Random Perturbations of Jordan Matrices and the Relevance of Directed Graphs

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The first part of this lecture is an account of the preprint of the same name by Mildred Hager and myself, of November 2006.

The later discussion of directed graphs is more recent and is work in progress.

Pseudospectra

The pseudospectral regions are defined by

$$\mathrm{Spec}_\varepsilon(A) = \{z : \|(zI - A)^{-1}\| > \varepsilon^{-1}\}.$$

and satisfy

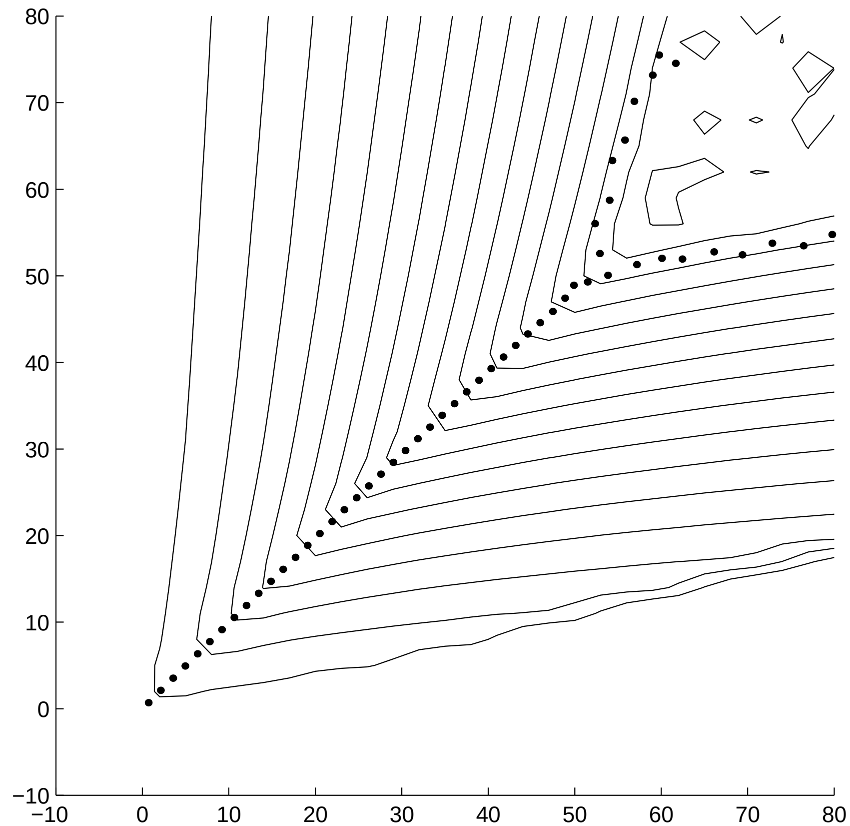
$$\mathrm{Spec}(A) \subseteq \mathrm{Spec}_\varepsilon(A).$$

