Wave propagation in networks of thin fibers: small diameter asymptotics.

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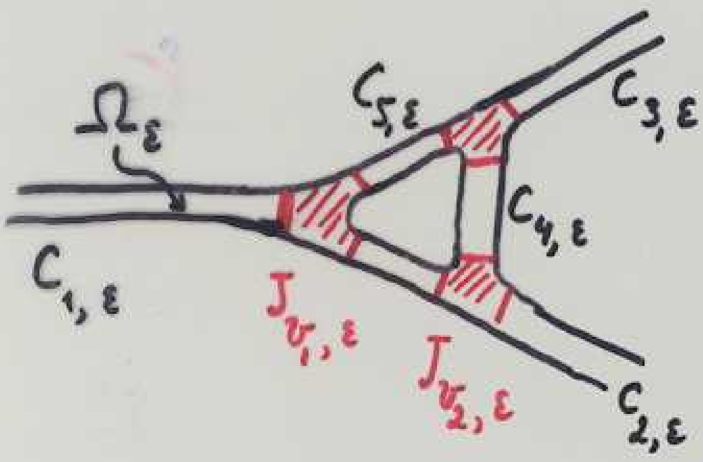
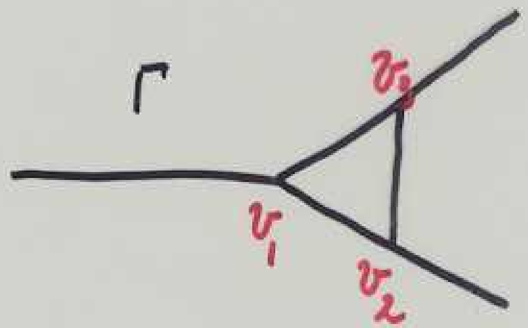
$-\varepsilon^2 \Delta u = \lambda u$ in Ω_ε

$Bu|_{\partial\Omega_\varepsilon} = 0$ (Dir., Neumann or Robin b.c.)

$\partial\Omega_\varepsilon \in C^\infty$.

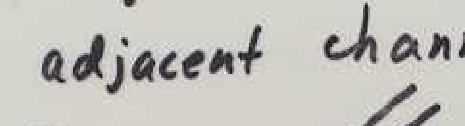




$C_{j,\varepsilon}$, $1 \le j \le m$, are semiinfinite channels

$C_{j,\varepsilon}$, $m < j \le N$, are channels of lenght $l_j < \infty$.

$\Omega_{v_j,\varepsilon}$ consists of one junction $J_{v_j,\varepsilon}$ and adjacent channels extended to inifinity



Assumption: $\exists v$, $\Omega_{v,\varepsilon}$ is self-similar (spider)

$$\Omega_{v,\varepsilon} = \{\hat{x}(\varepsilon) + \varepsilon x : x \in \Omega_v\}$$

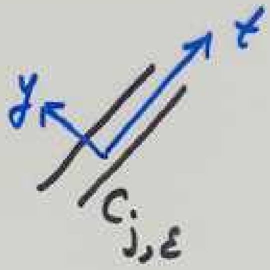
For simplicity: the cross-sections ω_ε are the same for all $C_{j,\varepsilon}$ $\Rightarrow$ $\omega_\varepsilon = \varepsilon\omega$

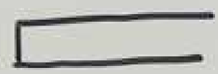
Consider $\begin{cases} -\Delta u = \lambda_j u & \text{in } \omega \\ Bu|_{\partial\omega} = 0 \end{cases}$

$\{\lambda_j\}$, $\{\varphi_j(y)\}$ are eigenvalues and eigenfunctions

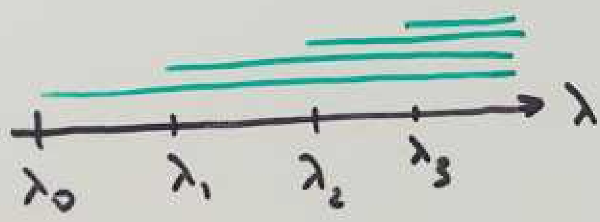
$\begin{cases} -\varepsilon^2\Delta u = \lambda_j u & \text{in } \omega_\varepsilon \\ Bu|_{\partial\omega_\varepsilon} = 0 \end{cases}$ $\quad \{\lambda_j\}$, $\{\varphi_j(y/\varepsilon)\}$

