Condition Number Estimates for Oscillatory Integral Operators

Simon Chandler-Wilde

University of Reading, UK

www.reading.ac.uk/~sms03snc

Joint work with: **Ivan Graham** (Bath)
Steve Langdon, Marko Lindner (Reading)
Peter Monk (Delaware)

INI HoP Programme, June 2007

Focus of Today's Talk

For

$$\Delta u + k^2 u = 0,$$

and boundary integral methods for its solution:

How does conditioning depend on k, the geometry and on 'coupling parameters' in the integral equation formulation?

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How does conditioning depend on k, ... ?

For an continuous linear operator $A : X \to Y$, the condition number is

$$\operatorname{cond} A := \|A\|_{X\to Y} \, \|A^{-1}\|_{Y\to X} \text{ where } \|A\|_{X\to Y} := \sup_{0\neq x\in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

Large condition numbers associated with, inter alia, **slow convergence of iterative solution methods**.

Precise Focus of Today's Talk

For

$$\Delta u + k^2 u = 0,$$

and boundary integral equation methods for its solution.

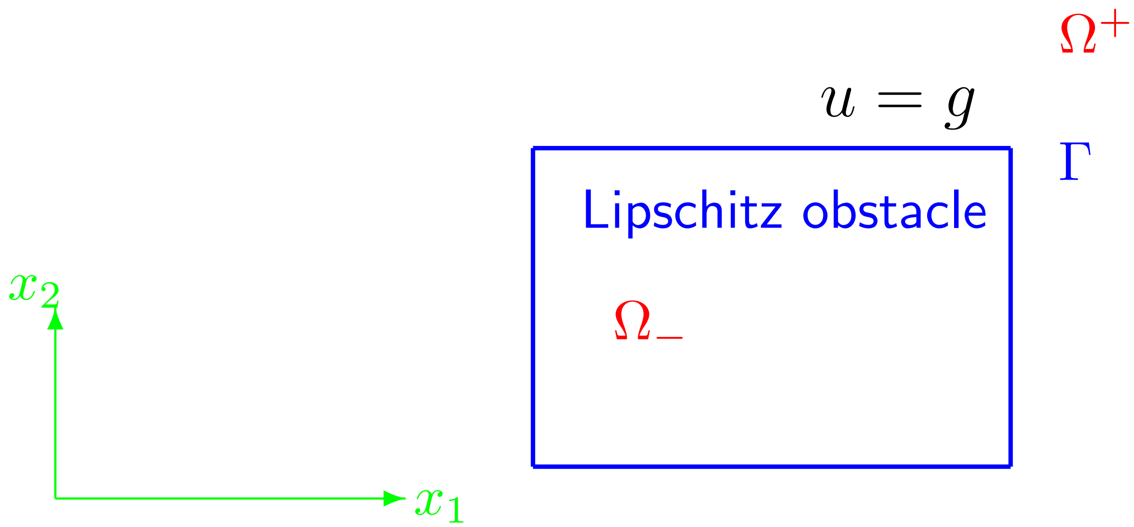
Precisely, we will study dependence of $\|A\|$ and $\|A^{-1}\|$ on k when A is a specific oscillatory boundary integral operator.

Contents

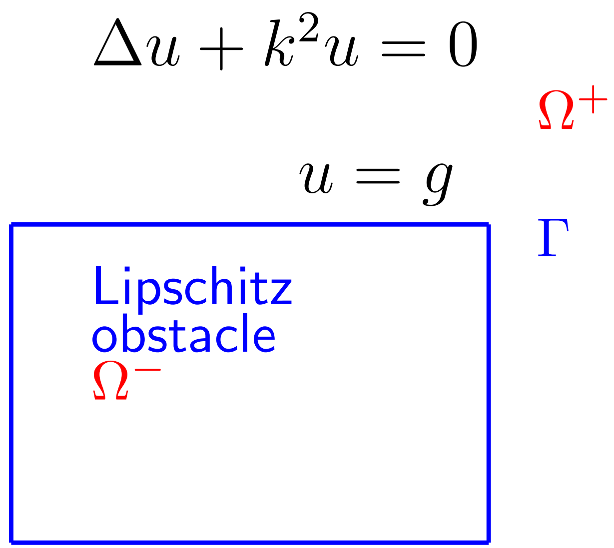
- Exterior Dirichlet problem for time-harmonic wave equation
- The standard combined potential integral equation formulation
- The known results (some very recent, 2007) for circle/sphere
- **New results for general Lipschitz obstacles**