The NSA harmonic oscillator

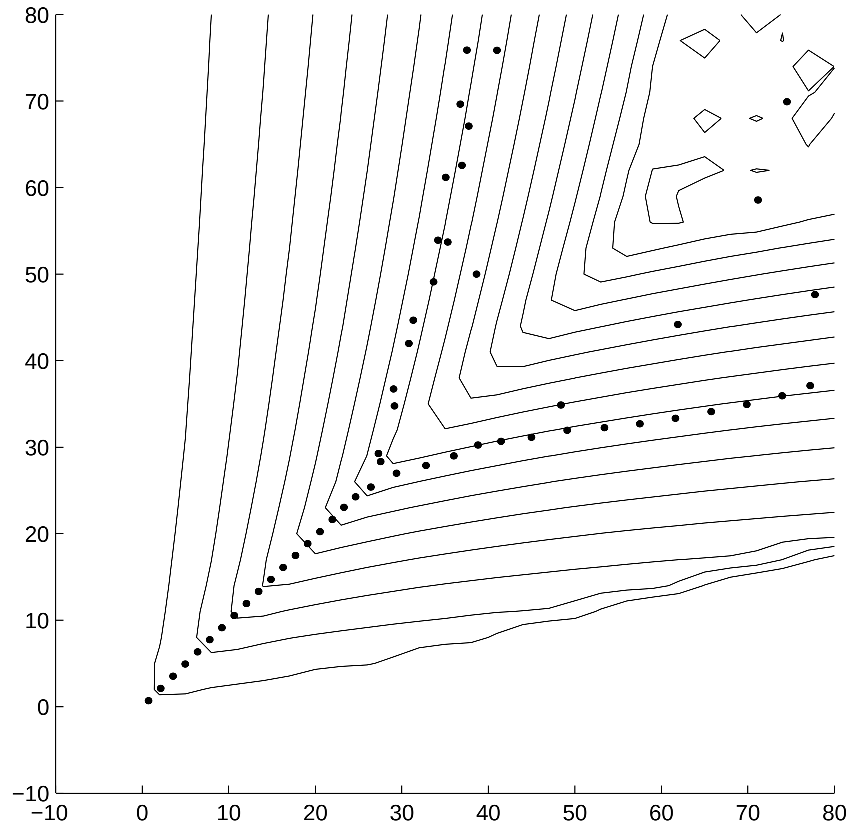
$$(Hf)(x) := -f''(x) + c^2x^2f(x)$$

acting in $L^2(\mathbf{R})$ has eigenvalues $\lambda_n := c(2n+1)$ where $n = 0, 1, \ldots.$

If c is complex then the norms of the spectral projections P_n increase at an exponential rate as $n \to \infty$.(EBD and Kuijlaars)



The contours correspond to $\varepsilon = 10^{-n}$ where $n = 0, 1, 2, ...$



With a perturbation of norm 10^{-6} the splitting of the eigenvalues along the pseudospectral contour is not due to model error or processor rounding errors.

The Jordan block

$$J_4 := \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Using the explicit formula for $(zI - J_n)^{-1}$ one immediately obtains

$$\|(zI - J_n)^{-1}\|_1 = \frac{|z|^{-n} - 1}{1 - |z|}$$

so the norm is exponentially large inside the unit circle.