Local coordinates in $C_{j,\varepsilon}$: (y,t)



Spectrum in 1) Ω_ε =  2) Ω_ε = 

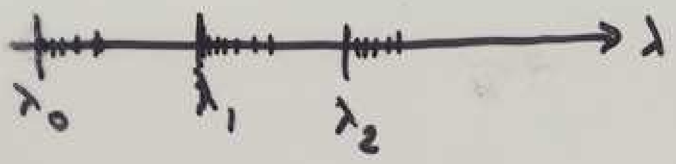
1)



$u_j = \varphi_j(y/\varepsilon)\sin\frac{\sqrt{\lambda-\lambda_j}}{\varepsilon}t$

$(u=0 \text{ as } t=0)$

2)



$\lambda_{n,m} = \lambda_n + \frac{\varepsilon^2 m^2\pi^2}{\ell^2}$

Consider $\lambda \geq \lambda_0$, let $\lambda \in (\lambda_0, \lambda_1)$ (for simplicity)

Scattering solution:

$$\Psi_{j,\varepsilon} = \begin{cases} \left(e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + t_{j,j}\, e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\right)\varphi_0(y/\varepsilon) + O(e^{-\delta t}), & x \in C_{j,\varepsilon} \\ t_{j,k}\, e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\,\varphi(y/\varepsilon) + O(e^{-\delta t}), & x \in C_{k,\varepsilon},\ k \neq j \end{cases}$$

$t = t_{j,k}(\lambda, \varepsilon)$, $j, k \leq m$

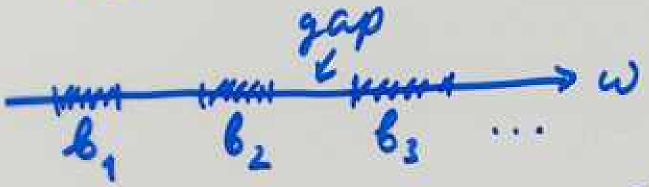
$T = [t_{j,k}]$ - scattering matrix, orthogonal, symmetric.

Th. Scattering solution exists and unique (for each j) if $\lambda \neq \{\lambda_{s,\varepsilon},\ 1 \leq s \leq N(\varepsilon)\}$

$\Psi_{j,\varepsilon} = ?$ $\varepsilon \to 0$

Slowing down of wave packets

$$H\psi = -\frac{d^2\psi}{dx^2} + V(x)\psi = \omega^2\psi, \qquad V(x+L) = V(x)$$



b_j are bands

$$\psi(x,\omega) = \varphi(x,\omega)\, e^{ik(\omega)\frac{x}{L}} \quad \text{- Bloch solution}$$

$$\varphi(x+L) = \varphi(x) \qquad \operatorname{Im} k(\omega) = 0,\ \omega \in \bigcup b_j$$

$$u = \psi(x,\omega)\, e^{-i\omega t} = \varphi(x,\omega)\, e^{i\left[k(\omega)\frac{x}{L} - \omega t\right]}$$

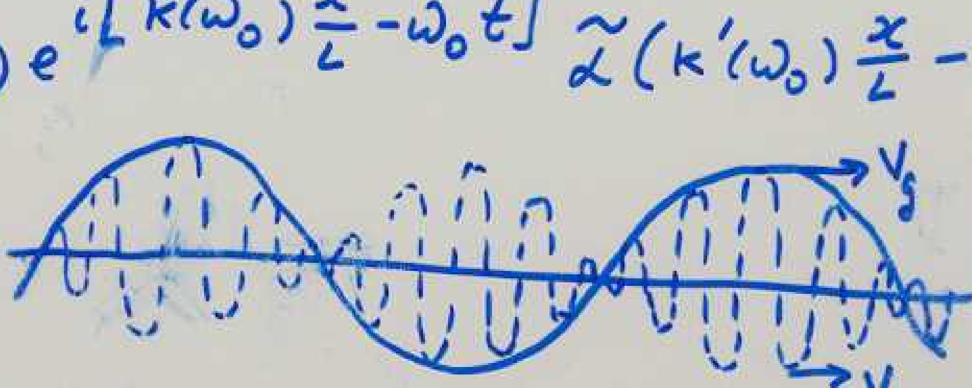
$$u_{tt} = -Hu \qquad V = \frac{\omega L}{k(\omega)} \quad \text{- phase velocity}$$

$$u = \int_{|\omega-\omega_0|<\varepsilon} \psi(x,\omega)\, e^{-i\omega t}\, \alpha(\omega-\omega_0)\, d\omega =$$

