

Plane Wave Discontinuous Galerkin Methods

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work in progress

1.1 Model Problems

☞ Propagation of time-harmonic acoustic waves in homogeneous medium

➤ **Helmholtz equation**

- Given:
- ▷ angular frequency (wave number) $\omega > 0$ ($[\omega] = \text{m}^{-1}$)
 - ▷ **bounded** computational domain $\Omega \subset \mathbb{R}^2$
 - ▷ source term $f \in L^2(\Omega)$ (supported in parts of Ω)

- 1D model problem:

$$-u'' - \omega^2 u = f \quad \text{in }]-1, 1[, \quad \pm u'(\pm 1) - i\omega u(\pm 1) = 0. \quad (1.1.1)$$

- 2D model problem: $\Omega \subset \mathbb{R}^2$ bounded polygon

$$\begin{aligned} -\Delta u - \omega^2 u &= f \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} - i\omega u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.1.2)$$

“Real world”:

$$\begin{aligned} -\nabla \cdot (\mathbf{A}(\mathbf{x}) \nabla u) - \omega^2 n(\mathbf{x}) u &= f \quad \text{in } \Omega, \\ \text{PML-ABC at cut-off boundary } \partial\Omega. \end{aligned}$$

1.2 Discontinuous Galerkin Discretization

(1.1.2) \triangleright first order system:

$$\begin{aligned}
 i\omega \boldsymbol{\sigma} &= \nabla u && \text{in } \Omega, \\
 i\omega u - \nabla \cdot \boldsymbol{\sigma} &= \frac{1}{i\omega} f && \text{in } \Omega, \\
 i\omega \boldsymbol{\sigma} \cdot \mathbf{n} + i\omega u &= g && \text{on } \partial\Omega.
 \end{aligned}
 \tag{1.2.1}$$

+ partitioning $\mathcal{T}_h = \{K\}$ of Ω into cells (\rightarrow finite element mesh)

 \leftarrow test & integrate by parts locally

$$\begin{aligned}
 \int_K i\omega \boldsymbol{\sigma} \cdot \overline{\boldsymbol{\tau}} dV + \int_K u \overline{\nabla \cdot \boldsymbol{\tau}} dV - \int_{\partial K} u \overline{\boldsymbol{\tau} \cdot \mathbf{n}} dS &= 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\text{div}; K) \\
 \int_K i\omega u \overline{v} dV + \int_K \boldsymbol{\sigma} \cdot \overline{\nabla v} dV - \int_{\partial K} \boldsymbol{\sigma} \cdot \mathbf{n} \overline{v} dS &= \frac{1}{i\omega} \int_K f \overline{v} dV \quad \forall v \in H^1(K).
 \end{aligned}
 \tag{1.2.2}$$

+ numerical fluxes $\hat{u}, \hat{\boldsymbol{\sigma}}$ on cell interfaces

$$\begin{aligned}
 \int_K i\omega \boldsymbol{\sigma}_h \cdot \overline{\boldsymbol{\tau}}_h dV + \int_K u_h \overline{\nabla \cdot \boldsymbol{\tau}}_h dV - \int_{\partial K} \hat{u}_h \overline{\boldsymbol{\tau}_h \cdot \mathbf{n}} dS &= 0 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h(K) \\
 \int_K i\omega u_h \overline{v}_h dV + \int_K \boldsymbol{\sigma}_h \cdot \overline{\nabla v}_h dV - \int_{\partial K} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \overline{v}_h dS &= \frac{1}{i\omega} \int_K f \overline{v}_h dV \quad \forall v_h \in V_h(K).
 \end{aligned}
 \tag{1.2.3}$$

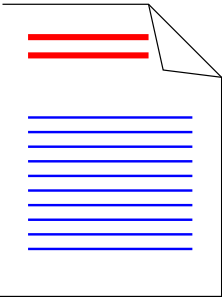
Here: $\Sigma_h(K) \rightarrow$ trial/test space for cell fluxes
 $V_h(K) \rightarrow$ local trial/test space for u

Note: interelement continuity conditions (\rightarrow discontinuous Galerkin)

Traditional choice: polynomial spaces $\Sigma_h(K), V_h(K)$ (fixed or variable degree)

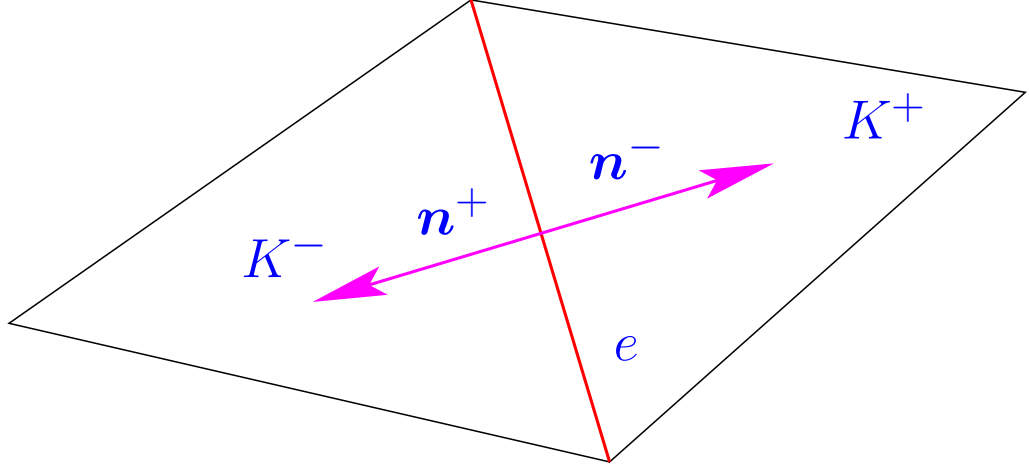
D. Arnold, F. Brezzi, B. Cockburn and L. Marini (2002), ‘Unified analysis of discontinuous Galerkin methods for elliptic problems’, *SIAM J. Numer. Anal.* **39**(5), 1749–1779.

P. Castillo, B. Cockburn, I. Perugia and D. Schötzau (2000), ‘An a priori error analysis of the local discontinuous galerkin method for elliptic problems’, *SIAM J. Numer. Anal.* **38**(5), 1676–1706.



“DG-speak”

averages: $\{u_h\} := \frac{1}{2}(u_h^+ + u_h^-),$
 $\{\sigma_h\} := \frac{1}{2}(\sigma_h^+ + \sigma_h^-),$
jumps: $[[u_h]] := u_h^+ \mathbf{n}^+ + u_h^- \mathbf{n}^-,$
 $[[\sigma_h]] := \sigma_h^+ \cdot \mathbf{n}^+ + \sigma_h^- \cdot \mathbf{n}^-.$



Common numerical fluxes (with polynomial spaces):

penalty parameter

interior penalty (IP) DG: $\hat{u}_h = \{u_h\}$, $\hat{\sigma}_h = \{\nabla u_h\} - \alpha(\mathbf{x})[u_h]$,

local DG (LDG): $\hat{u}_h = \{u_h\} - \beta[u_h]$, $\hat{\sigma}_h = \{\sigma_h\} - \beta[u_h] - \alpha[\sigma_h]$.

mixed DG: $\hat{u}_h = \{u_h\} + \gamma \cdot [u_h] - \beta[\sigma_h]$, $\hat{\sigma}_h = \{\sigma_h\} - \alpha[u_h] - \gamma[\sigma_h]$.

Terminology: $\hat{u}_h = \hat{u}_h(\{u_h\}, [u_h])$ ➤ primal DG

1.3 The Ultra-weak Connection

Ultra-weak variational formulation ➤ March 2007 HOP programme presentation by Peter Monk

O. Cessenat and B. Despres (1998), ‘Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation’, *SIAM J. Numer. Anal.* **35**(1), 255–299.