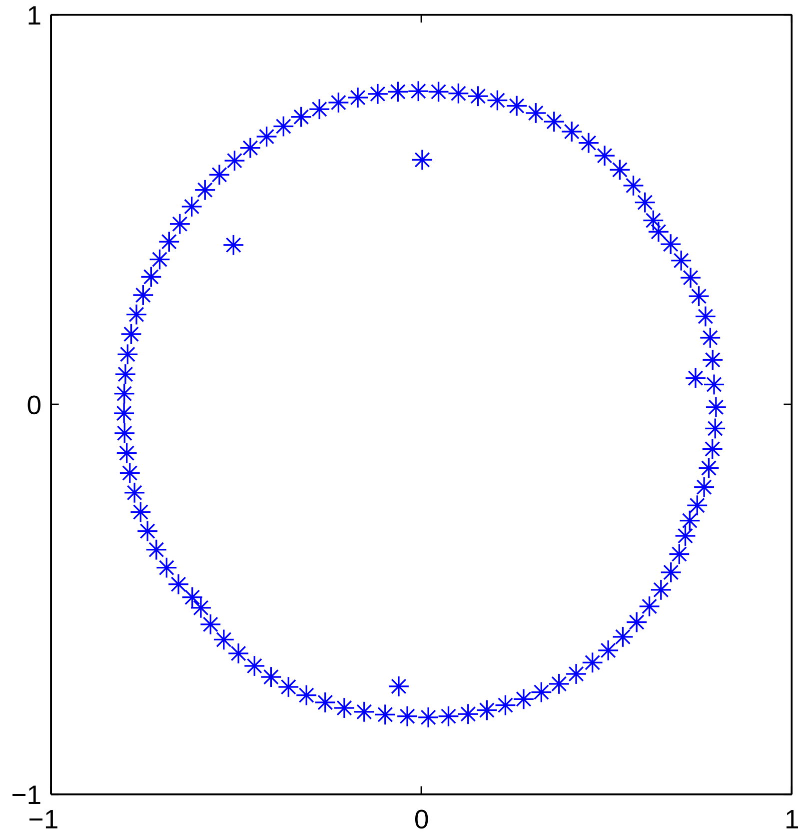
Perturbations of The Jordan block

If $\|B\| \le 1$ and $0 < c < 1$ then

$$\mathrm{Spec}(J_n + c^n B) \subseteq \{z : |z| \le c\}.$$

If B is chosen randomly one might expect the spectrum to be randomly distributed within this ball.

Mildred Hager showed that this was not correct. I got involved in looking with her in some detail at this problem.



The result of adding a small random perturbation to the Jordan matrix is to move most of the eigenvalues to the Lidskii circle, but a few are left at random positions inside the circle.

Theorem 1 *Let* $M = J + c^n K$ *where* J *is the standard* $n \times n$ *Jordan matrix,* $0 < c < 1$ *and* K *is a random matrix with independent Gaussian entries.*

Then for any $\varepsilon > 0$ *with probability that converges to* 1 *as* $n \to \infty$*, the proportion of the eigenvalues that lie in any annulus*

$$\{z : c - \varepsilon < |z| < c + \varepsilon\}$$

converges to 1*.*

The remaining eigenvalues lie inside the annulus.

Proof: Reduce the problem to finding the solutions of an equation of the form

$$w^n = f(w), \qquad w = z/c.$$

The analysis of the spectrum involves using theorems such as the following, and proving that the bounds hold with high probability.

Proposition 2 (The Poisson-Jensen formula) *Let f be a holomorphic function that does not vanish anywhere on the boundary of $D(0, R)$, where $0 < R < \infty$. Let M be the number of zeros of f in $D(0, Re^{-\sigma})$ for some positive constant σ. Then*

$$M \leq \frac{1}{\sigma}\left(-\ln \frac{|f(0)|}{\|f\|_{L^\infty(D(0,R))}}\right). \tag{1}$$

