Self-similar graphs, algebras and fractals

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It describes the process of adding 1 to a diadic integer.

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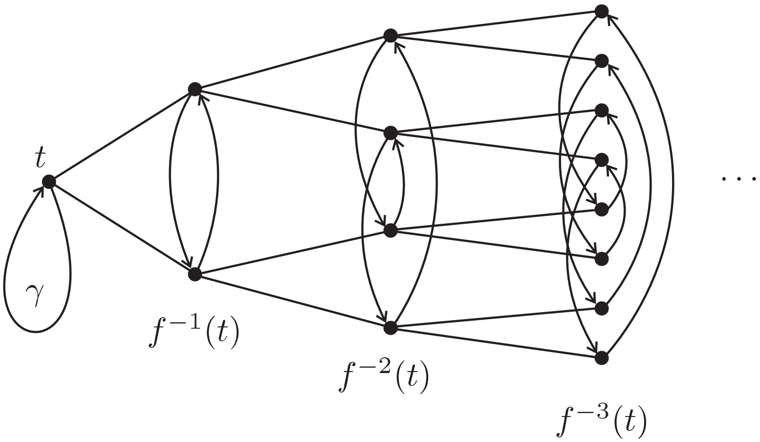
It is isomorphic to the *iterated monodromy group of* $z^2 - 1$.

Iterated monodromy groups



Julia sets

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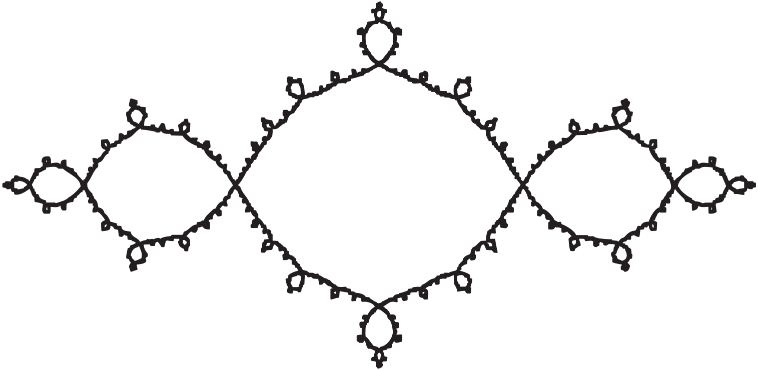
Basic example: hyperbolic complex rational functions.



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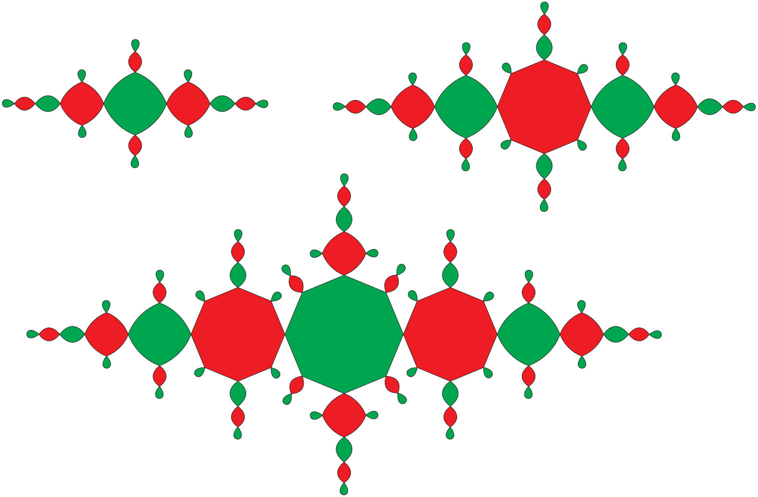
The set of vertices of Γ_n is X^n and v is connected to $s(v)$ for $s \in S$.



