The Dirichlet to Neumann Map for the Helmholtz and Modified Helmholtz Equations with Complex Boundary Data

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Overview

- Introduction
- Fokas' boundary integral representation
- Numerical solution of modified Helmholtz equation in a polygon
- The values $q_n^{(j)}(\pm\pi)$
- Spectral discretization

- Choice of collocation points
- Numerical results
- Numerical solution of Helmholtz equation in a polygon
- Numerical results
- Current/future work

Introduction

We consider the PDE

$$q_{xx} + q_{yy} = 4\lambda q, \quad (x, y) \in \Omega$$

in a bounded, convex domain Ω with

$\lambda > 0$: modified Helmholtz equation

$\lambda < 0$: Helmholtz equation

and known Dirichlet boundary data $q(x, y)$, $(x, y) \in \partial\Omega$.

We approximate the Neumann data $q_n(x, y)$, $(x, y) \in \partial\Omega$,

without solving the PDE on the interior of Ω.

We transform to the complex plane via $z = x + iy$.

The PDE becomes

$$q_{z\bar{z}} = \lambda q, \quad z \in \Omega$$

with known Dirichlet boundary data $q(z)$, $z \in \partial\Omega$.

We approximate the Neumann data $q_n(z)$, $z \in \partial\Omega$.

Fokas' Boundary Integral Representation

If q, $\tilde{q}$ satisfy the PDEs

$$q_{z\bar{z}} = \lambda q \quad \text{and} \quad \tilde{q}_{z\bar{z}} = \lambda \tilde{q},$$

then

$$\frac{\partial}{\partial \bar{z}} \left(\tilde{q} q_z - q \tilde{q}_z \right) - \frac{\partial}{\partial z} \left(q \tilde{q}_{\bar{z}} - \tilde{q} q_{\bar{z}} \right) = 0.$$

The complex form of Green's theorem gives

$$\int_{\partial \tilde{\Omega}} \left[\left(\tilde{q} q_z - q \tilde{q}_z \right) dz + \left(q \tilde{q}_{\bar{z}} - \tilde{q} q_{\bar{z}} \right) d\bar{z} \right] = 0.$$

Separation of variables gives

$$\tilde{q} = e^{\pm\left(ikz + \frac{\lambda}{ik}\bar{z}\right)}, \quad \tilde{q} = e^{\pm\left(ik\bar{z} + \frac{\lambda}{ik}z\right)}.$$

Letting $\tilde{q} = e^{-\left(ikz+\frac{\lambda}{ik}\bar{z}\right)}$ gives

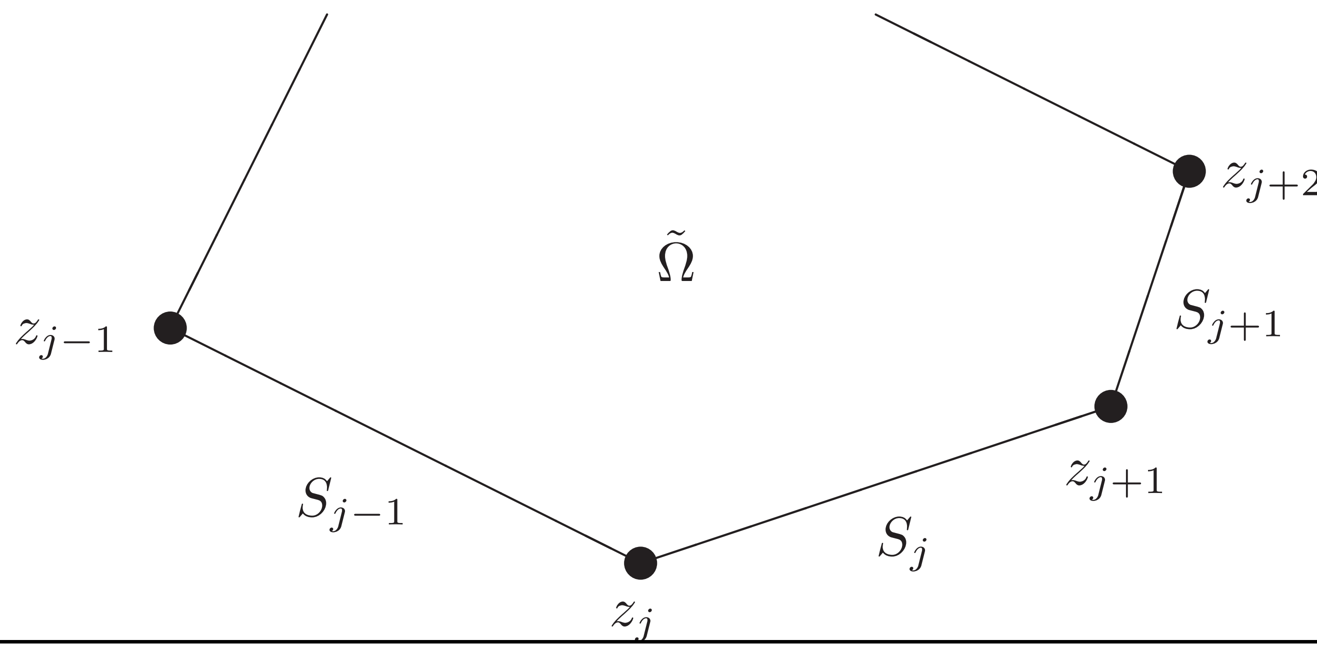
$$\int_{\partial\tilde{\Omega}} e^{-\left(ikz+\frac{\lambda}{ik}\bar{z}\right)} \left[(q_z + ikq)\,dz - \left(\frac{\lambda}{ik}q + q_{\bar{z}}\right)d\bar{z}\right] = 0 \quad \forall k \in \mathbb{C}.$$

Similarly, we can deduce the Schwarz conjugate global relation:

$$\int_{\partial\tilde{\Omega}} e^{\left(ik\bar{z}+\frac{\lambda}{ik}z\right)} \left[\left(\frac{\lambda}{ik}q - q_z\right)dz + (q_{\bar{z}} - ikq)\,d\bar{z}\right] = 0 \quad \forall k \in \mathbb{C}.$$

Numerical solution of modified Helmholtz equation ($\lambda > 0$) in a polygon

Take $\tilde{\Omega}$ to be a bounded, convex n-sided polygon with vertices $z_1, z_2, \ldots, z_n$ and sides $S_1, S_2, \ldots, S_n$ (indexed anticlockwise, modulo n):



On S_j,

- we parametrize z by

$$z = m_j + sh_j, \ s \in [-\pi, \pi], \quad m_j := \frac{z_j + z_{j+1}}{2}, \quad h_j := \frac{z_{j+1} - z_j}{2\pi};$$

- we reinterpret q as a function $q^{(j)}(s)$ of $s \in [-\pi, \pi]$;

- we have that

$$q_z dz = \frac{1}{2}\left(q_s^{(j)}(s) + iq_n^{(j)}(s)\right) ds, \ q_{\bar{z}} d\bar{z} = \frac{1}{2}\left(q_s^{(j)}(s) - iq_n^{(j)}(s)\right) ds,$$

where $q_s^{(j)}(s)$ and $q_n^{(j)}(s)$ are the derivatives of q tangential and normal to S_j.

