspan="5"></td></tr>
<tr><td>$\mathrm{mult}_0(z)$</td><td colspan="8">1</td><td colspan="9">2</td><td colspan="5"></td></tr>
<tr><td>$z \in \sigma(\Delta_1)$</td><td colspan="3">$\mathbf{0}$</td><td colspan="4">$\mathbf{1}$</td><td>$\frac{3\pm\sqrt{2}}{4}$</td><td colspan="4">$\mathbf{\frac{1}{4}}$</td><td colspan="4">$\mathbf{\frac{3}{4}}$</td><td>$\frac{3\pm\sqrt{5}}{4}$</td><td colspan="3">$\mathbf{\frac{3}{2}}$</td><td colspan="2"></td></tr>
<tr><td>$\mathrm{mult}_1(z)$</td><td colspan="3">1</td><td colspan="4">1</td><td></td><td colspan="4">2</td><td colspan="4">2</td><td></td><td colspan="3">6</td><td colspan="2"></td></tr>
<tr><td>$z \in \sigma(\Delta_2)$</td><td>$\mathbf{0}$</td><td>$\mathbf{1}$</td><td>$\frac{3\pm\sqrt{2}}{4}$</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td></td><td>$\mathbf{\frac{1}{4}}$</td><td>$\mathbf{\frac{3}{4}}$</td><td>$\frac{3\pm\sqrt{5}}{4}$</td><td>$\mathbf{\frac{3}{2}}$</td><td>$\frac{3\pm\sqrt{2}}{4}$</td></tr>
<tr><td>$\mathrm{mult}_2(z)$</td><td>1</td><td>1</td><td></td><td>1</td><td>1</td><td>1</td><td>1</td><td></td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td>6</td><td>6</td><td></td><td>30</td><td>1 1</td></tr>
</table>

TABLE 5. Ancestor-offspring structure of the eigenvalues on the Hexagasket.

Proposition 2.1(3):

$$\mathrm{mult}_1(\tfrac{1}{4}(3 \pm \sqrt{2})) = 6^0 \cdot 2 - 3 + 1 = 0,$$
$$\mathrm{mult}_2(\tfrac{1}{4}(3 \pm \sqrt{2})) = 6^1 \cdot 2 - 12 + 1 = 1,$$
$$\mathrm{mult}_1(\tfrac{1}{4}(3 \pm \sqrt{5})) = 6^0 \cdot 1 - 3 + 2 = 0,$$
$$\mathrm{mult}_2(\tfrac{1}{4}(3 \pm \sqrt{5})) = 6^1 \cdot 1 - 12 + 6 = 0.$$

Proposition 2.1(5):

$$\mathrm{mult}_1(\tfrac{3}{2}) = 6^0 \cdot 3 + 0 + 3 = 6,$$
$$\mathrm{mult}_2(\tfrac{3}{2}) = 6^1 \cdot 3 + 0 + 12 = 30.$$

8. One dimensional interval as a self-similar set.

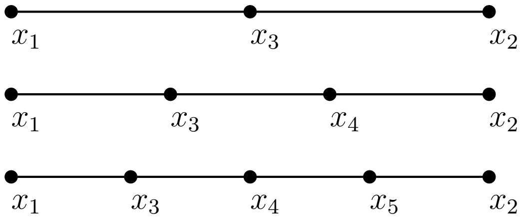
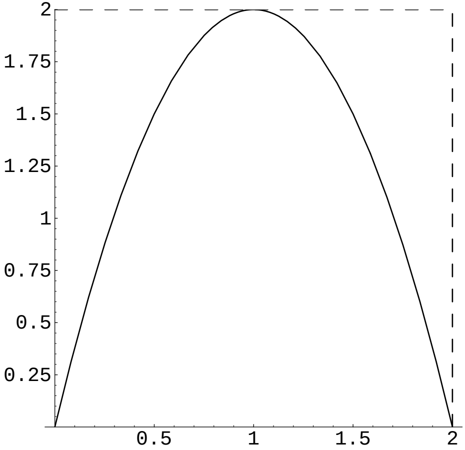
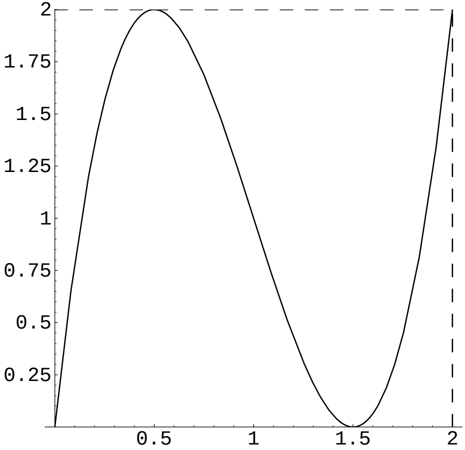
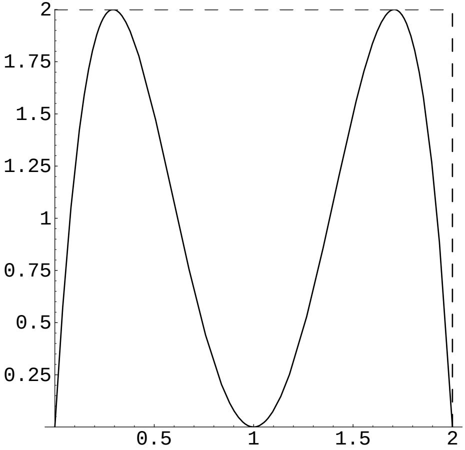


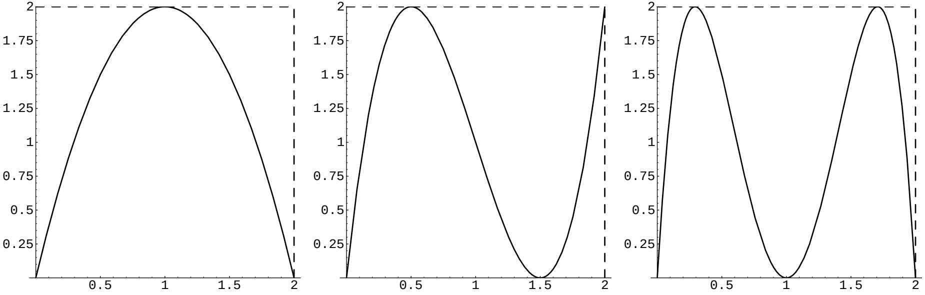
Figure 1. V_1 networks for the interval, $m = 2, 3, 4$ respectively.


2
1.75
1.5
1.25
1
0.75
0.5
0.25
0.5
1
1.5
2



2
1.75
1.5
1.25
1
0.75
0.5
0.25
0.5
1
1.5
2


2
1.75
1.5
1.25
1
0.75
0.5
0.25
0.5
1
1.5
2


2
1.75
1.5
1.25
1
0.75
0.5
0.25
0.5
1
1.5
2