O. Cessenat and B. Despres (2003), ‘Using plane waves as base functions for solving time harmonic equations with the ultra weak variational formulation’, *J. Computational Acoustics* **11**, 227–238.

T. Huttunen, P. Monk and J. Kaipio (2002), ‘Computational aspects of the ultra-weak variational formulation’, *J. Comp. Phys.* **182**(1), 27–46.

$$\int_K i\omega \boldsymbol{\sigma}_h \cdot \overline{\boldsymbol{\tau}}_h dV + \int_K u_h \overline{\nabla \cdot \boldsymbol{\tau}}_h dV - \int_{\partial K} \hat{u}_h \overline{\boldsymbol{\tau}_h \cdot \mathbf{n}} dS = 0 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h(K) \quad (1.2.3)$$

$$\int_K i\omega u_h \overline{v}_h dV + \int_K \boldsymbol{\sigma}_h \cdot \overline{\nabla v}_h dV - \int_{\partial K} \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \overline{v}_h dS = \frac{1}{i\omega} \int_K f \overline{v}_h dV \quad \forall v_h \in V_h(K) .$$

 ← another integration by parts

$$\int_K \boldsymbol{\sigma}_h \cdot \overline{\boldsymbol{\tau}}_h dV = \frac{1}{i\omega} \int_K \nabla u_h \cdot \overline{\boldsymbol{\tau}}_h dV - \frac{1}{i\omega} \int_{\partial K} (u_h - \hat{u}_h) \overline{\boldsymbol{\tau}_h \cdot \mathbf{n}} dS . \quad (1.3.1)$$

Assume $\nabla_h V_h \subseteq \Sigma_h$

$$\int_K \nabla u_h \cdot \nabla \bar{v}_h - \omega^2 u_h \bar{v}_h dV - \int_{\partial K} (u_h - \hat{u}_h) \overline{\nabla v_h \cdot \mathbf{n}} dS - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \bar{v}_h dS = \int_K f \bar{v}_h dV . \quad (1.3.2)$$

← another integration by parts

$$\int_K (-\Delta v_h - \omega^2 v_h) u_h dV + \int_{\partial K} \hat{u}_h \overline{\nabla v_h \cdot \mathbf{n}} dS - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \bar{v}_h dS = \int_K f \bar{v}_h dV . \quad (1.3.3)$$

Assume

$$-\Delta v_h - \omega^2 v_h = 0 \quad \forall v_h \in V_h(K) , \quad \forall K .$$

$$(1.3.3) \Rightarrow \int_{\partial K} \hat{u}_h \overline{\nabla v_h \cdot \mathbf{n}} dS - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \bar{v}_h dS = \int_K f \bar{v}_h d\mathbf{x} . \quad (1.3.4)$$

Ultra-weak fluxes:

$$\begin{aligned} \hat{\boldsymbol{\sigma}}_h &= \frac{1}{i\omega} \{ \nabla_h u_h \} - \frac{1}{2} [u_h] , \\ \hat{u}_h &= \{ u_h \} - \frac{1}{2i\omega} [\nabla_h u_h] . \end{aligned} \quad (1.3.5)$$

Note: special boundary fluxes required

Note: non-standard fluxes $(\hat{u}_h = \{\{u_h\} - \frac{1}{2i\omega} \llbracket \nabla_h u_h \rrbracket)$

Variational problem on skeleton of \mathcal{T}_h ($\mathcal{F}^{\mathcal{I}} \hat{=}$ interior edges):

$$\begin{aligned} u_h \in V_h: & \sum_{K \in \mathcal{T}_h} \int_{\partial K} (-\nabla u_h \cdot \mathbf{n} + i\omega u_h) \overline{(-\nabla v_h \cdot \mathbf{n} + i\omega v_h)} \, dS \\ & - \int_{\mathcal{F}^{\mathcal{I}}} (-\nabla u_h^- \cdot \mathbf{n}^- + i\omega u_h^-) \overline{(\nabla v_h^+ \cdot \mathbf{n}^+ + i\omega v_h^+)} \, dS \\ & - \int_{\mathcal{F}^{\mathcal{I}}} (-\nabla u_h^+ \cdot \mathbf{n}^+ + i\omega u_h^+) \overline{(\nabla v_h^- \cdot \mathbf{n}^- + i\omega v_h^-)} \, dS \\ & = -2i\omega (f, v_h) + \int_{\mathcal{F}_h^{\mathcal{B}}} g \overline{(\nabla v_h \cdot \mathbf{n} + i\omega v_h)} \, dS, \quad \forall v_h \in V_h. \end{aligned}$$

Trial/test space for the original ultra-weak formulation:

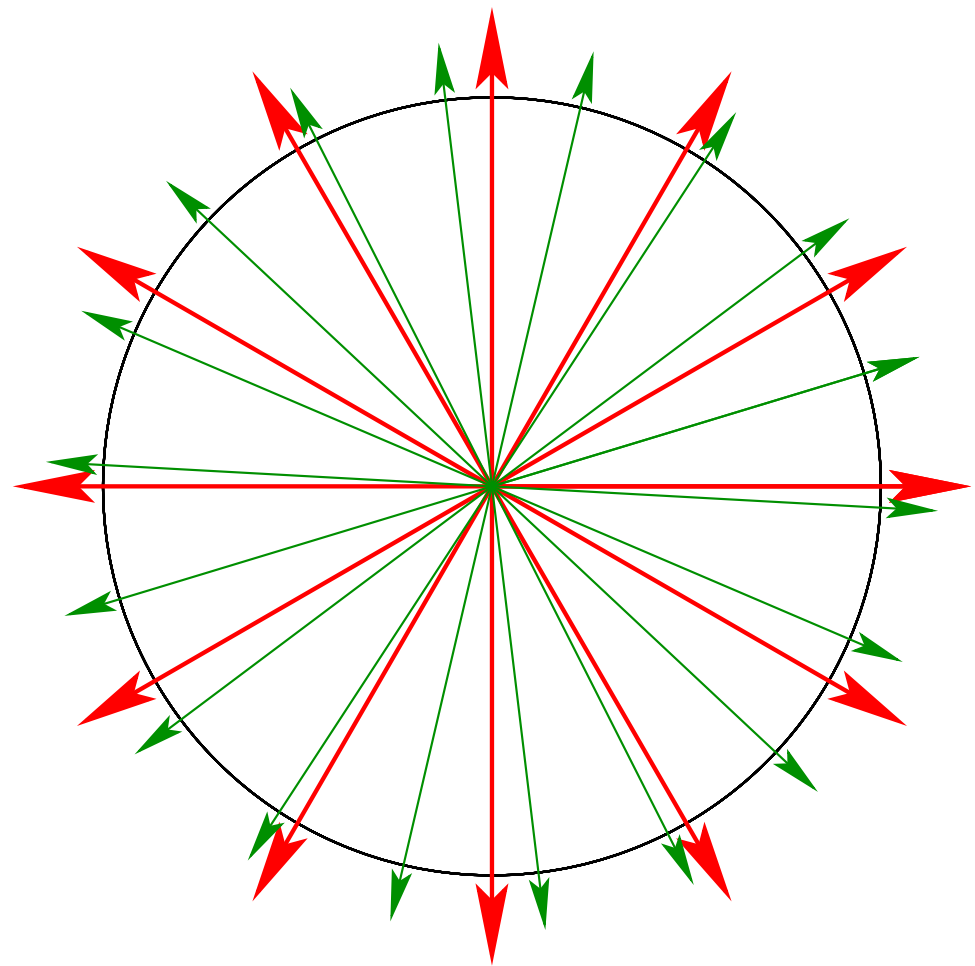
$\mathcal{T}_h \hat{=}$ triangular finite element mesh

plane wave space:

$$V_h(K) := \underbrace{\text{Span} \{ \mathbf{x} \mapsto \exp(i\omega \mathbf{d}_j \cdot \mathbf{x}) \}_{j=1}^p}_{=: W_p},$$