The global relations become

$$\sum_{j=1}^{n} e^{i\left(\frac{\lambda}{k}\overline{m_j}-km_j\right)}\widehat{q_n^{(j)}}\left(kh_j-\frac{\lambda}{k}\overline{h_j}\right)$$
$$=-\sum_{j=1}^{n}\left(kh_j+\frac{\lambda}{k}\overline{h_j}\right)e^{i\left(\frac{\lambda}{k}\overline{m_j}-km_j\right)}\widehat{q^{(j)}}\left(kh_j-\frac{\lambda}{k}\overline{h_j}\right)\ \forall k\in\mathbb{C}$$

and

$$\sum_{j=1}^{n} e^{i\left(k\overline{m_j}-\frac{\lambda}{k}m_j\right)}\widehat{q_n^{(j)}}\left(-k\overline{h_j}+\frac{\lambda}{k}h_j\right)$$
$$=-\sum_{j=1}^{n}\left(k\overline{h_j}+\frac{\lambda}{k}h_j\right)e^{i\left(k\overline{m_j}-\frac{\lambda}{k}m_j\right)}\widehat{q^{(j)}}\left(-k\overline{h_j}+\frac{\lambda}{k}h_j\right)\ \forall k\in\mathbb{C},$$

where $\hat{h}$ is the Fourier transform of h over $(-\pi,\pi)$:

$$\hat{h}(k):=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-iks}h(s)ds\quad\forall k\in\mathbb{C}.$$

Consider the first global relation.

The term $e^{-i\left(kh_j-\frac{\lambda}{k}\overline{h_j}\right)s}$ in the integrand of $\widehat{q_n^{(j)}}\left(kh_j-\frac{\lambda}{k}\overline{h_j}\right)$ is bounded as $|k|\to\infty$ if, and only if, $kh_j-\frac{\lambda}{k}\overline{h_j}\in\mathbb{R}$.

Suppose $kh_j-\frac{\lambda}{k}\overline{h_j}=-l\in\mathbb{R}$. Then $k=k_\pm=\frac{-l\pm\sqrt{l^2+4\lambda|h_j|^2}}{2h_j}$.

For $l>0$ we discard k_+ because $k_+\to 0$ as $l\to\infty$. So $k=k_-=\frac{-l-\sqrt{l^2+4\lambda|h_j|^2}}{2h_j}$.

For $l<0$ we discard k_- and take $k=k_+=\frac{-l+\sqrt{l^2+4\lambda|h_j|^2}}{2h_j}$.

So $k = -\frac{\tilde{l}}{h_j}$ with

$$\tilde{l} = \begin{cases} \frac{l+\sqrt{l^2+4\lambda|h_j|^2}}{2} & \text{if } l > 0, \\ \frac{l-\sqrt{l^2+4\lambda|h_j|^2}}{2} & \text{if } l < 0. \end{cases}$$

As $l \to -\infty$, $e^{i\left(\frac{\lambda}{k}\overline{m_j} - km_j\right)} \to 0$ and so the unknowns become weakly coupled.

But as $l \to \infty$, $e^{i\left(\frac{\lambda}{k}\overline{m_j} - km_j\right)} \to \infty$ and so the unknowns remain strongly coupled.

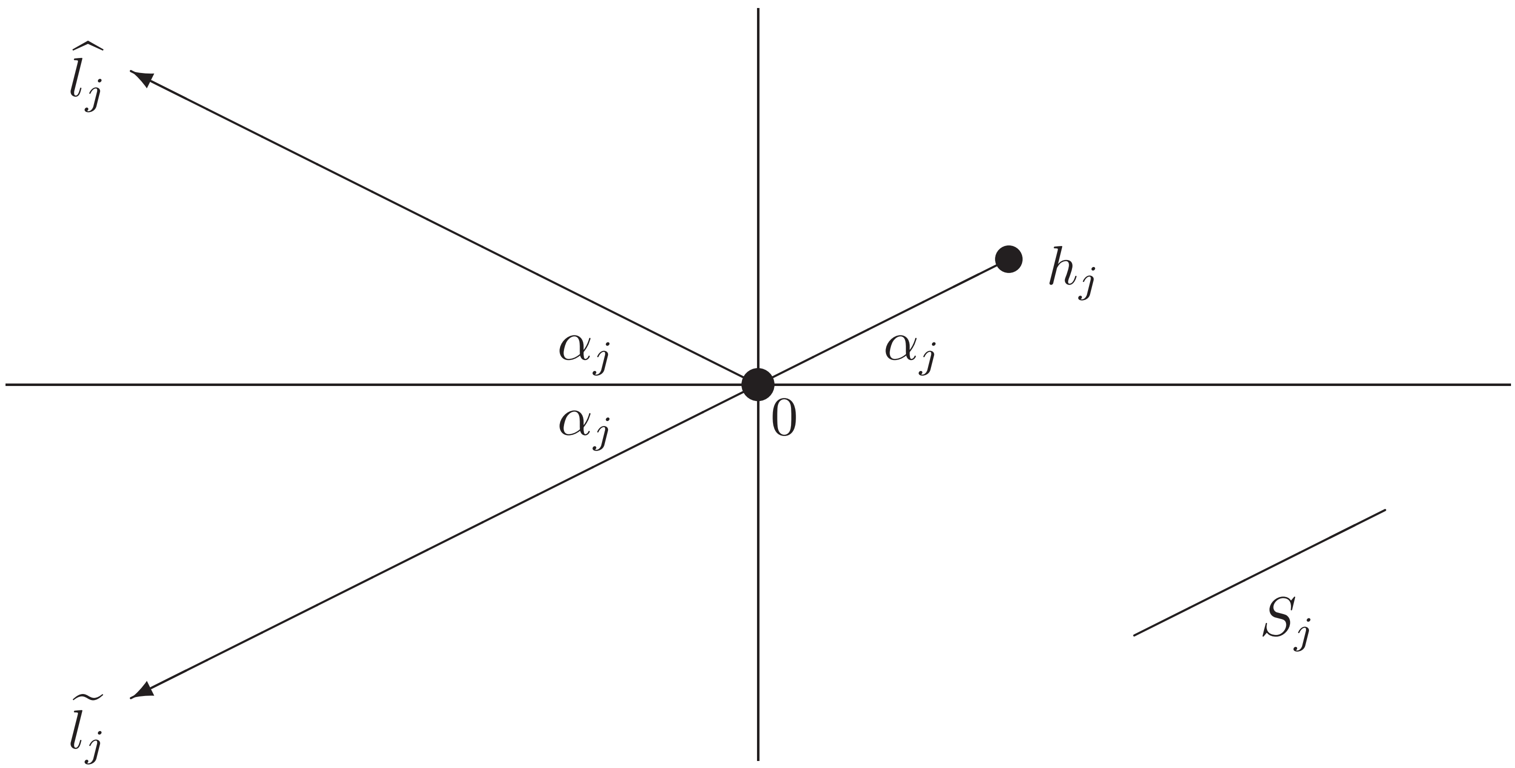
So we take $l > 0$ and $\tilde{l} = \frac{l+\sqrt{l^2+4\lambda|h_j|^2}}{2}$.