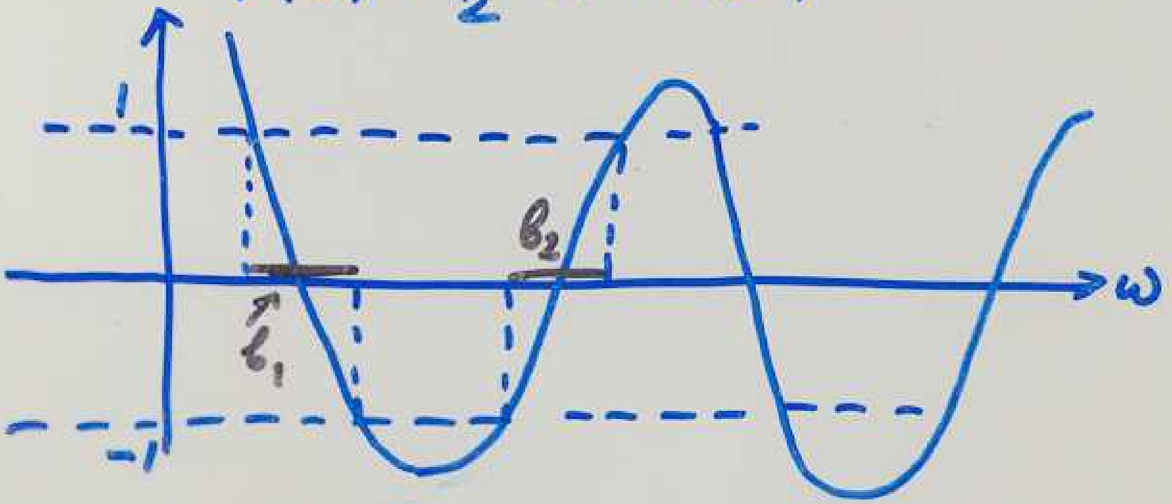
$$= \varphi(x,\omega_0)\, e^{i\left[k(\omega_0)\frac{x}{L} - \omega_0 t\right]} \int \alpha(\Delta\omega)\, e^{i\Delta\omega\left[k'(\omega_0)\frac{x}{L} - t\right]}\, d\omega$$

$$= \varphi(x,\omega_0)\, e^{i\left[k(\omega_0)\frac{x}{L} - \omega_0 t\right]}\, \tilde{\alpha}\!\left(k'(\omega_0)\frac{x}{L} - t\right)$$