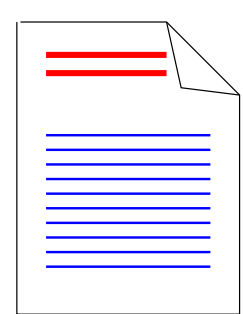
$$\mathbf{d}_j = \begin{pmatrix} \cos\left(\frac{2\pi}{p}(j-1)\right) \\ \sin\left(\frac{2\pi}{p}(j-1)\right) \end{pmatrix}, \quad j = 1, \dots, p.$$

$p \in \mathbb{N} \hat{=}$ no. of plane wave directions



1.4 Plane waves

1.4.1 Approximation of homogeneous Helmholtz solutions



J. Melenk (1995), On Generalized Finite Element Methods, PhD thesis, University of Maryland, USA.

notation: plane wave space $W_p := \text{Span} \{ \mathbf{x} \mapsto \exp(i\omega \mathbf{d}_j \cdot \mathbf{x}) \}_{j=1}^p, p \in \mathbb{N}$

Theorem 1.4.1 (Proposition 8.4.14 in (Melenk 1995)).

D star-shaped Lipschitz domain, exterior angles $\geq \lambda\pi$, $u \in H^s(D)$, $s > 1$, $-\Delta u - \omega^2 u = 0$

$$\inf_{w \in W_p} \|u - w\|_{1,D} \leq C(D, \omega, s) \left(\frac{\log^2 p}{p} \right)^{\lambda(s-1)} \|u\|_{s,D} .$$

Theorem 1.4.2 (Sect. 8.6 of (Melenk 1995)).

$u \hat{=}$ homogeneous Helmholtz solution in bounded domain D , $D_0 \subset\subset D$ compact,

$$\exists C > 0, 0 < q < 1: \quad \inf_{w \in W_p} \|u - w\|_\infty \leq Cq^p \quad \forall p \in \mathbb{N}.$$

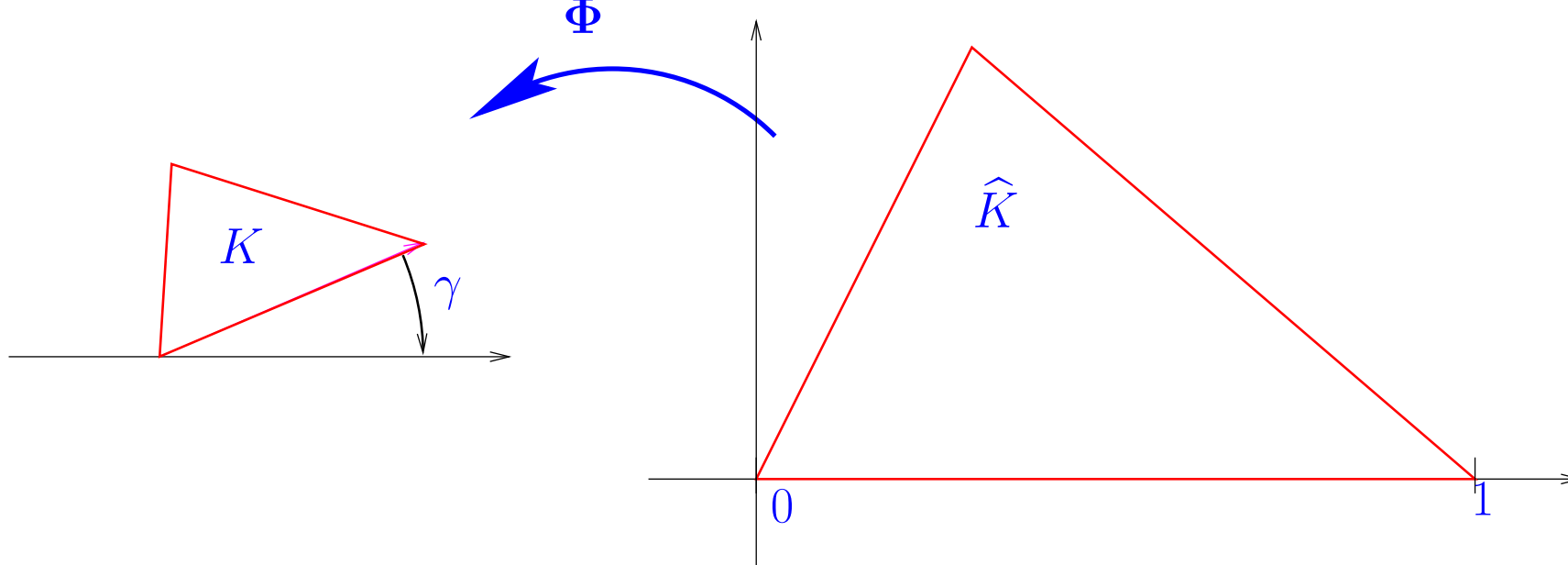
Approximation of
Helmholtz solutions : plane waves \sim spectral polynomial approximation

1.4.2 Estimates for low frequencies

👉 relevant for **h-version** of plane-wave DG

👉 focus: triangular meshes

① Technical tool: *similarity* transformation techniques (cf. finite element theory)



Note: change of frequency $\omega \rightarrow h_K \omega$ under similarity pullback

② A $\omega \rightarrow 0$ **stable** basis of W_p , p **odd**

$$\langle b_1, \dots, b_p \rangle = \underbrace{\langle \mathbf{x} \mapsto \exp(i\omega \mathbf{d}_1 \mathbf{x}), \dots, \mathbf{x} \mapsto \exp(i\omega \mathbf{d}_p \mathbf{x}) \rangle}_{\text{unstable basis}}, \quad r = 1, \dots, p.$$

$$b_j(\mathbf{x}) := (i\omega)^{-[j/2]} \sum_{k=1}^p (\mathbf{C}^{-1})_{kj} \exp(i\omega \mathbf{d}_k \cdot \mathbf{x}), \quad (1.4.1)$$

$$\mathbf{C} := \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ \cos(\varphi_1) & \cos(\varphi_2) & \cos(\varphi_3) & \cdots & \cdots & \cos(\varphi_p) \\ \sin(\varphi_1) & \sin(\varphi_2) & \sin(\varphi_3) & \cdots & \cdots & \sin(\varphi_p) \\ \cos(2\varphi_1) & \cos(2\varphi_2) & \cos(2\varphi_3) & \cdots & \cdots & \cos(2\varphi_p) \\ \sin(2\varphi_1) & \sin(2\varphi_2) & \sin(2\varphi_3) & \cdots & \cdots & \sin(2\varphi_p) \\ \vdots & \vdots & \vdots & & & \vdots \\ \cos(m\varphi_1) & \cos(m\varphi_2) & \cos(m\varphi_3) & \cdots & \cdots & \cos(m\varphi_p) \\ \sin(m\varphi_1) & \sin(m\varphi_2) & \sin(m\varphi_3) & \cdots & \cdots & \sin(m\varphi_p) \end{pmatrix},$$

directional angles: $\mathbf{d}_j := \begin{pmatrix} \cos(\varphi_j) \\ \sin(\varphi_j) \end{pmatrix}, j = 1, \dots, p$

Observation: coefficients $(\mathbf{C}^{-1})_{kj}$ do not depend on ω !

$$b_j(\mathbf{x}) = \frac{2^{-[j/2]}}{[j/2]!} \begin{Bmatrix} \text{Re} \\ \text{Im} \end{Bmatrix} (x + iy)^{[j/2]} + \omega R_j(\omega, \mathbf{x}) \quad \begin{array}{l} \text{for even } j, \\ \text{for odd } j, \end{array} \quad (1.4.2)$$

$R_j(\omega, \mathbf{x})$ uniformly bounded on $\mathbb{R}^+ \times \Omega$.