The Exterior Problem in $\mathbb{R}^d$ ($d = 2$ or 3)

$$\Delta u + k^2 u = 0$$



BVP. Find u which satisfies the Helmholtz equation in Ω_+, $u = g$ on Γ, and the Sommerfeld radiation condition $\dfrac{\partial u}{\partial r} - \mathrm{i}ku = o\left(r^{-(d-1)/2}\right)$ as $r = |x| \to \infty$.

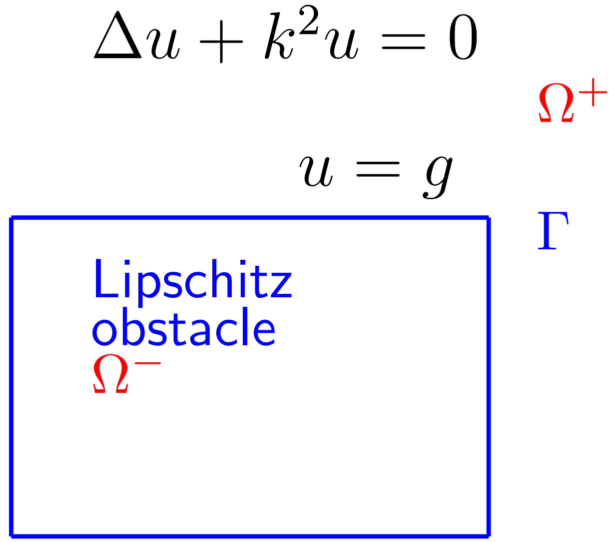


Following Brakhage & Werner, Leis, Panich (1965), look for a solution as a **combined single and double-layer potential**, with density φ and **coupling parameter** $\eta \in \mathbb{R}$, i.e. in the form

$$u(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)} \varphi(y) ds(y) - \mathrm{i}\eta \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Omega_+,$$

where

$$\Phi(x,y) := \tfrac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \approx C \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{(k|x-y|)^{1/2}} \text{ (2D)}, \quad := \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$



$$u(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)} \varphi(y) ds(y) - \mathrm{i}\eta \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Omega_+.$$

satisfies the BVP iff **the trace of** u **on** Γ **is** g, i.e.

$$g(x) = \frac{1}{2}u(x) + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial n(y)} \varphi(y) ds(y) - \mathrm{i}\eta \int_\Gamma \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \Gamma,$$

in operator form

$$A\varphi = 2g, \quad A := I + D - \mathrm{i}\eta S.$$

$$\boxed{A := I + D - \mathrm{i}\eta S}$$

where I is the identity operator, $\eta \in \mathbb{R}$ the coupling parameter, and, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

$$S\varphi(x) = 2\int_\Gamma \Phi(x,y)\varphi(y)\,ds(y), \quad D\varphi(x) = 2\int_\Gamma \frac{\partial\Phi(x,y)}{\partial n(y)}\varphi(y)ds(y).$$

Boundedness and Invertibility.

$$A : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$$

and is bounded for $|s| \leq 1/2$; further this mapping is a bijection, for $\eta \neq 0$ (Mitrea (1996), C-W & Langdon (2007)).

$$\boxed{A := I + D - \mathrm{i}\eta S}$$

Wave Number Dependence. But how do $\|A\|$ and $\|A^{-1}\|$ depend on k, especially as $k \to \infty$, and how should we choose η?

Engineering practice is to take $\eta = k$, but is this optimal in terms of minimising $\mathrm{cond}\, A$?

We will address these questions for A as an operator on $H^0(\Gamma) = L^2(\Gamma)$.

The Case of a Circle/Sphere (Kress & Spassov '83, Kress '85)

Spherical harmonics are the eigenfunctions of A and A is normal.