$$V_g = \frac{L}{k'(\omega_0)}$$

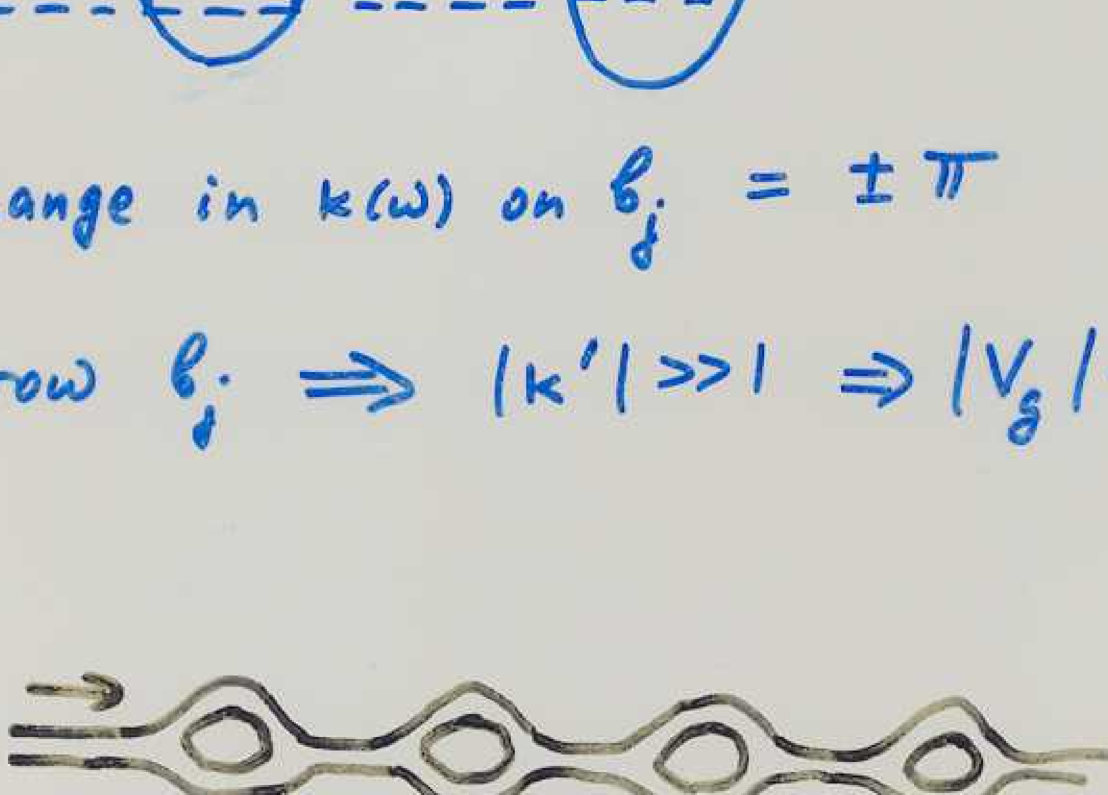


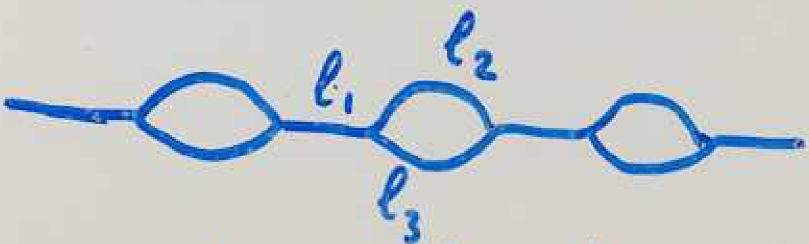
$$F(\omega) = \frac{1}{2} \operatorname{Tr} M(\omega)$$

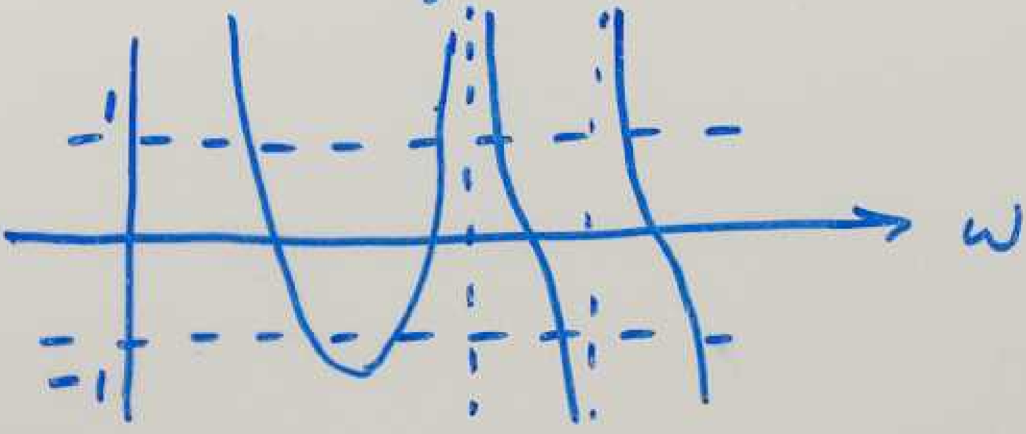


Change in $k(\omega)$ on $b_j = \pm\pi$

Narrow $b_j \Longrightarrow |k'| \gg 1 \Rightarrow |V_g| \ll 1$







Th $\Psi_{\gamma,\varepsilon} = \zeta_\varepsilon(\gamma)\varphi_0(y/\varepsilon) + O\left(e^{-\frac{\beta d(\gamma)}{\varepsilon}}\right)$

when $x \in \cup C_{\gamma,\varepsilon}$, $|x| < C$, $\lambda \notin \Lambda_\varepsilon$.