For $\omega \rightarrow 0$ plane waves approach **harmonic** polynomials

Lemma 1.4.3 (Inverse trace inequality for plane waves).

Assume “shape regularity” of triangle K : minimal angle condition $\alpha_{\min}(K) \geq \alpha_0$

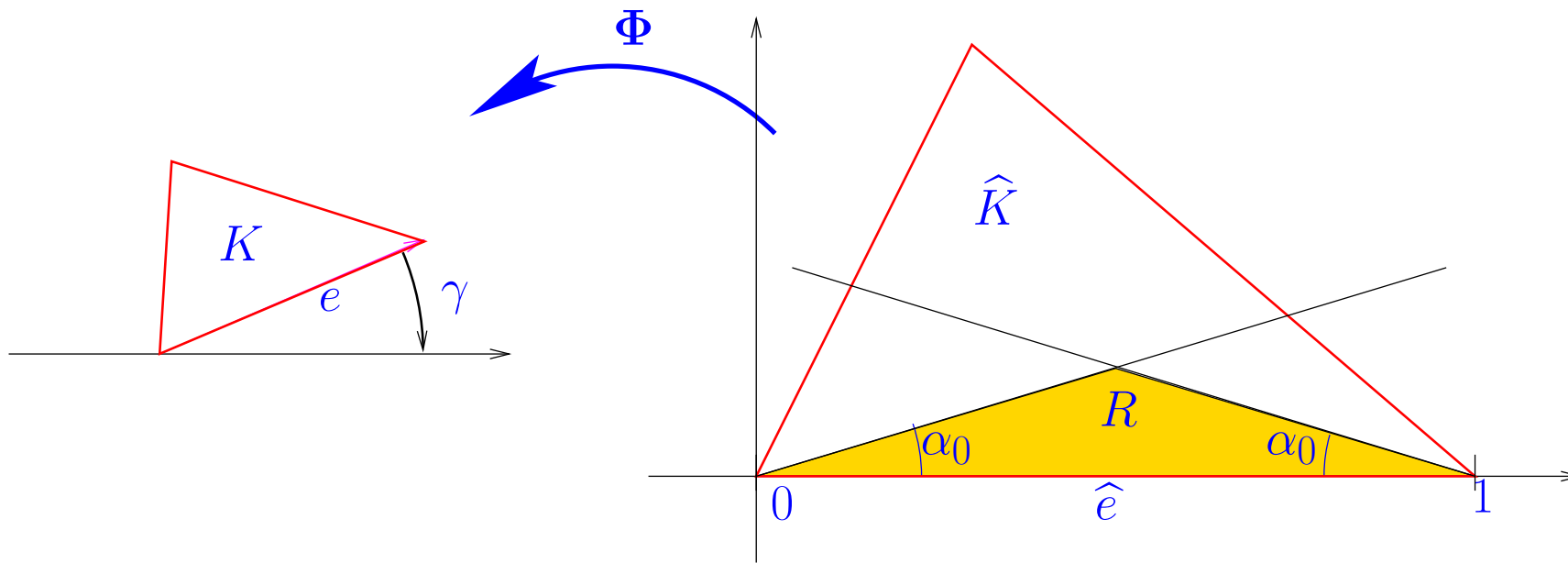
Then $\exists C > 0$ only depending on p

$$\|v\|_{0,\partial K} \leq Ch_K^{-1/2} \|v\|_{0,K} \quad \forall v \in W_p(K), \quad \forall \omega.$$

Proof. ($e \hat{=}$ edge of K)

① similarity mapping $\Phi : \hat{K} \mapsto K$, $\Phi^{-1}(e) = \hat{e} := \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$

② prove inverse trace estimate for \hat{e} and R



③ C^2 = largest eigenvalue $\lambda_{\max}(\omega)$ of generalized eigenvalue problem

$$\mathbf{T}x = \lambda \mathbf{M}x ,$$

$$\mathbf{T}_{jk} = \int_{\hat{e}} w_j(\mathbf{x}) \bar{w}_k(\mathbf{x}) dS \quad , \quad \mathbf{M}_{jk} = \int_R w_j(\mathbf{x}) \bar{w}_k(\mathbf{x}) d\mathbf{x} ,$$

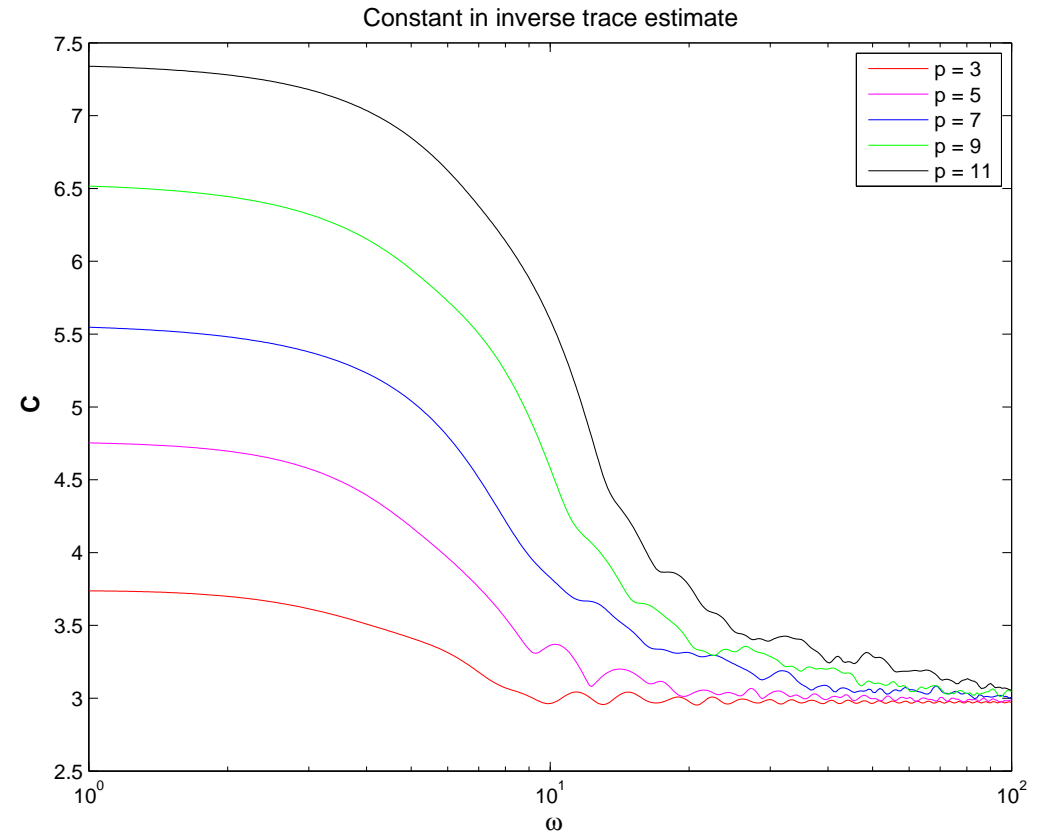
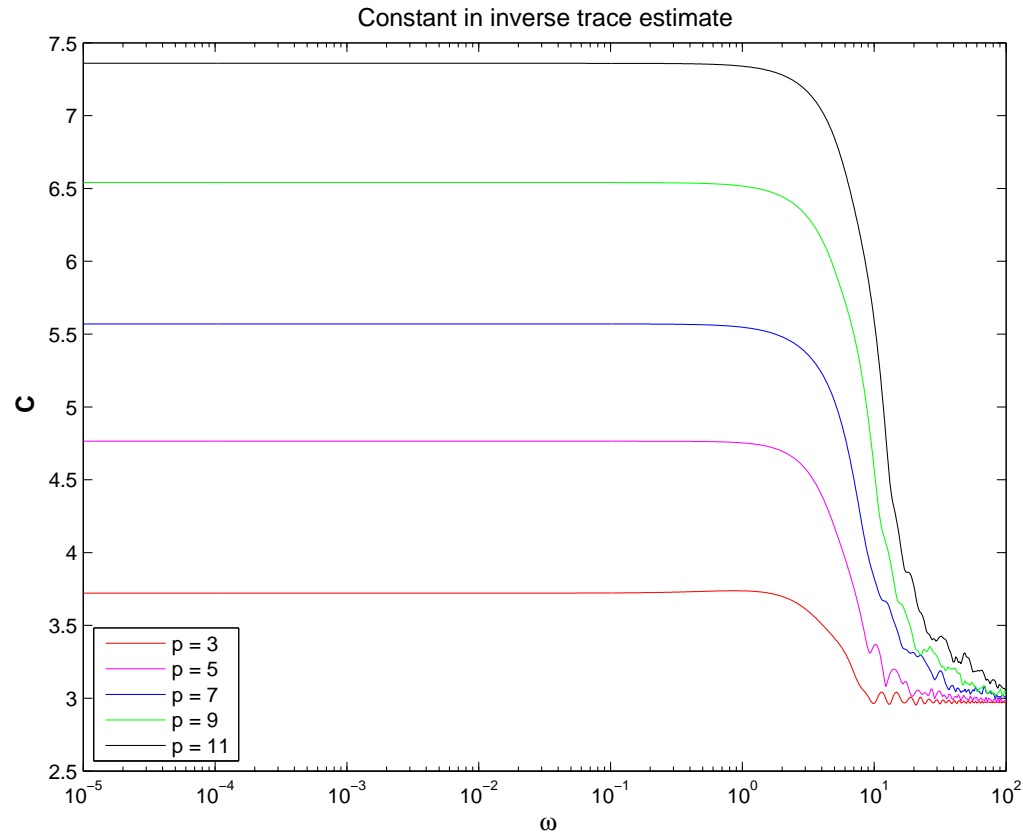
$\{w_1, \dots, w_p\} \hat{=}$ any basis of W_p (\rightarrow does not affect eigenvalues !)