The eigenvalues for the unit circle (Kress 1985) are

$$\lambda_m = \pi \mathrm{i} H_m^{(1)}(k)[kJ_m'(k) - \mathrm{i}\eta J_m(k)], \quad m = 0, 1, ...,$$

and, as operators on $L^2(\Gamma)$,

$$\|A\| = \sup_m |\lambda_m|, \;\; \|A^{-1}\| = \frac{1}{\inf_m |\lambda_m|}, \;\; \mathrm{cond}\, A = \frac{\sup_m |\lambda_m|}{\inf_m |\lambda_m|}.$$

Dominguez, Graham, Smyshlyaev, Numer. Math., 2007 If Γ is a circle, and $\eta = k$, then, for all sufficiently large k, A is elliptic on $L^2(\Gamma)$, precisely

$$\inf_m \Re\lambda_m \geq 1 \;\Rightarrow\; \Re(A\phi, \phi) \geq \|\phi\|^2,$$

so that $\|A^{-1}\| \leq 1$.

C-W & Monk to appear SIAM J. Math. Anal. $1 \lesssim \|A^{-1}\| \lesssim 1 + \dfrac{k}{\eta}$.

Banjai, Sauter, SIAM J. Numer. Anal., 2007

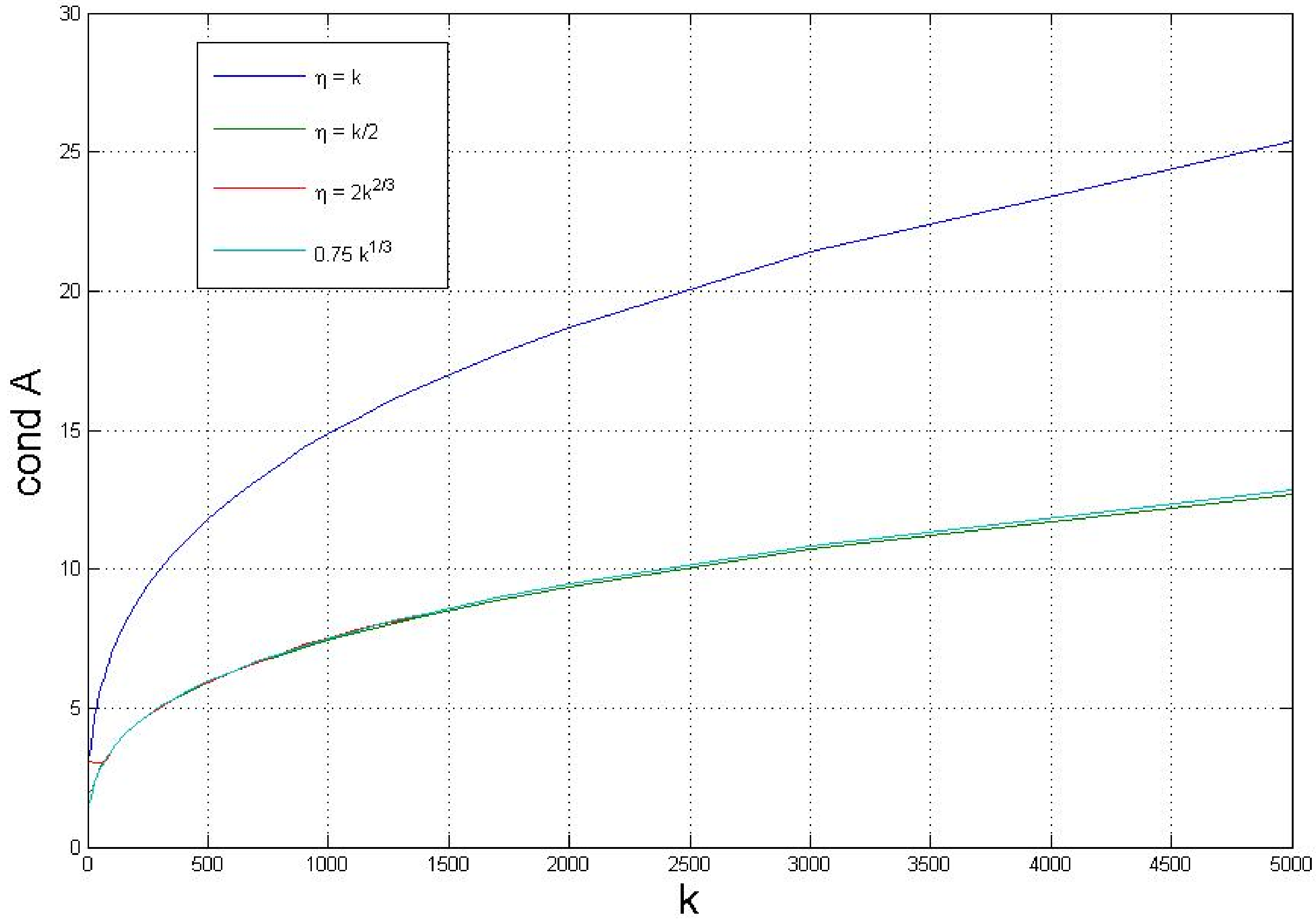
$$\|D\| \approx 1, \quad \|S\| \approx k^{-2/3} \text{ so } \|A\| \lesssim 1 + \eta k^{-2/3}.$$