So $k \in \widehat{l_j} := \{k \in \mathbb{C} : \arg(k) = \pi - \arg(h_j)\}$.

For the second global relation we use

$$k = -\frac{\tilde{l}}{\overline{h_j}} \in \widetilde{l_j}, \quad l > 0, \quad \tilde{l} = \frac{l + \sqrt{l^2 + 4\lambda|h_j|^2}}{2},$$

where $\widetilde{l_j} := \{k \in \mathbb{C} : \arg(k) = \arg(h_j) - \pi\}$.



The Values $q_n^{(j)}(\pm\pi)$

On S_j,

$$q_z^{(j)}(s) = \frac{1}{2h_j}\left(q_s^{(j)}(s) + iq_n^{(j)}(s)\right) = \frac{\pi}{l_j}e^{-i\alpha_j}\left(q_s^{(j)}(s) + iq_n^{(j)}(s)\right),$$

where $l_j = |S_j|$.

Two representations of q_z at z_j: $q_z^{(j)}(\pi)$ and $q_z^{(j-1)}(-\pi)$.

Matching these gives

$$\frac{\pi}{l_j}e^{-i\alpha_j}\left(q_s^{(j)}(\pi) + iq_n^{(j)}(\pi)\right) = \frac{\pi}{l_{j-1}}e^{-i\alpha_{j-1}}\left(q_s^{(j-1)}(-\pi) + iq_n^{(j-1)}(-\pi)\right).$$

Taking real and imaginary parts and solving gives $q_n^{(j)}(\pi)$ and $q_n^{(j-1)}(-\pi)$.

For each $j \in \{1, 2, \ldots, n\}$, we set

$$Q_n^{(j)}(s) = q_n^{(j)}(s) - q_{n,\star}^{(j)}(s),$$

where $q_{n,\star}^{(j)}(s)$ is linear on $[-\pi, \pi]$ s.t. $q_{n,\star}^{(j)}(\pm\pi) = q_n^{(j)}(\pm\pi)$.

Then $Q_n^{(j)}(\pm\pi) = 0$.

Spectral Discretization

We associate with each side S_j of the polygon the same set of linearly independent functions $\{\phi_r(s)\}_{r=1}^{N} \subset C^0[-\pi, \pi]$, with N even.

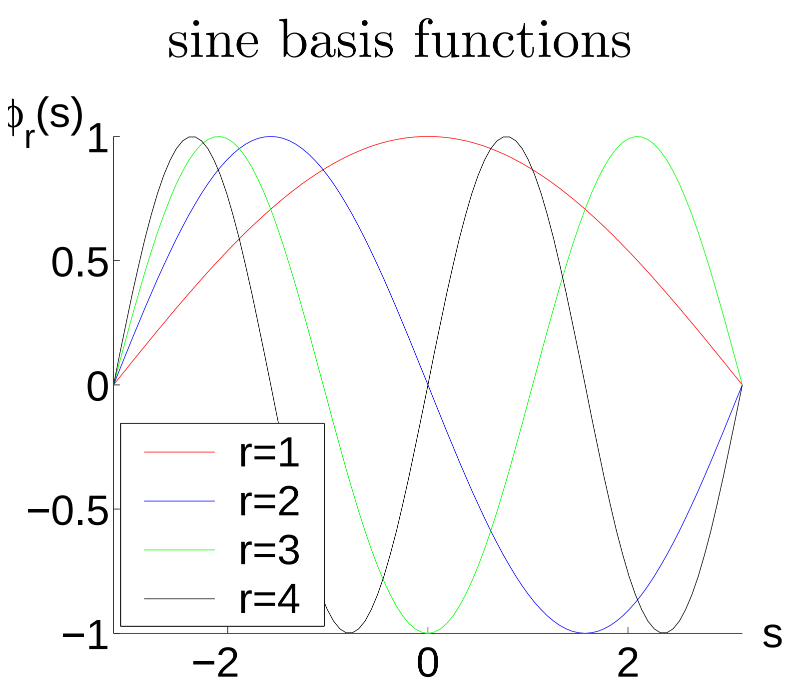
Sine basis functions: $\phi_r(s) := \sin\left[r\left(\frac{s+\pi}{2}\right)\right]$.

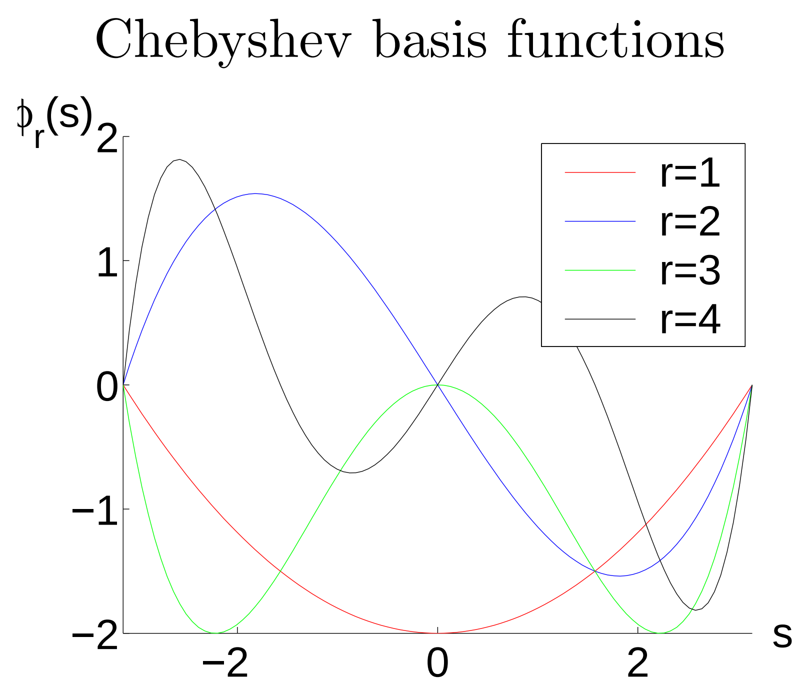
Chebyshev basis functions:

$$\phi_r(s) := \begin{cases} T_{r+1}\left(\frac{s}{\pi}\right) - T_0\left(\frac{s}{\pi}\right) & \text{if } r \text{ is odd,} \\ T_{r+1}\left(\frac{s}{\pi}\right) - T_1\left(\frac{s}{\pi}\right) & \text{if } r \text{ is even,} \end{cases}$$

constructed from the Chebyshev polynomials of the first kind

$$T_n(x) = \cos(n\cos^{-1}(x)) \quad \forall n \in \mathbb{N}.$$





For each $j \in \{1, 2, \ldots, n\}$, we approximate the function $Q_n^{(j)}(s)$ by a linear combination

$$Q_{n,N}^{(j)}(s) = \sum_{r=1}^{N} c_r^{(j)} \phi_r^{(j)}(s) \quad \forall s \in [-\pi, \pi].$$