④ use stable basis for small ω : \triangleright positive limit $\lim_{\omega \rightarrow 0} \lambda_{\max}(\omega)$

⑤ use exponential basis for large ω : \triangleright $\lim_{\omega \rightarrow 0} \lambda_{\max}(\omega) < \infty$

⑥ $\omega \rightarrow \lambda_{\max}(\omega)$ continuous $\Rightarrow C^2 := \sup_{\omega \geq 0} \lambda_{\max}(\omega) < \infty$ □

Numerical experiment: inverse trace estimate for unit triangle



Theorem 1.4.4 (Inverse estimate for plane waves).

$$\exists C = C(\alpha_0, p): \quad \|\nabla v\|_{0,K} \leq Ch_K^{-1} \|v\|_{0,K} \quad \forall v \in W_p.$$

Proof. transformation & estimate (using inverse trace estimate) on reference triangle K

$$\begin{aligned}
 \int_{\hat{K}} |\nabla v|^2 d\mathbf{x} &= - \int_{\hat{K}} \Delta v \cdot \bar{v} d\mathbf{x} + \int_{\partial\hat{K}} \nabla v \cdot \mathbf{n} \bar{v} dS \\
 &\leq \omega^2 \int_{\hat{K}} |v|^2 d\mathbf{x} + \|\nabla v\|_{0,\partial\hat{K}} \|v\|_{0,\partial\hat{K}} \\
 &\leq \omega^2 \|v\|_{0,\hat{K}}^2 + C \|\nabla v\|_{0,\hat{K}} \|v\|_{0,\hat{K}} \\
 &\leq \left(\omega^2 + \frac{C^2}{2} \right) \|v\|_{0,\hat{K}}^2 + \frac{1}{2} \|\nabla v\|_{0,\hat{K}}^2 .
 \end{aligned}$$



Theorem 1.4.5 (Projection estimates for plane waves).

$Q_\omega \hat{=} L^2(K)$ -orthogonal projection onto $V_h(K)$. For odd $p \geq 5$

$$\begin{aligned}
 \exists C = C(p, \alpha_0): \quad &\|(Id - Q_\omega)u\|_{0,K} \leq Ch_K(|u|_{1,K} + \omega^2 \|u\|_{0,K}) \quad \forall u \in H^1(K) , \\
 &|(Id - Q_\omega)u|_{1,K} \leq Ch_K(|u|_{2,K} + \omega^2 \|u\|_{0,K}) \quad \forall u \in H^2(K) .
 \end{aligned}$$

1.5 Coercivity

Now: $V_h \hat{=}$ piecewise plane wave space on triangular mesh \mathcal{T}_h

$$\int_K \nabla u_h \cdot \nabla \bar{v}_h - \omega^2 u_h \bar{v}_h dV - \int_{\partial K} (u_h - \hat{u}_h) \overline{\nabla v_h \cdot \mathbf{n}} dS - \int_{\partial K} i\omega \hat{\boldsymbol{\sigma}}_h \cdot \mathbf{n} \bar{v}_h dS = \int_K f \bar{v}_h dV . \quad (1.3.2)$$

+ generalized UWF fluxes:
(boundary fluxes ignored)

$$\hat{\boldsymbol{\sigma}}_h = \frac{1}{i\omega} \{ \nabla_h u_h \} - \alpha [u_h] , \quad \alpha, \beta > 0 . \quad (1.5.1)$$

$$\hat{u}_h = \{ u_h \} - \frac{\beta}{i\omega} [\nabla_h u_h] .$$

+ “DG magic formula”

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial \mathbf{n}_K} v dS = \int_{\mathcal{F}_h} \{ \nabla u \} [v] dS + \int_{\mathcal{F}_h^I} [\nabla u] \{ v \} dS . \quad (1.5.2)$$

notation: $\mathcal{F}_h / \mathcal{F}_h^I \hat{=}$ edges/interior edges of \mathcal{T}_h

$$(1.3.2) \Leftrightarrow u_h \in V_h: a_h(u_h, v_h) - \omega^2 (u_h, v_h) = (f, v_h) + \{ \text{boundary terms} \} \quad \forall v_h \in V_h .$$

$$\begin{aligned}
a_h(u, v) = & (\nabla_h u, \nabla_h v) - \int_{\mathcal{F}_h^I} [u] \cdot \{\{\overline{\nabla_h v}\}\} dS - \int_{\mathcal{F}_h^I} \{\{\nabla_h u\}\} \cdot [\bar{v}] dS \\
& - \int_{\mathcal{F}_h^B} (1 - \delta) u \overline{\nabla_h v} dS - \int_{\mathcal{F}_h^B} (1 - \delta) \nabla_h u \cdot \mathbf{n} \bar{v} dS \\
& - \frac{1}{i\omega} \int_{\mathcal{F}_h^I} \beta [\nabla_h u] [\overline{\nabla_h v}] dS - \frac{1}{i\omega} \int_{\mathcal{F}_h^B} (1 - \delta) \nabla_h u \cdot \mathbf{n} \overline{\nabla_h v \cdot \mathbf{n}} dS \\
& + i \int_{\mathcal{F}_h^I} \alpha [u] \cdot [\bar{v}] dS + i\omega \int_{\mathcal{F}_h^B} \delta u \bar{v} dS, \quad \alpha, \beta, \delta > 0.
\end{aligned}$$

$u_h \mapsto |\operatorname{Im} a_h(u_h, u_h)|$ is a norm on V_h

► existence/uniqueness of solutions of discretized problem

Lemma 1.4.3. [Inverse trace estimate for plane waves]

$$\exists C_{\text{inv}} > 0: \quad \|\nabla v_h\|_{0, \partial K} \leq C_{\text{inv}} h_k^{-\frac{1}{2}} \|\nabla v_h\|_{0, K} \quad \forall v_h \in W_p, \forall K \in \mathcal{T}_h,$$

with C_{inv} only depending on shape regularity of K .