so, for $\eta \approx k^{\epsilon}$, $\frac{2}{3} \leq \epsilon \leq 1$, it holds that

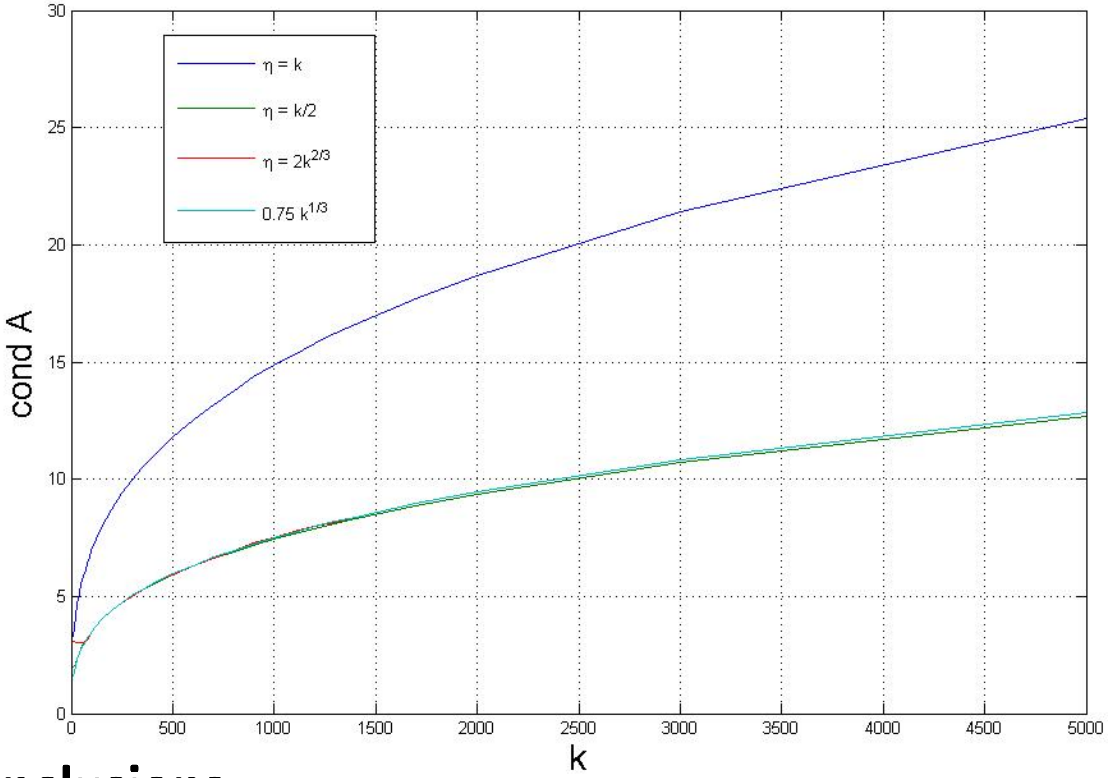
$$\mathrm{cond}\, A \lesssim 1 + k^{1/3},$$

with equality for $\eta \approx k$.

Unit Circle: Plot of $\operatorname{cond} A$ against k for choices of η



Unit Circle: Plot of $\mathrm{cond}\, A$ against k for choices of η



Conclusions.

(i) $\mathrm{cond}\, A$ grows only slowly (like $k^{1/3}$) for circle sphere: $\mathrm{cond}\, A \approx 100$ for $k \approx 5 \times 10^6$.