The global relations become: for $p \in \{1, 2, \ldots, n\}$ and $l > 0$,

$$\sum_{j=1}^{n} \sum_{r=1}^{N} \tau_{p,j}^{\frac{1}{\tilde{l}}} \sigma_{p,j}^{\tilde{l}} \widehat{\phi_r^{(j)}}(z_{l,p,j,1}) c_r^{(j)} = \sum_{j=1}^{n} \tau_{p,j}^{\frac{1}{\tilde{l}}} \sigma_{p,j}^{\tilde{l}} \left[z_{l,p,j,2} \widehat{q^{(j)}}(z_{l,p,j,1}) - \widehat{q_{n,\star}^{(j)}}(z_{l,p,j,1}) \right] \quad (1)$$

and

$$\sum_{j=1}^{n} \sum_{r=1}^{N} \left(\overline{\tau_{p,j}}\right)^{\frac{1}{\tilde{l}}} \left(\overline{\sigma_{p,j}}\right)^{\tilde{l}} \widehat{\phi_r^{(j)}}(-\overline{z_{l,p,j,1}}) c_r^{(j)} = \sum_{j=1}^{n} \left(\overline{\tau_{p,j}}\right)^{\frac{1}{\tilde{l}}} \left(\overline{\sigma_{p,j}}\right)^{\tilde{l}} \left[\overline{z_{l,p,j,2}} \widehat{q^{(j)}}(-\overline{z_{l,p,j,1}}) - \widehat{q_{n,\star}^{(j)}}(-\overline{z_{l,p,j,1}}) \right], \quad (2)$$

where $z_{l,p,j,1} = -\tilde{l} \frac{h_j}{h_p} + \frac{\lambda}{\tilde{l}} \overline{h_j} h_p$, $z_{l,p,j,2} = \tilde{l} \frac{h_j}{h_p} + \frac{\lambda}{\tilde{l}} \overline{h_j} h_p$,
$\tau_{p,j} = e^{-i\lambda(\overline{m_j} - \overline{m_p})h_p}$ and $\sigma_{p,j} = e^{i \frac{m_j - m_p}{h_p}}$.

Choice of Collocation Points

For real-valued boundary data $q(z)$, we only use (1).

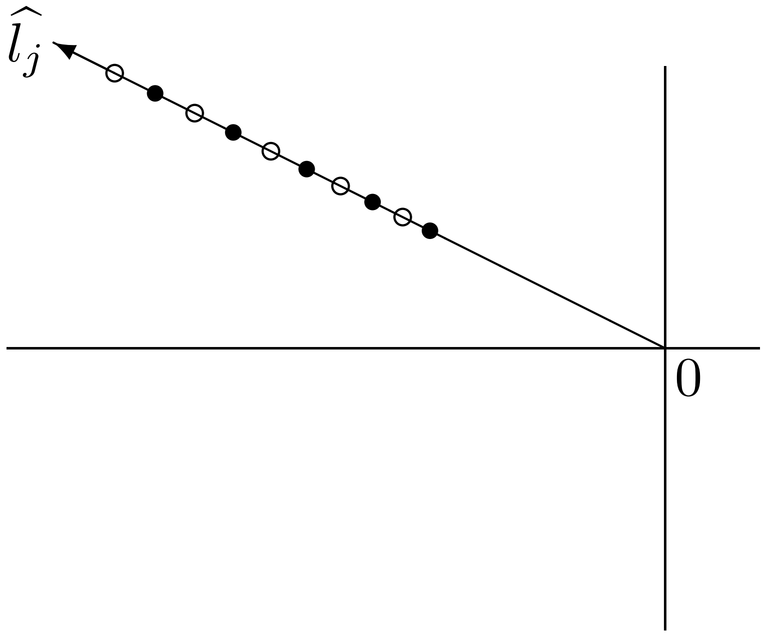
For each $p \in \{1, 2, \ldots, n\}$, we take the real and imaginary parts of (1) at the collocation points $l \in \left\{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{N-1}{2}\right\}$ and $l \in \left\{1, 2, \ldots, \frac{N}{2}\right\}$ respectively.

This yields a system of nN equations for $\{\{c_r^{(j)}\}_{r=1}^{N}\}_{j=1}^{n} \subset \mathbb{R}$.

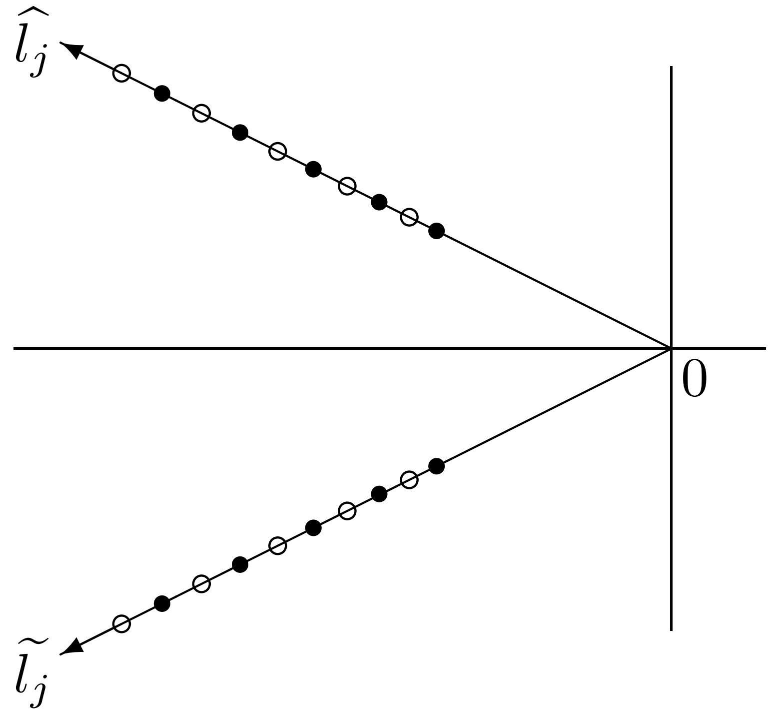
For complex-valued boundary data $q(z)$, we consider the difference and sum of (1) and (2) at the collocation points $l \in \left\{1, 2, \ldots, \frac{N}{2}\right\}$ and $l \in \left\{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{N-1}{2}\right\}$ respectively.