A simpler Example

Consider $A = J_n + c^n K$ where

$$K = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

and C is a fixed $k \times k$ matrix, for example

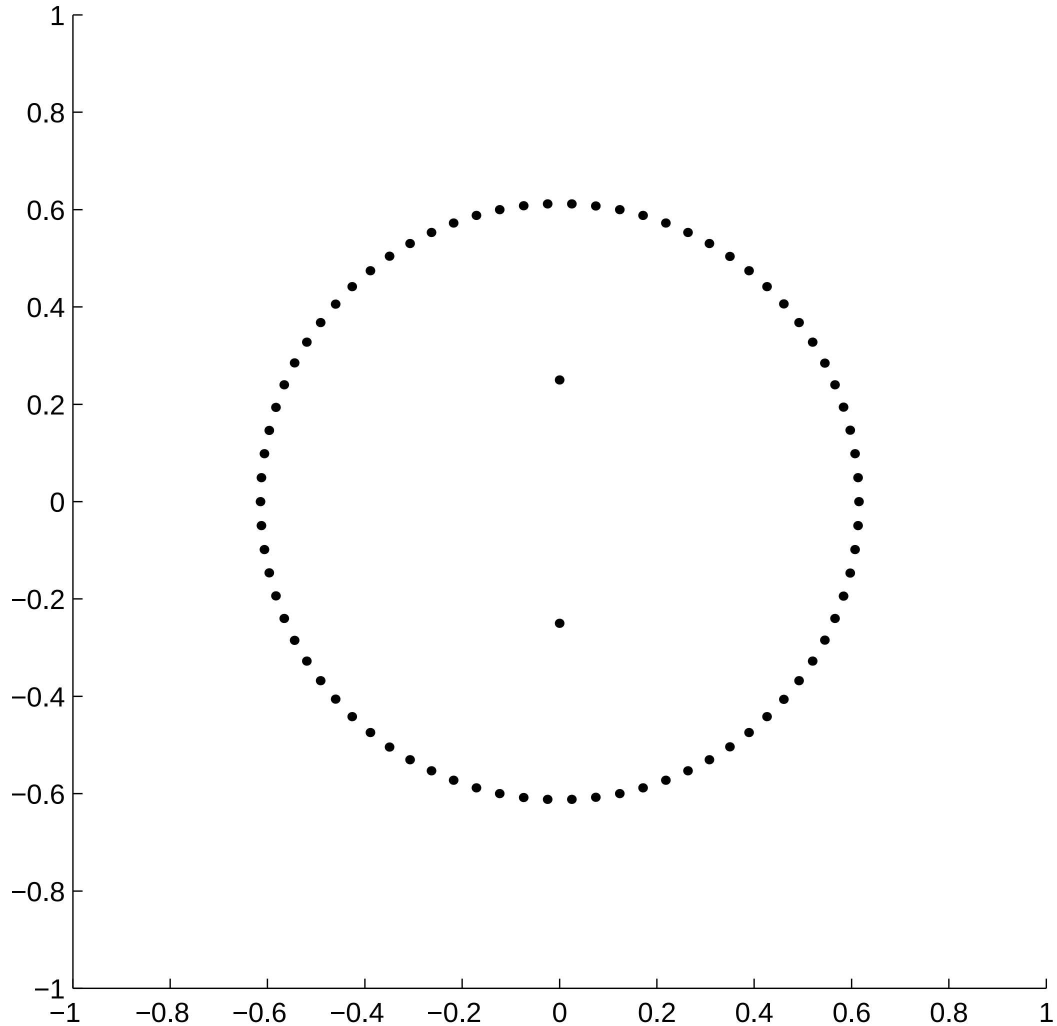
$$C = \begin{pmatrix} 8 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & -2 & 3 \end{pmatrix}.$$

THEOREM If $0 < c < \infty$ then $z \in \mathrm{Spec}(J_n + c^n K)$ if and only if

$$(z/c)^n = p(z)$$

where p is a fixed (i.e. n-independent) polynomial of degree $2k$. There is a large family of solutions for which $|z/c|$ is close to 1. If $|z/c| < 1$ then there are other solutions close to the zeros of $p(z)$.

The resulting spectrum is shown in the next figure.