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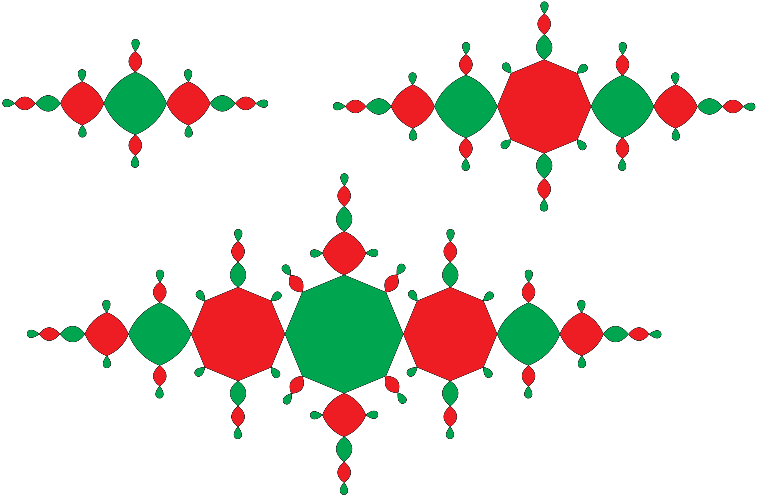
Pointed Schreier graphs Γ_n converge to infinite *Schreier graphs* of the action of the group on the boundary X^ω of the tree X^*.



Operator algebras

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For instance, one can take the groupoid generated by the germs of f and consider the convolution algebra of this groupoid.

Let us denote it $\mathcal{O}_f$. It contains $C(J)$ as a natural "diagonal" subalgebra.

An alternative approach is to use the Schreier graphs of the action of $\mathrm{IMG}(f)$ and consider adjacency relation and self-similarities.

The *Cuntz-Pimsner algebra* $\mathcal{O}_G = \mathcal{O}_{(G,\mathsf{X})}$ of a self-similar group (G, X) is the universal C^*-algebra generated by G and S_x (for $x \in \mathsf{X}$) satisfying

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3. for all $g \in G, x \in \mathsf{X}$:

$$g \cdot S_x = S_y \cdot h$$

whenever $g(xw) = yh(w)$ for all w.

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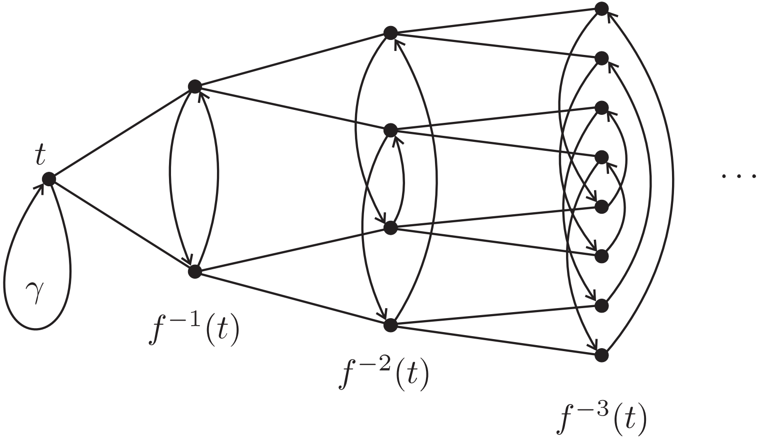
We have $\widetilde{\mathcal{O}}_f = \mathcal{O}_{\mathrm{IMG}(f)}$.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$$a = S_1 S_0^* + S_0(1 - S_v S_v^* + S_v a S_v^*) S_1^*.$$

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Then

$$K_0(\widetilde{\mathcal{M}}_f) = \mathbb{Z}[1/d], \qquad K_1(\widetilde{\mathcal{M}}_f) = \mathbb{Z}^{k-1}$$

and

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Δ induces an automorphism of the direct limit.

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Then the intersection of D_∞ with the asymptotic center is the algebra of continuous functions on the limit space of G.

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$$\inf\{p > 0 \;:\; [\pi(g), \rho(z)] \in \mathcal{L}^p\}$$

is equal to the Hausdorff dimension of J_f.