This yields a system of nN equations for $\{\{c_r^{(j)}\}_{r=1}^{N}\}_{j=1}^{n} \subset \mathbb{C}$.

real boundary data



complex boundary data



• Real part of (1)

∘ Imaginary part of (1)

• (1)+(2) ∘ (1)-(2)

Numerical Results

We take $\lambda = 100$ and use the analytical solution

$$q(z, \bar{z}) = e^{11z + \frac{100}{11}\bar{z}}$$

to generate complex valued boundary data $\left\{ q^{(j)}(s),\, q_n^{(j)}(s) \right\}_{j=1}^{n}$.

We compare the numerical approximation $\left\{ q_{n,N}^{(j)}(s) \right\}_{j=1}^{n}$ to the exact data $\left\{ q_n^{(j)}(s) \right\}_{j=1}^{n}$ by taking 10001 evenly spaced points

$$-\pi = s_1 < s_2 < \ldots < s_{10000} < s_{10001} = \pi$$

and calculating the error

$$E_\infty := \frac{||q_n - q_{n,N}||_\infty}{||q_n||_\infty},$$

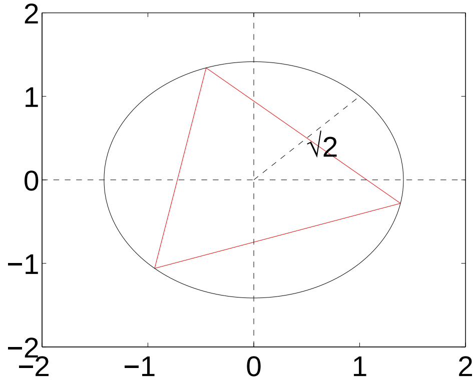
where

$$||q_n||_\infty = \max_{j\in\{1,2,\ldots,n\}} \left\{ \max_{k\in\{1,2,\ldots,10001\}} |q_n^{(j)}(s_k)| \right\}$$

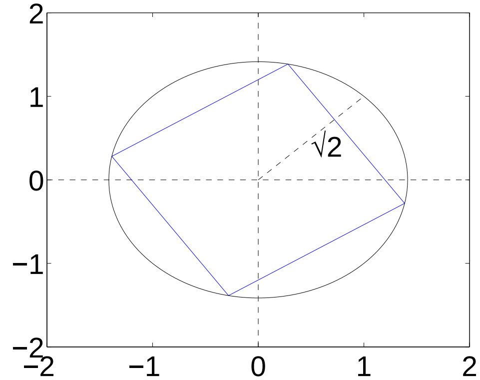
and

$$||q_n - q_{n,N}||_\infty = \max_{j\in\{1,2,\ldots,n\}} \left\{ \max_{k\in\{1,2,\ldots,10001\}} |q_n^{(j)}(s_k) - q_{n,N}^{(j)}(s_k)| \right\}.$$

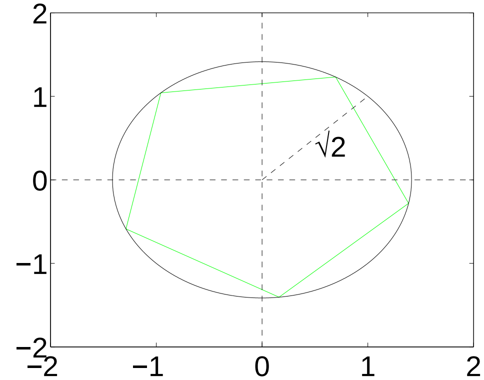
Regular Polygons



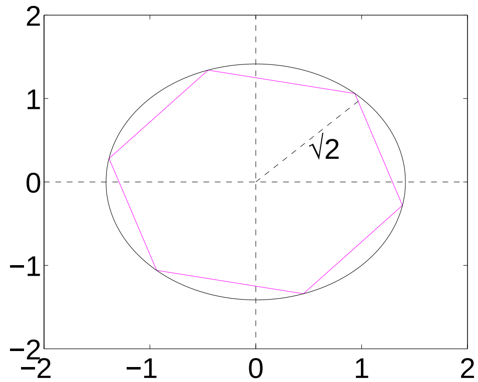
$n = 3$



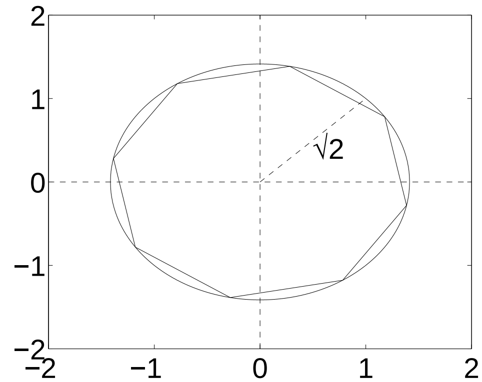
$n = 4$



$n = 5$

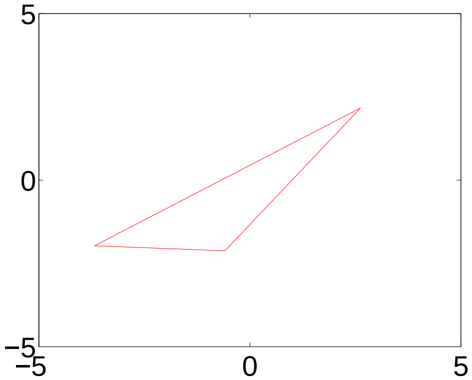


$n = 6$



$n = 8$

Irregular Polygons



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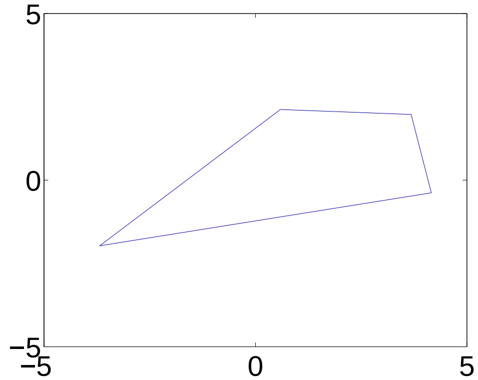