Here $\gamma \in \Gamma$, $\beta > 0$

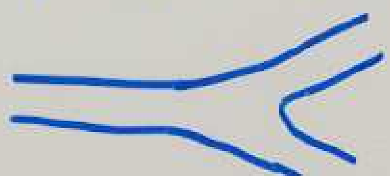
$d(\gamma) = \mathrm{dist}(\gamma, V)$, $V = \cup v_j$ - set of verticies

Function $\zeta_\varepsilon = \zeta_\varepsilon(\gamma)$, $\gamma \in \Gamma$, satisfies the eq-n:

$$-\varepsilon^2 \frac{d^2}{dt^2} \zeta + (\lambda - \lambda_0)\zeta = 0$$

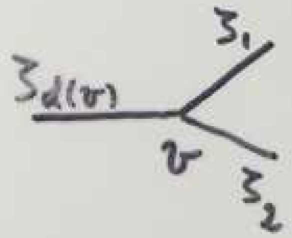
and conditions at ∞ similar to those for the scattering solutions.

It satisfies specific gluing conditions at verticies which are defined by the scattering matrix for the auxiliary problem in the spider domains $\Omega_{v,\varepsilon}$ (with one vertex)



(4)

Gluing conditions:



$$\xi^v = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{d(v)} \end{pmatrix}$$

$T_v(\lambda)$ - scattering matrix for $\Omega_{v,\varepsilon}$ 

$$i\varepsilon\left[I + T_v(\lambda)\right]\frac{d}{dt}\xi^v(0) - \sqrt{\lambda-\lambda_0}\left[I - T_v(\lambda)\right]\xi^v(0) = 0$$

Λ_ε can be covered by c_1 intervals of the lenght δ and $c_2\varepsilon^{-1}$, $c_2 = c_2(\delta)$, intervals of the lenght $e^{-\beta/\varepsilon}$

Asymptotics of the resolvent.

Let $\operatorname{supp} f \in C_{s,\varepsilon}$

$$-\varepsilon^2\Delta u - \lambda u = f, \quad Bu|_{\partial\Omega_\varepsilon} = 0$$

$$-\varepsilon^2\xi_{tt} - (\lambda-\lambda_0)\xi = f_0, \quad f_0 = f_0(\gamma,\varepsilon) = \langle f, \varphi_0(y/\varepsilon)\rangle$$

Th. $\left|R_\lambda f - G_\varepsilon f_0 \cdot \varphi_0(y/\varepsilon)\right| = O\left(e^{-\beta\frac{d(\gamma)}{\varepsilon}}\right), \quad \lambda \notin \Lambda_\varepsilon$

$$d = \operatorname{dist}(\gamma, V \cup \operatorname{supp} f)$$

$$\lambda = \lambda_0 + O(\varepsilon^2)$$

Freidlin-Wentzel, Exner (+ Seba, Duclos...)

Kuchment (+ Zeng), Rubinstain, Schatzman

Post, Dell'Antonio-Tenuta

Alberverio, Cacciapuoti, Finco

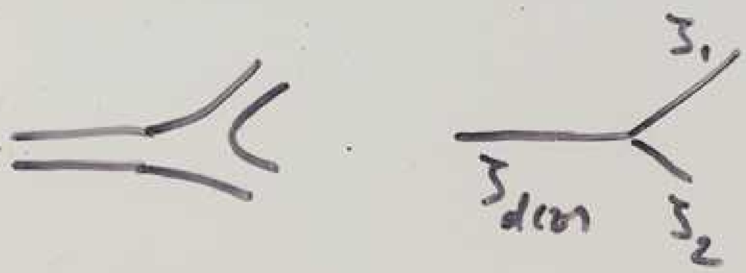
Neumann b.c. on $\partial\Omega_\varepsilon$. (1) $\lambda_0 = 0$

$u_t = \Delta u$. Eigenvalue μ/ε^2 of $-\varepsilon^2\Delta$ in Ω_ε

contributes $u = C e^{-\frac{\mu}{\varepsilon^2} t} \varphi_0(y/\varepsilon)$

$\lambda_0 = 0$ provides the existence of a non-trivial solution.