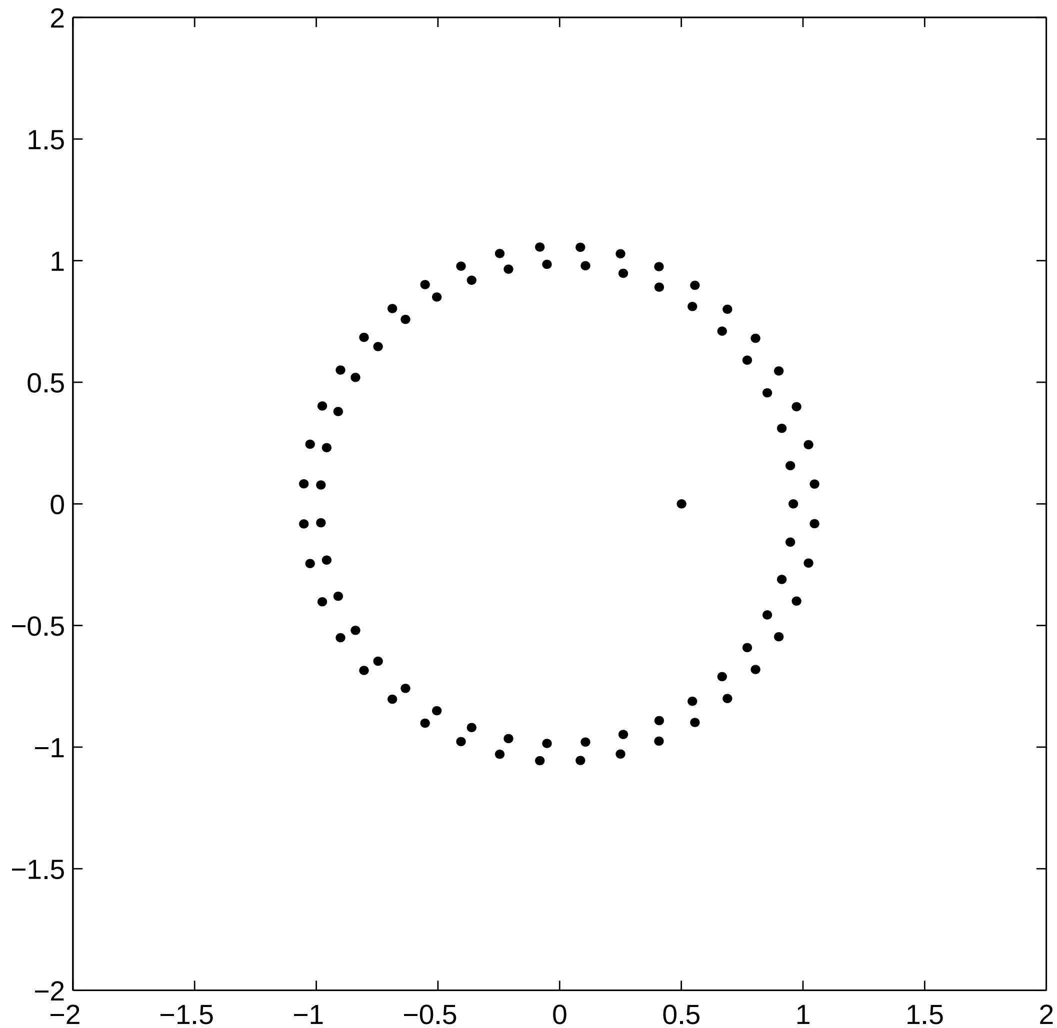


$A = J_n + c^n K$ where $n = 80$ and $c = 0.6$

More complex problems may lead to equations of the type

$$z^{2n} + p(z)z^n + q(z) = 0$$

or polynomial equations of higher order. The zeros of such equations are as shown in the following figure.


2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

Directed Graphs

Every $n \times n$ matrix has an associated directed graph, defined by putting $i \to j$ if $A_{i,j} \neq 0$.

Conversely every directed graph has an adjacency matrix.

Irreducible graphs have channels and junctions.

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What does this imply about the eigenvalues of the matrix?

A Figure Eight Graph

$$A = \left(\begin{array}{c|ccc|ccc}
\gamma & \beta_1 & & & \beta_2 & & \\
\hline
 & \alpha_1 & \beta_1 & & & & \\
 & & \alpha_1 & \beta_1 & & & \\
\beta_1 & & & \alpha_1 & & & \\
\hline
 & & & & \alpha_2 & \beta_2 & \\
 & & & & & \alpha_2 & \beta_2 \\
\beta_2 & & & & & & \alpha_2
\end{array}\right)$$