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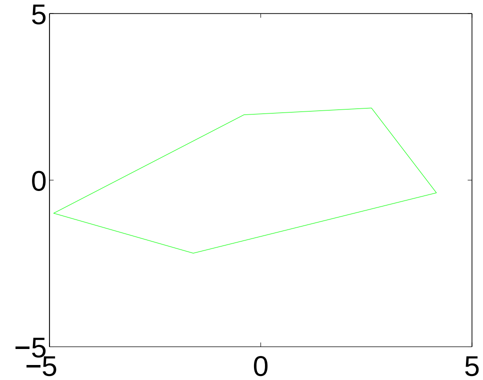

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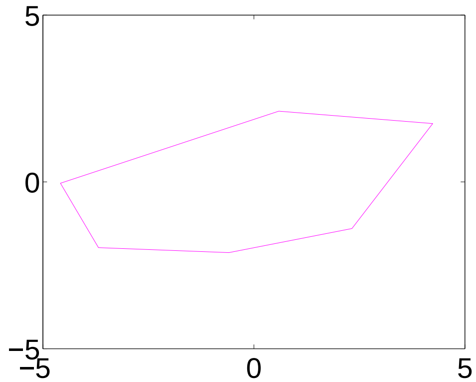
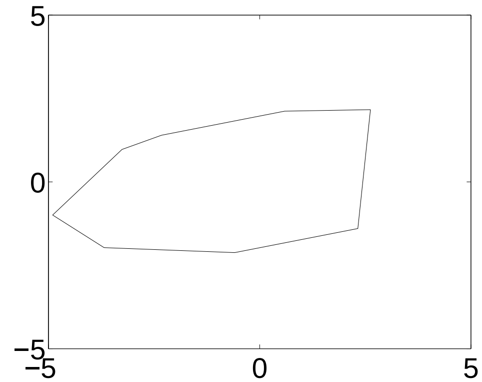