(ii) $\mathrm{cond}\, A$ is almost minimised by every η between $0.75k^{2/3}$ and $k/2$.

What Happens for Geometries Other than Circle/Sphere: Some New Results

(C-W, Graham, Langdon, Lindner, in preparation.)

Simple Upper Bounds, by interpolation, which ignore oscillation. E.g.

$$\|S\|_2 \leq \max(\|S\|_1, \|S\|_\infty) = \text{ess sup}_{x\in\Gamma} \int_\Gamma |\Phi(x,y)|\, ds(y) \lesssim 1$$

in **3D**, since

$$|\Phi(x,y)| = \left| \frac{\exp(\mathrm{i}k|x-y|)}{4\pi|x-y|} \right| = \frac{1}{4\pi|x-y|}.$$

Similarly, in 3D,

$$\|D\|_2 \lesssim k,$$

which compares with

$$\|S\| \lesssim k^{-2/3}, \quad \|D\| \lesssim 1$$

for a sphere.

In **2D**

$$\|S\|_2 \leq \max(\|S\|_1, \|S\|_\infty) = \text{ess sup}_{x\in\Gamma} \int_\Gamma |\Phi(x,y)|\, ds(y) \lesssim k^{-1/2}$$

since

$$|\Phi(x,y)| = \frac{1}{4}|H_0^{(1)}(k|x-y|)| \leq Ck^{-1/2}|x-y|^{-1/2}.$$

Similarly,

$$\|D\|_2 \lesssim k^{1/2},$$

compared to

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for a circle.

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Similarly,

$$\|D\|_2 \lesssim k^{1/2},$$

compared to

$$\|S\| \lesssim k^{-2/3}, \quad \|D\| \lesssim 1$$

for a circle. **But these simple bounds can be sharp.**

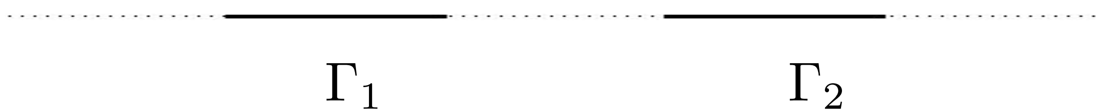
Lemma. If Γ contains a straight line segment, then

$$\|S\| \approx k^{-1/2}$$

so that, for $\eta \approx k$,

$$\|A\| = \|I + D - \mathrm{i}\eta S\| \approx 1 + k^{1/2}.$$

Proof.



Take $\varphi(y) = \exp(\mathrm{i}ky_1)$ on Γ_1, $= 0$ elsewhere. Then, for $x \in \Gamma_2$,

$$S\varphi(x) = \int_{\Gamma_1} \Phi(x,y)\exp(\mathrm{i}ky_1)dy_1 \approx C\int_{\Gamma_1} \frac{\exp(\mathrm{i}k(x_1-y_1))}{k^{1/2}(x_1-y_1)^{1/2}}\exp(\mathrm{i}ky_1)dy_1.$$

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Take $\varphi(y) = \exp(\mathrm{i}ky_1)$ on Γ_1, $= 0$ elsewhere. Then, for $x \in \Gamma_2$,

$$|S\varphi(x)| = \left| \int_{\Gamma_1} \Phi(x, y) \exp(\mathrm{i}ky_1) dy_1 \right| \geq Ck^{-1/2}.$$