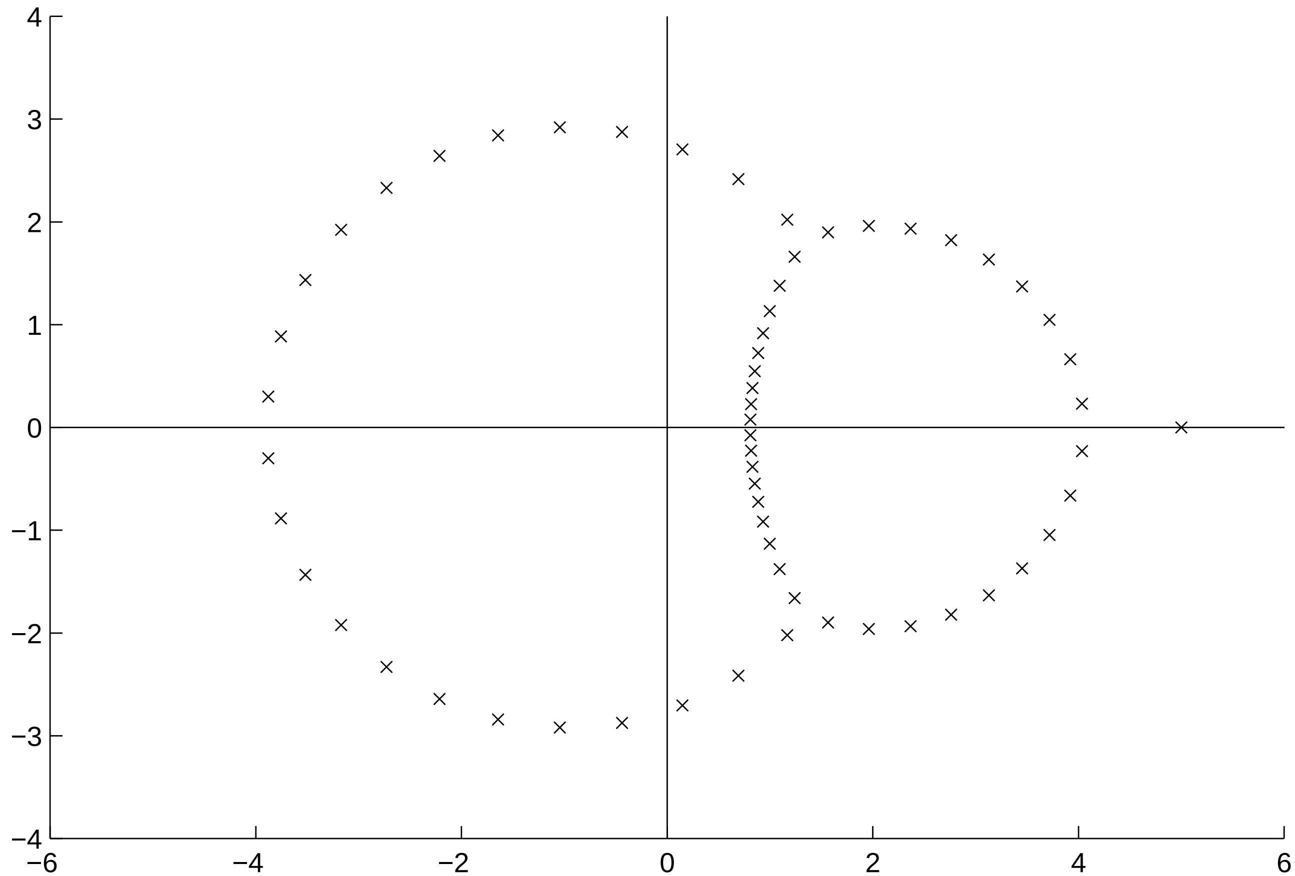


Figure 4. Eigenvalues of a matrix with $h = 2$ and $k = 1$.