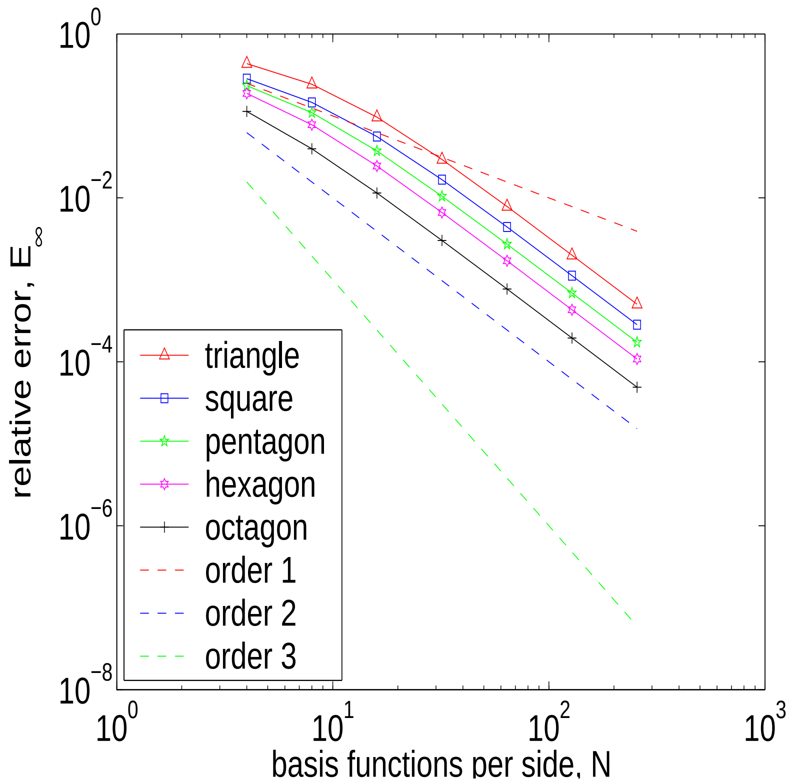
$n = 6$



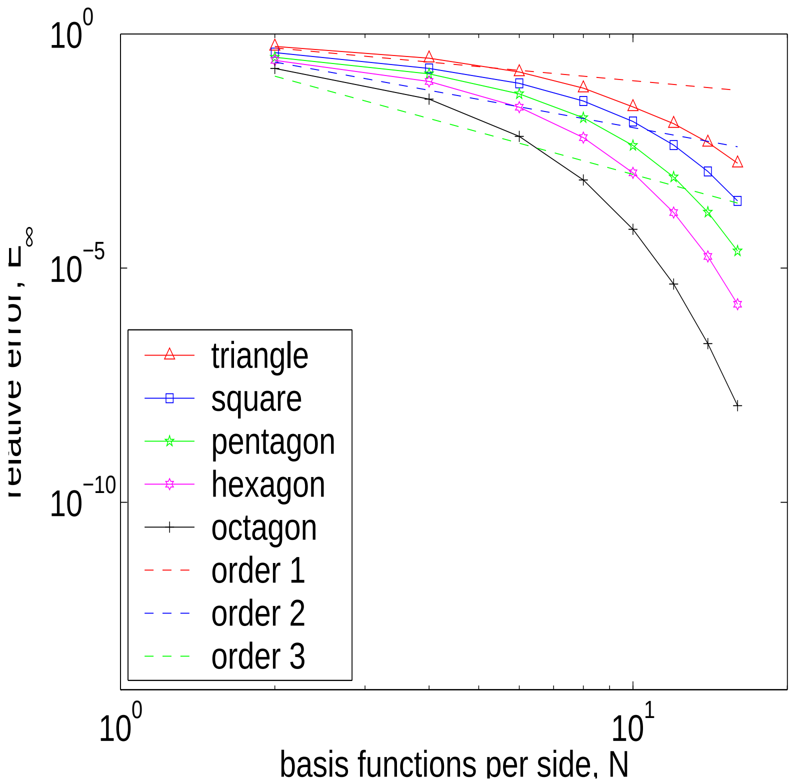
$n = 8$

Regular Polygons

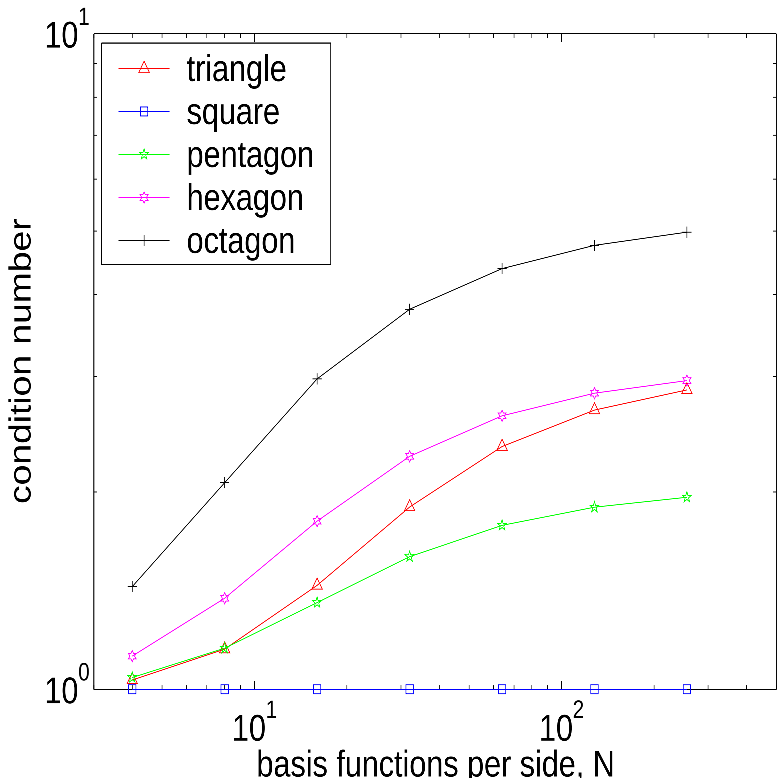
sine basis



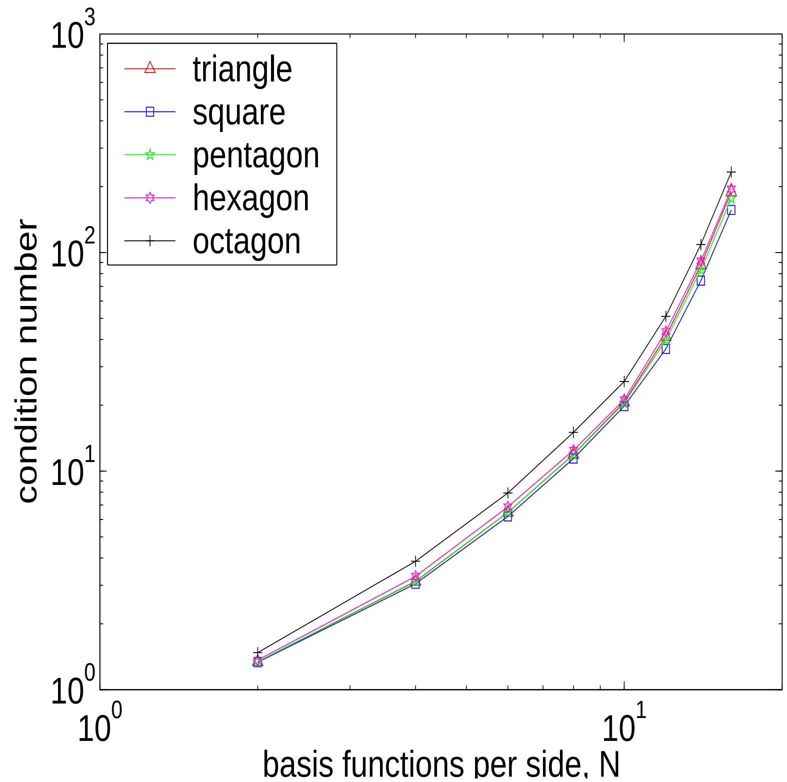
Chebyshev basis



sine basis

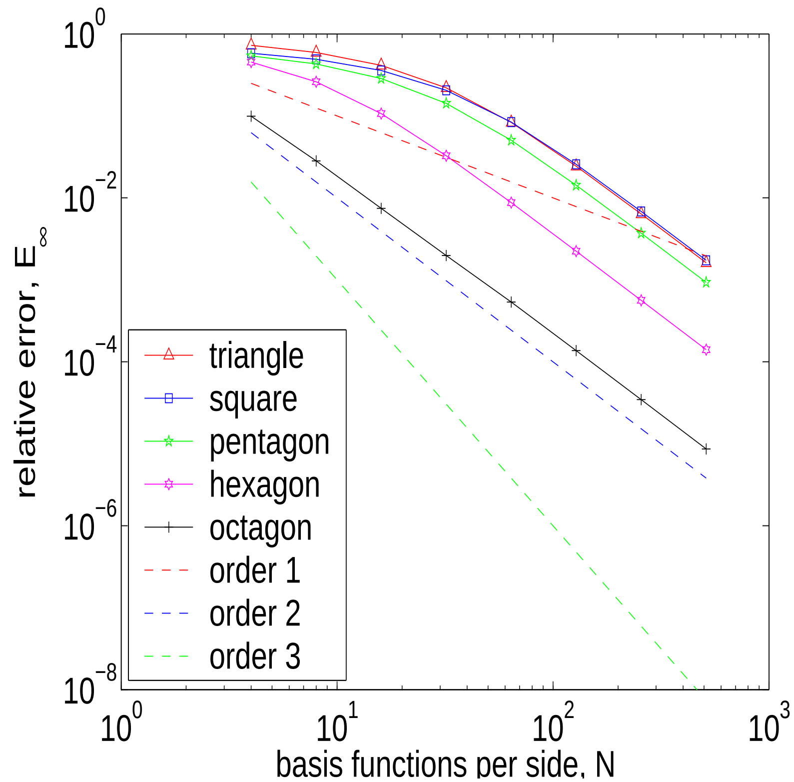


Chebyshev basis

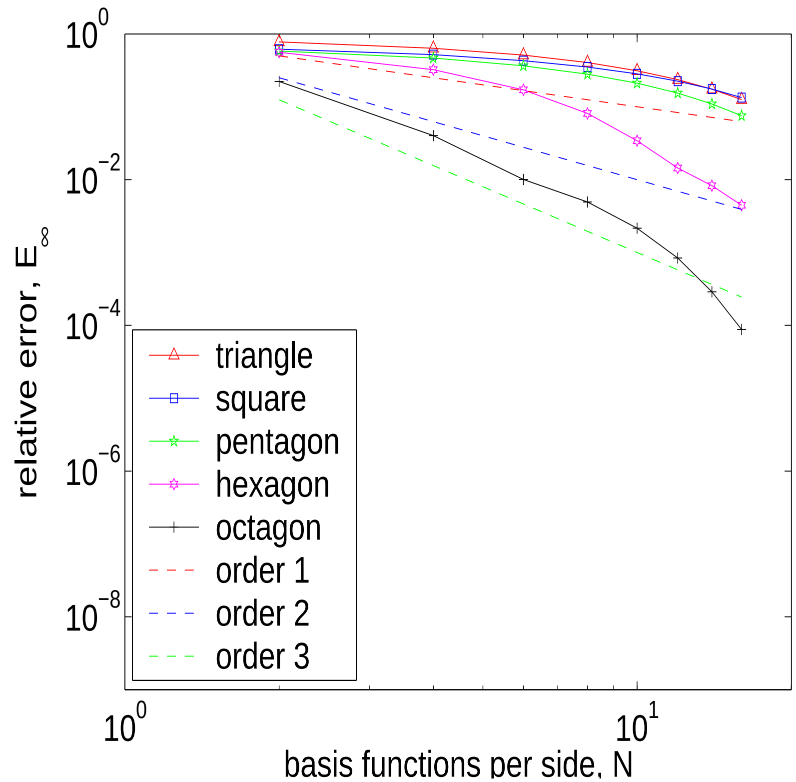


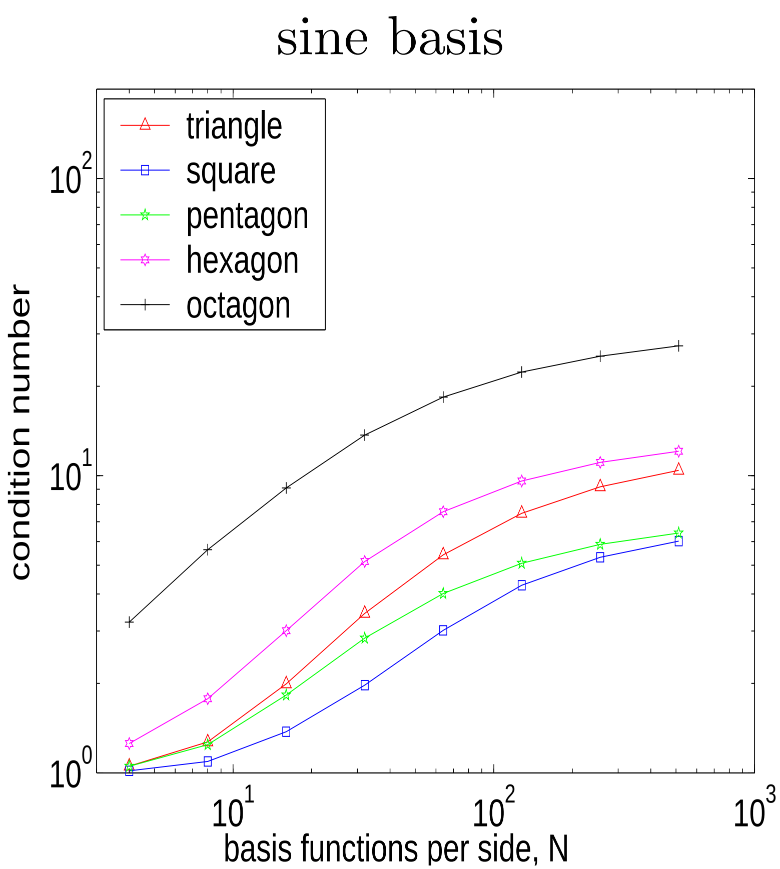
Irregular Polygons

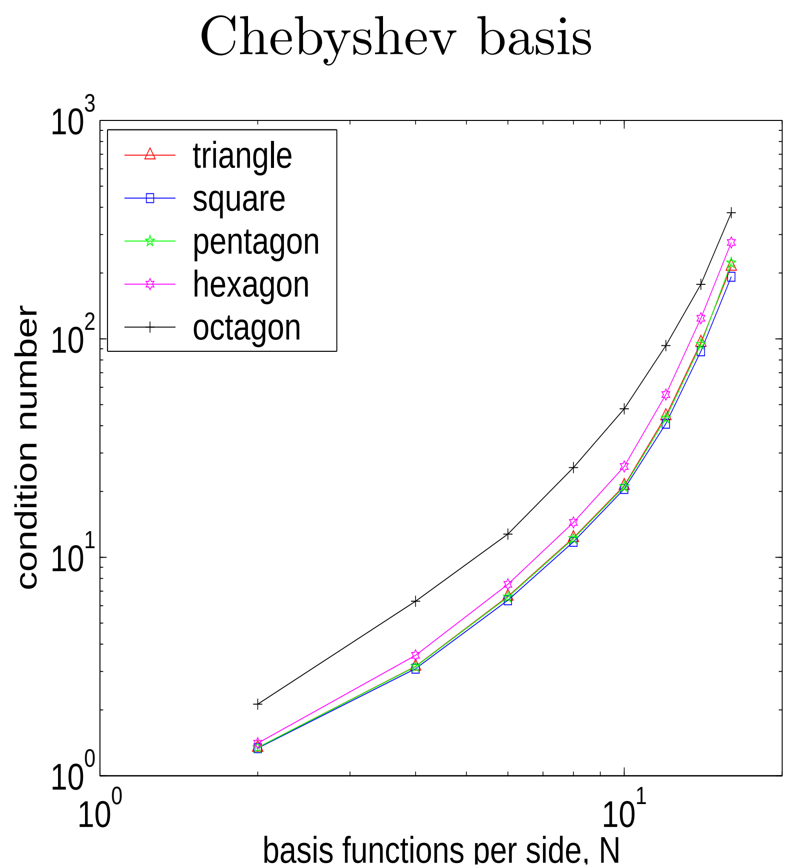
sine basis



Chebyshev basis




sine basis
triangle
square
pentagon
hexagon
octagon
condition number
10^0
10^1
10^2
10^1
10^2
10^3
basis functions per side, N


Chebyshev basis
triangle
square
pentagon
hexagon
octagon
condition number
10^0
10^1
10^2
10^3
10^0
10^1
basis functions per side, N

Numerical solution of Helmholtz equation ($\lambda < 0$) in a polygon

For the modified Helmholtz equation, we took

$$k = -\frac{\tilde{l}}{h_j} \in \widehat{l_j}, \quad l > 0, \quad \tilde{l} = \frac{l + \sqrt{l^2 + 4\lambda|h_j|^2}}{2}$$