Lemma. In **2D**, if Γ contains a $C^{1,1}$ line element then

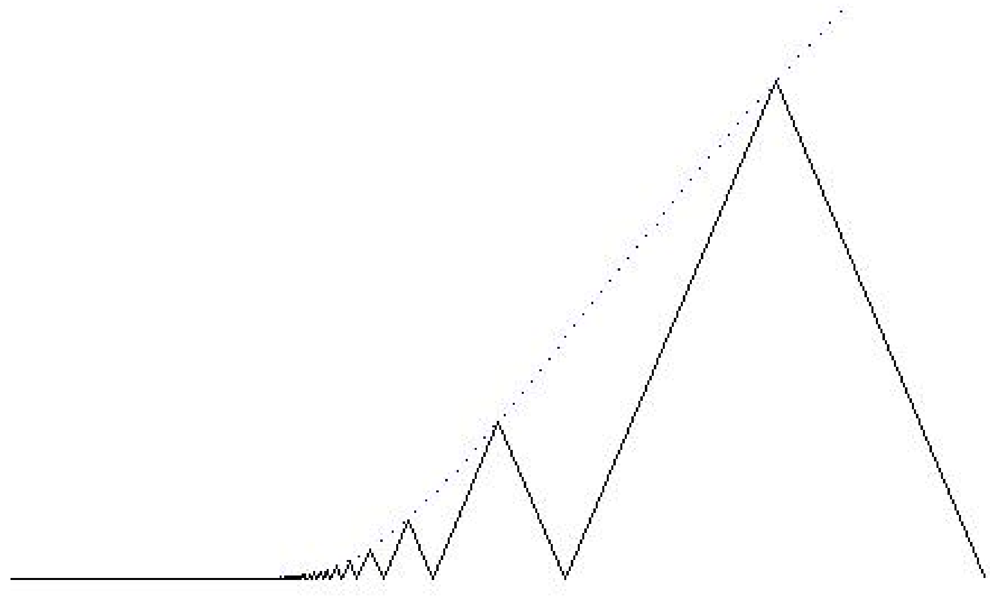
$$k^{-2/3} \lesssim \|S\| \lesssim k^{-1/2}.$$

If a $C^{2,1}$ line element with a point where the curvature vanishes, then

$$k^{-3/5} \lesssim \|S\| \lesssim k^{-1/2},$$

etc.

Lemma. If Γ has the form

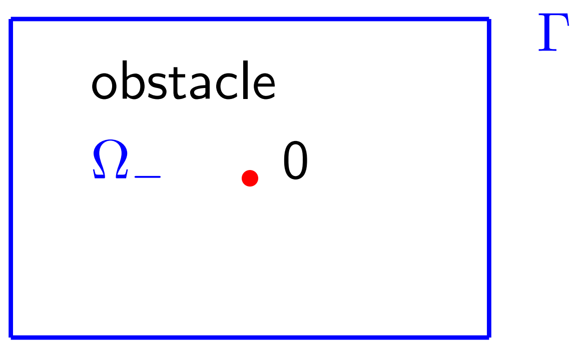


then, for every $\epsilon > 0$,

$$k^{1/2-\epsilon} \lesssim \|D\| \lesssim k^{1/2},$$

while if Γ is a polygon then $k^{1/4} \lesssim \|D\| \lesssim k^{1/2}$.

Bounding $\|A^{-1}\|$ for a Starlike Obstacle



Let $n(x)$ denote the outward unit normal at $x \in \Gamma$, and

$$R_0 := \max_{x \in \Gamma} |x|, \quad \delta_- := \text{ess.} \inf_{x \in \Gamma} \; x \cdot n(x).$$

Let $n(x)$ denote the outward unit normal at $x \in \Gamma$, and

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Theorem. (C-W & Monk, to appear SIAM J. Math. Anal.) If Ω^- is a polyhedron which is starlike with respect to the origin (i.e. $\delta_- > 0$), or a more general piecewise smooth, Lipschitz, starlike domain, and if $kR_0 \geq 1$, then

$$\|A^{-1}\| \leq 1 + 2\theta + 2\theta^2 + \frac{k}{\eta}(3\theta + \theta^2),$$

where $\theta := R_0/\delta_-$.

Examples.

Circle/sphere: $\theta = 1$, $\|A^{-1}\| \leq 5 + 4\dfrac{k}{\eta}$.

Cube: $\theta = \sqrt{3}$, $\|A^{-1}\| \leq 10 + 8\dfrac{k}{\eta}$.