(2) Existence of the ground-state $u_0 \equiv 1$



Gluing conditions are Kirchoff's cond.

$\xi \in C$, $\quad \sum \xi'_{j_t} \rho_j = 0$

1) Post (2005) Dirichlet b.c. on $\partial\Omega_\varepsilon$ + junctions are more narrow than channels

$\Rightarrow$ Gluing cond. are Dirichlet cond.

2) Dell'Antonio - Tenuta (2006) - potentials with a deep minimum on Γ, and the width of the walls $\to 0$.

3) Albeverio - Cacciapuoti - Finco - Exner (2007)

(*) $$i\varepsilon\left[I+T_v(\lambda)\right]\frac{d}{dt}\xi^v(0)-\sqrt{\lambda-\lambda_0}\left[1-T_v(\lambda)\right]\xi^v(0)=0$$

$T_v(\lambda)$ is analytic in $\sqrt{\lambda-\lambda_0}$, unitary, symmetric

Eigenvalues of $T_v(\lambda_0)$ are ± 1.

They are -1 for generic Ω_ε.

Let $+1$ have multiplicity d_+, $-1 \leftrightarrow d_-$.

Then there is an orthogonal projection P

(*) $$P\xi^v(0)+O(\varepsilon)\frac{d}{dt}\xi^v(0)=0, \qquad P^\perp\frac{d}{dt}\xi^v(0)+O(\varepsilon)\xi^v(0)=0$$

$$\mathrm{Rank}\, P = d_-$$

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Kirchoff's condition: $\xi \in C$, $\sum \rho_j \frac{d\xi_j}{dt} = 0$, corresponds to the case of $d_- = d(v) - 1$.

In fact, let $\sum \rho_j^2 = 1$ and $\hat{\xi} = B\xi$ where $B = \begin{bmatrix} \rho_1 \dots \rho_{d(v)} \\ ? \end{bmatrix}$ is an orthogonal matrix with the first row $(\rho_1 \dots \rho_{d(v)})$.

Then K. condition $\Leftrightarrow \frac{d}{dt}\hat{\xi}_1(0) = 0$, $\hat{\xi}_j(0) = 0$, $j > 1$.

Generically, the problem $-\varepsilon^2 \Delta u = \lambda_0 u$, $Bu|_\Gamma = 0$ does not have a bounded solution $\Rightarrow$

$T(\lambda_0) = -I$ since otherwise

$$\Psi_j = \begin{cases} \left(e^{i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t} + t_{jj} e^{-i\frac{\sqrt{\lambda-\lambda_0}}{\varepsilon}t}\right)\varphi_0(y/\varepsilon) + O(e^{-\frac{\delta}{\varepsilon}}), & x \in C_{j,\varepsilon} \\ t_{jk} e^{-i\frac{\sqrt{\ }}{\varepsilon}t}\varphi_0(y/\varepsilon) + O(e^{-\delta t}), & x \in C_k,\ k \neq j \end{cases}$$

is a bounded solution.

Th. For any b.c. on $\partial\Omega_\varepsilon$, $Bu|_{\partial\Omega_\varepsilon}=0$, if there is a ground state at $\lambda=\lambda_0$ (a positive bounded solution with $\lim\limits_{x\in C_j,\ |x|\to\infty} u = \rho_j > 0$)

then the b.c. are Kirchoff's conditions

and solution of the problem

$$u_t = \Delta u \quad \text{in } \Omega_\varepsilon, \quad Bu|_{\partial\Omega_\varepsilon}=0$$

$$u|_{t=0} = \psi(\gamma)\varphi_0(y/\varepsilon)$$

has the form

$$u \sim e^{-\frac{\lambda_0 t}{\varepsilon^2}} w(\tau,\gamma)\varphi_0(y/\varepsilon)$$

There exists an effective matrix potential $V(t)=V^*(t)$ such that the problem $-\psi'' + [V(t)-I]\psi = 0$ on Γ has the same scattering data on $(-\infty,\lambda_1)$ as the original problem in Ω_ε (For spider domains).