Tool: mesh dependent “DG-norm”:

$$\|v\|_{DG}^2 = \|\nabla_h v\|_{0,\Omega}^2 + \omega^{-1} \left\| \beta^{1/2} [\nabla_h v] \right\|_{0,\mathcal{F}_h^I}^2 + \left\| \alpha^{1/2} [v]_N \right\|_{0,\mathcal{F}_h^I}^2 + \omega^2 \|v\|_{0,\Omega}^2$$

does not match original UWVF !

Theorem 1.5.1. If $\alpha := \frac{a}{h}$ ($h \hat{=}$ local meshwidth), $a > C_{\text{inv}}$, $\delta > \frac{1}{2}$

$$\exists C > 0: \quad |a_h(u_h, u_h) + \omega^2(u_h, u_h)| \geq C \|u_h\|_{DG}^2 \quad \forall u_h \in V_h,$$

with $C > 0$ only depending on shape regularity of \mathcal{T}_h .

1.6 Duality techniques

Tool: strong mesh dependent “DG norm”

$$\|v\|_{DG+}^2 = \|v\|_{DG}^2 + \omega \left\| \beta^{-1/2} \{v\} \right\|_{0,\mathcal{F}_h^I}^2 + \left\| a^{-1/2} h^{1/2} \{ \nabla_h v \} \right\|_{0,\mathcal{F}_h^I}^2.$$

A. Schatz (1974), 'An observation concerning Ritz-Galerkin methods with indefinite bilinear forms', *Math. Comp.* **28**, 959–962.

L. Banjai and S. Sauter (2006), A refined Galerkin error and stability analysis for highly indefinite variational problems, Report 03-06, Institut für Mathematik, Universität Zürich, Zürich, Switzerland.

Lemma 1.6.1. With $C > 0$ only depending on shape regularity and p

$$\|u - u_h\|_{DG} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{DG} + C\omega \sup_{0 \neq w_h \in V_h} \frac{|(u - u_h, w_h)|}{\|w_h\|_{0,\Omega}}.$$

attack with **duality estimates**

adjoint problem : $w_h \in V_h$:
$$\begin{aligned} -\Delta\phi - \omega^2\phi &= w_h && \text{in } \Omega, \\ -\nabla\phi \cdot \mathbf{n} + i\omega\phi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.6.1}$$

[consistency & adjoint consistency]



⋮



$$|(u - u_h, w_h)| \leq C \|u - u_h\|_{DG} \|\phi - Q_\omega \phi\|_{DG^+} + |(f - Q_\omega f, \phi - Q_\omega \phi)| .$$

$Q_\omega \hat{=} L^2(\Omega)$ -orthogonal projection onto V_h (piecewise definition)

Theorem 1.6.2 (Proposition 8.1.4 in (Melenk 1995)).

$$\Omega \text{ convex} \Rightarrow \begin{aligned} |\phi|_{1,\Omega} + \omega \|\phi\|_{0,\omega} &\leq C(\Omega) \|w_h\|_{0,\Omega} , \\ |\phi|_{2,\Omega} &\leq C(\Omega)(1 + \omega) \|w_h\|_{0,\Omega} . \end{aligned}$$

conjecture



$$\|\phi - Q_\omega \phi\|_{DG^+} \leq Ch\omega \|w_h\|_{0,\Omega} .$$

$$[f = 0]: \quad Ch\omega^2 < 1 \Rightarrow \|u - u_h\|_{DG} \leq \inf_{v_h \in V_h} \|u - v_h\|_{DG} .$$

inacceptable threshold

1.6.1 1D plane wave duality estimate

- 1D adjoint problem:

$$-\phi'' - \omega^2 \phi = w_h \quad \text{in }]-1, 1[, \quad \pm \phi'(\pm 1) - i\omega \phi(\pm 1) = 0. \quad (1.1.1)$$

- 1D mesh: $\mathcal{T}_h = \{]x_{j-1}, x_j[\}_{j=1}^M$, $M \in \mathbb{N}$, meshwidth h

- 1D plane wave space $V_h(K) = \langle x \mapsto \cos(\omega x), x \mapsto \sin(\omega x) \rangle$

Fix $K =]x_{j-1}, x_j[$, $h_K := x_j - x_{j-1}$:

$$-\phi'' - \omega^2 \phi = \chi_K(x) (\beta \cos(\omega x) + \gamma \sin(\omega x)) \quad \text{in }]-1, 1[, \quad \pm \phi'(\pm 1) - i\omega \phi(\pm 1) = 0.$$

$$\blacktriangleright \begin{cases} \phi(x) \in \beta x \frac{\sin(\omega x)}{2\omega} - \gamma x \frac{\cos(\omega x)}{2\omega} + V_h(K) & \text{for } x \in K, \\ \phi \in \langle \cos(\omega x), \sin(\omega x) \rangle & \text{for } x \notin K \end{cases}$$

$$w_h(x) = \sum_{j=1}^N \chi_{[x_{j-1}, x_j]}(x) (\beta_j \cos(\omega x) + \gamma_j \sin(\omega x)) , \quad -1 \leq x \leq 1$$

↓

$$(Id - Q_\omega)\phi = \sum_{j=1}^N \chi_{[x_{j-1}, x_j]}(Id - Q_\omega^j) \left(\beta_j \left[x \frac{\sin(\omega x)}{2\omega} \right] + \gamma_j \left[x \frac{\cos(\omega x)}{2\omega} \right] \right) .$$

study local projection estimates for functions $x \mapsto x \frac{\sin(\omega x)}{2\omega}$, $x \mapsto x \frac{\cos(\omega x)}{2\omega}$!