The Main Ingredients in the Proof

1. Green's theorem and a Rellich(-Payne-Weinberger-Nečas) type identity.

2. A property of radiating solutions of the Helmholtz equation, that, if v is radiating and Γ_R is the boundary of the sphere of radius R, then

$$\Re \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r}\, ds + R \int_{\Gamma_R} \left(k^2 |v|^2 + \left| \frac{\partial v}{\partial r} \right|^2 - |\nabla_T v|^2 \right) ds \leq 2kR\, \Im \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r}\, ds.$$

Summary of What is Known About $A = I + D - \mathrm{i}\eta S$

1. Circle/Sphere

For $\eta \approx k$ and $k \gtrsim 1$,

$$\|A\| \approx k^{1/3}, \quad \|A^{-1}\| \approx 1, \quad \mathrm{cond}\, A \approx k^{1/3}$$

and $\mathrm{cond}\, A \lesssim k^{1/3}$ also for $\eta \approx k^{\epsilon}$, $\frac{2}{3} \leq \epsilon \leq 1$.

2. General Piecewise Smooth Lipschitz Domains

$$1 \leq \|A^{-1}\| \lesssim 1 + \frac{k}{\eta} \quad \textbf{(starlike case)}$$

and, in 3D,

$$\|A\| \lesssim k + \eta.$$

so for $\eta \approx k$ and Γ starlike,

$$\mathrm{cond}\, A \lesssim k.$$

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2. General Piecewise Smooth Lipschitz Domains

$$1 \leq \|A^{-1}\| \lesssim 1 + \frac{k}{\eta} \quad \textbf{(starlike case)}$$

and, in 2D,

$$\|A\| \lesssim k^{1/2} + k^{-1/2}\eta.$$

so for $\eta \approx k$ and Γ starlike,

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and $\mathrm{cond}\, A \lesssim k^{1/3}$ also for $\eta \approx k^{\epsilon}$, $\frac{2}{3} \leq \epsilon \leq 1$.

2. Polygon

$$1 \leq \|A^{-1}\| \lesssim 1 + \frac{k}{\eta} \quad \textbf{(starlike case)}$$

and, in 2D, for $\eta \approx k$

$$\|A\| \approx k^{1/2}.$$

so for $\eta \approx k$ and Γ starlike,

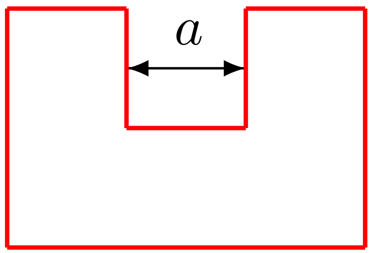
$$\mathrm{cond}\, A \approx k^{1/2}.$$

What if Γ is not starlike?

In **2D**, if Γ contains two parallel line segments separated by the domain of propagation, distance a apart, then

$$ka \in \pi\mathbb{N} \Rightarrow \|A^{-1}\| \gtrsim k^{3/4} \text{ so } \mathrm{cond}\, A \gtrsim k^{5/4} \text{ for } \eta \approx k\ .$$

Cf. $\mathrm{cond}\, A \approx k^{1/3}$ for a circle/sphere.



k	$\|A\|$	p	$\|A^{-1}\|$	p
5	2.96	0.21	0.94	0.00
10	3.41	0.33	0.94	0.00
20	4.30	0.38	0.94	0.00
40	5.58	0.42	0.94	0.00
80	7.46	0.44	0.94	-0.00
160	10.12		0.94	

Square.

k	$\|A\|$	p	$\|A^{-1}\|$	p
5	4.58	0.04	1.84	0.54
10	4.70	0.16	2.68	0.57
20	5.27	0.18	3.97	0.65
40	5.98	0.24	6.23	0.72
80	7.05	0.27	10.24	0.72
160	8.49		16.92	

Trapping domain.

Open Problems

1. For smooth, strictly convex obstacle in 2D $k^{-2/3} \lesssim \|S\| \lesssim k^{-1/2}$. Can we sharpen this, e.g. by

$$\|S\|_2 \leq \|S^* S\|_2^{1/2} \leq \max(\|S^* S\|_1^{1/2}, \|S^* S\|_\infty^{1/2})$$

   (cf. Stein, Harmonic Analysis, 1993).

2. Sharper estimates than $\|S\| \lesssim 1$ for the 3D case ($\|S\| \lesssim k^{-2/3}$ for sphere).

3. Any wave-number-explicit upper bounds on $\|A^{-1}\|$ for non starlike obstacles.

4. Any wave-number-explicit bounds in the discrete case.

5. The case where η is replaced by an operator (see work of Antoine, Buffa, Bruno, Hiptmair, Levadoux, Sauter, etc.).