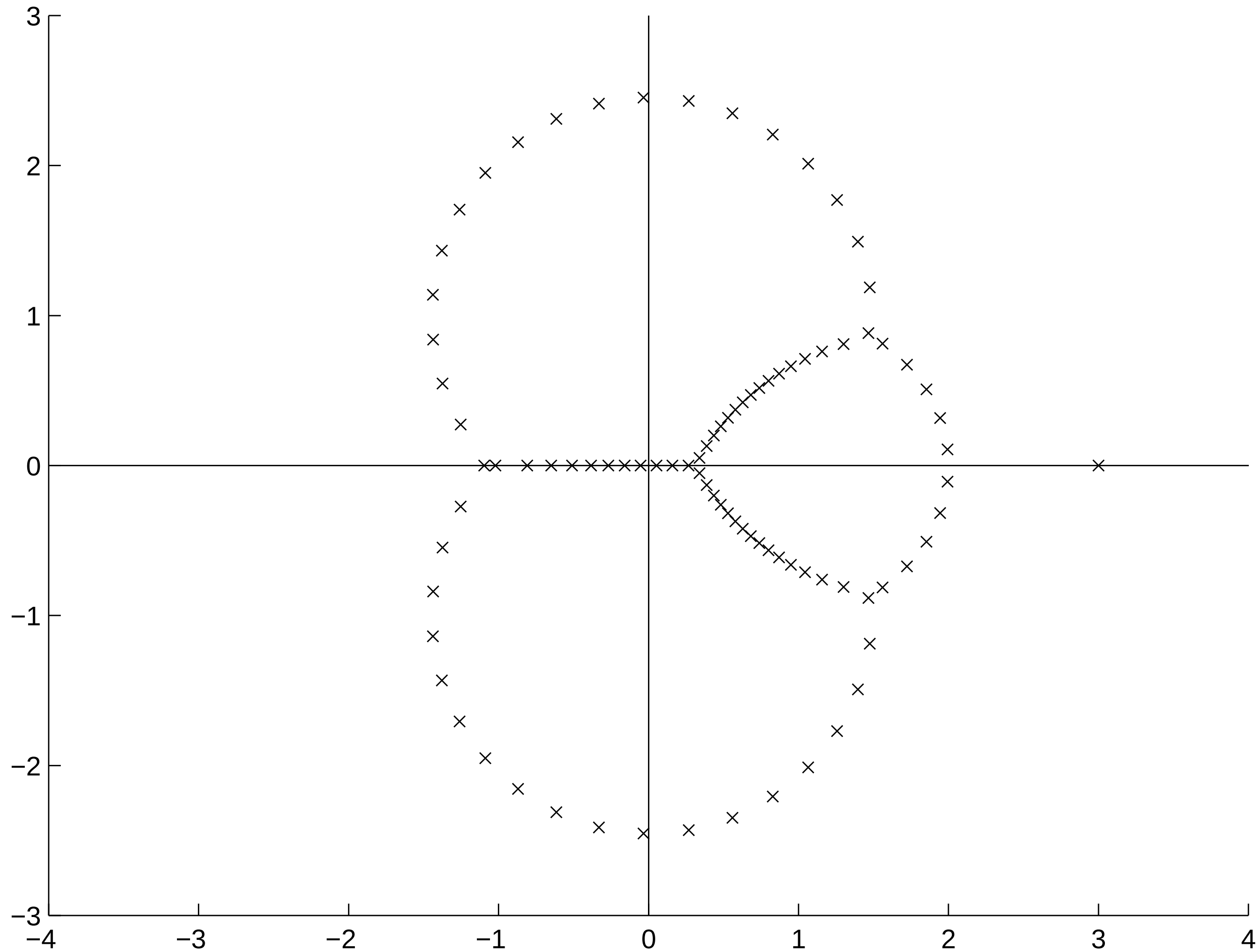
Theorem 3 *If*

$$P_n(z) = \sum_{r=1}^{k} a_r(z) f_r(z)^n$$

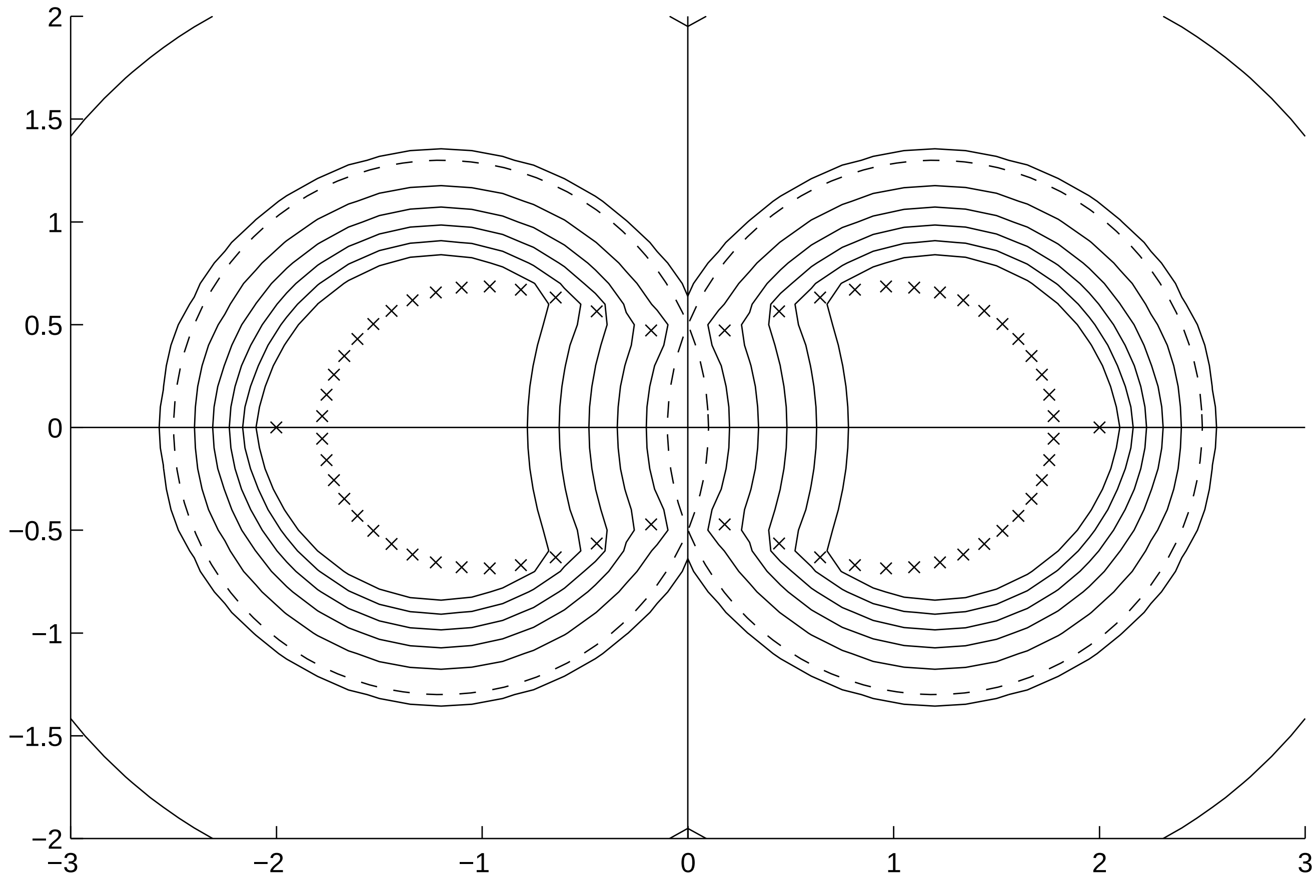
then most of the solutions of $P_n(z) = 0$ *converge to one of the sets*

$$\{z : |f_r(z)| = |f_s(z)|\}.$$

Only certain points on these sets are relevant.



Eigenvalues of a matrix with $h = 3$ and $k = 1$.



Eigenvalues of a matrix with $h = 2$, $k = 2$ and $n = 30$.