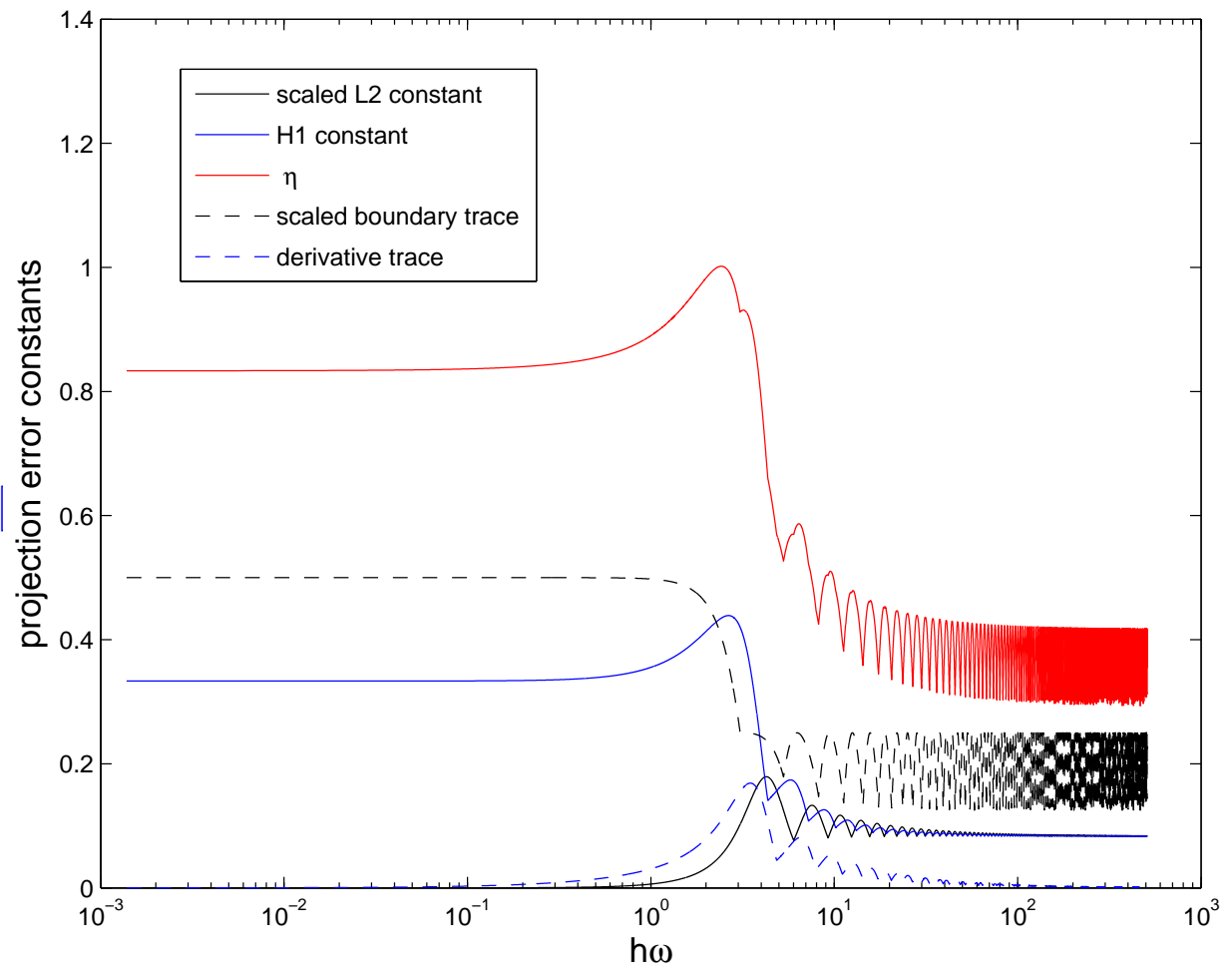


← MAPLE & MATLAB

$$\|\phi - Q_\omega \phi\|_{DG^+}^2 \leq \eta(h\omega) h^2 \|w_h\|_{0,[-1,1]}^2$$

with bounded $\eta : \mathbb{R}^+ \mapsto \mathbb{R}^+$.

$\eta \triangleright$

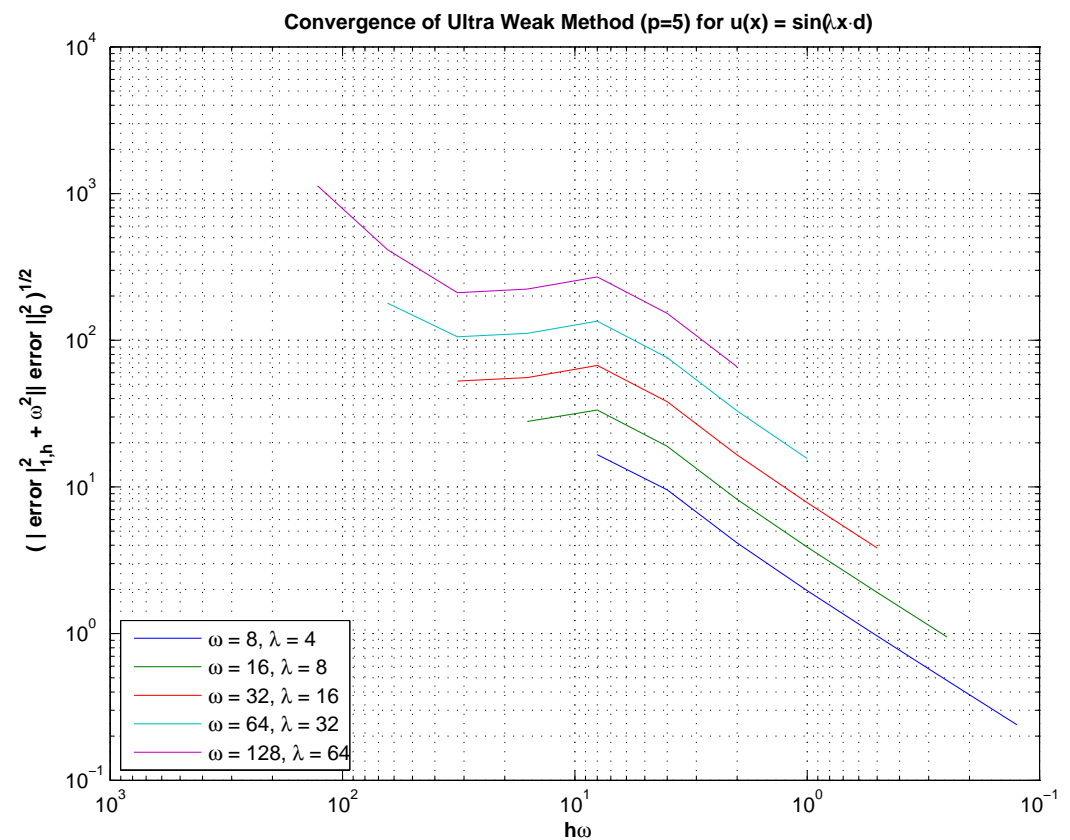
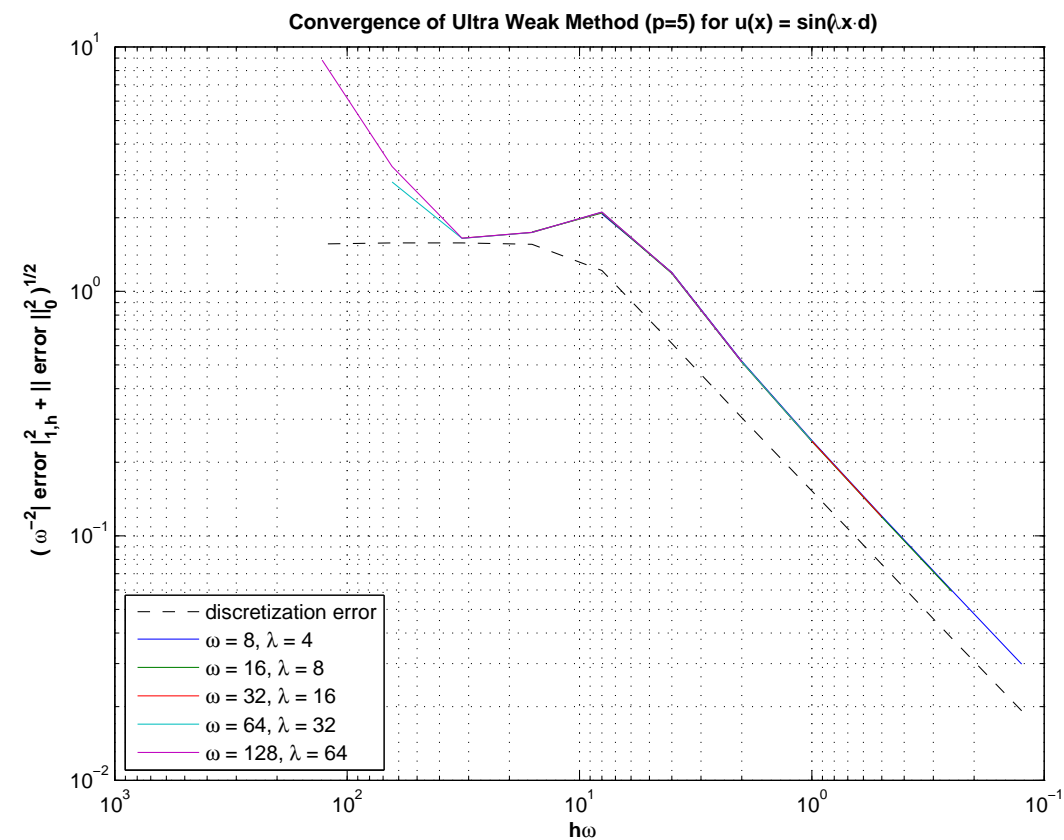


“quasi-optimality threshold” $Ch\omega < 1$

Challenge: extension to higher dimensions

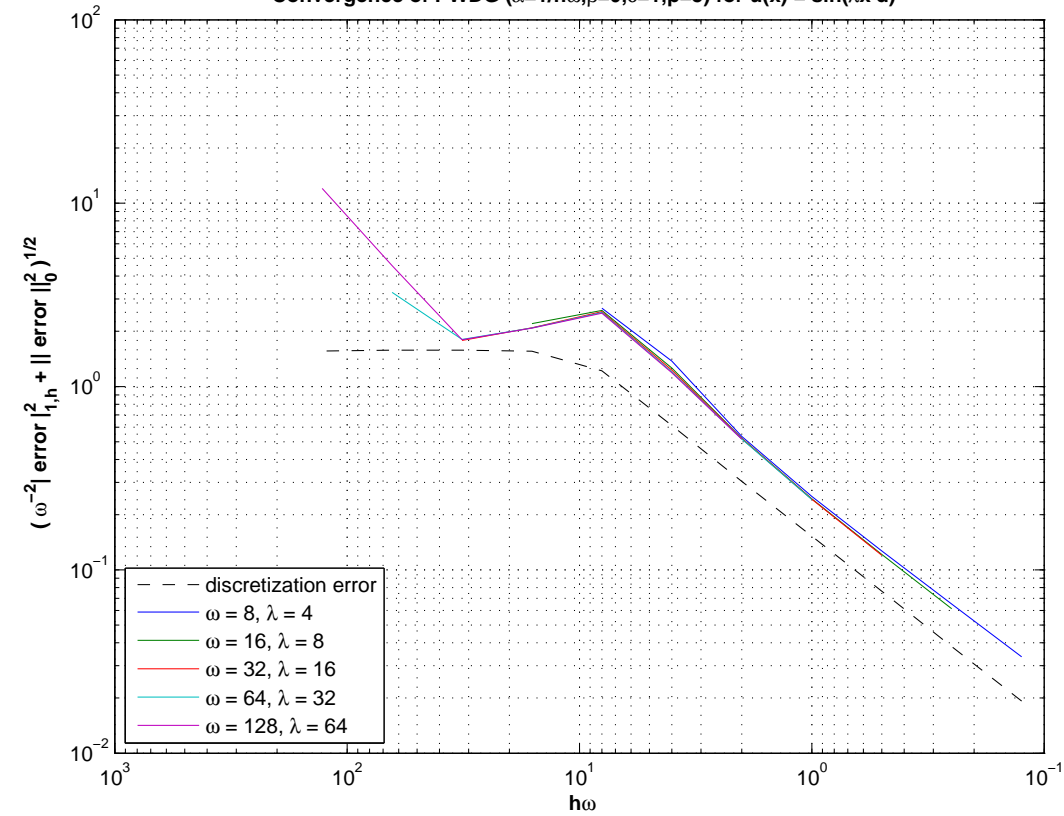
1.7 Numerical experiments

- $\Omega =]0, 1[^2$, unstructured triangular grids, $p = 7$
- boundary conditions & right hand side \rightarrow analytically prescribed u

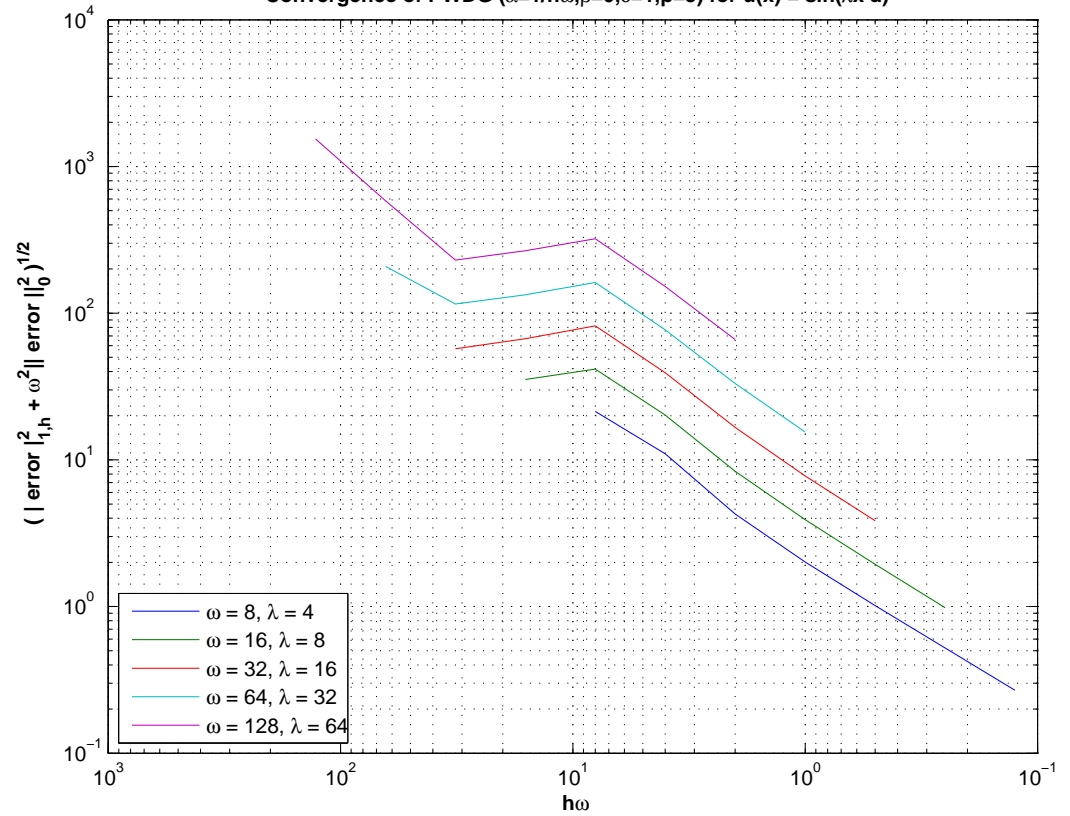


Discretization error of the original ultra-weak discretization of Cessenat and Despres

Convergence of PWDG ($\alpha=1/h\omega, \beta=0, \delta=1, p=5$) for $u(x) = \sin(\lambda x \cdot d)$



Convergence of PWDG ($\alpha=1/h\omega, \beta=0, \delta=1, p=5$) for $u(x) = \sin(\lambda x \cdot d)$



Discretization error of the plane wave DG method with $\alpha = (h\omega)^{-1}, \beta = 0, \delta = 1$

Bibliography

J. Melenk (1995), On Generalized Finite Element Methods, PhD thesis, University of Maryland, USA.