in the first global relation.

For $l \geq 2\beta|h_j|$ ($\beta := \sqrt{-\lambda}$), we continue to do this.

When $l = 2\beta|h_j|$, $\tilde{l} = \beta|h_j|$ and so $k = -\beta\frac{|h_j|}{h_j}$. Hence $|k| = \beta$.

For $l < 2\beta|h_j|$, we take k on the circle with centre 0, radius β.

Recall the first global relation:

$$\sum_{j=1}^{n} e^{i\left(\frac{\lambda}{k}\overline{m_j}-km_j\right)}\widehat{q_n^{(j)}}\left(kh_j-\frac{\lambda}{k}\overline{h_j}\right)$$

$$=-\sum_{j=1}^{n}\left(kh_j+\frac{\lambda}{k}\overline{h_j}\right)e^{i\left(\frac{\lambda}{k}\overline{m_j}-km_j\right)}\widehat{q^{(j)}}\left(kh_j-\frac{\lambda}{k}\overline{h_j}\right)\ \forall k\in\mathbb{C}.$$

If $k=|k|e^{i\arg(k)}=\beta e^{i\phi}$ and $h_j=|h_j|e^{i\arg(h_j)}=|h_j|e^{i\alpha_j}$, then

$$kh_j-\frac{\lambda}{k}\overline{h_j}=2\beta|h_j|\cos\left(\phi+\alpha_j\right).$$

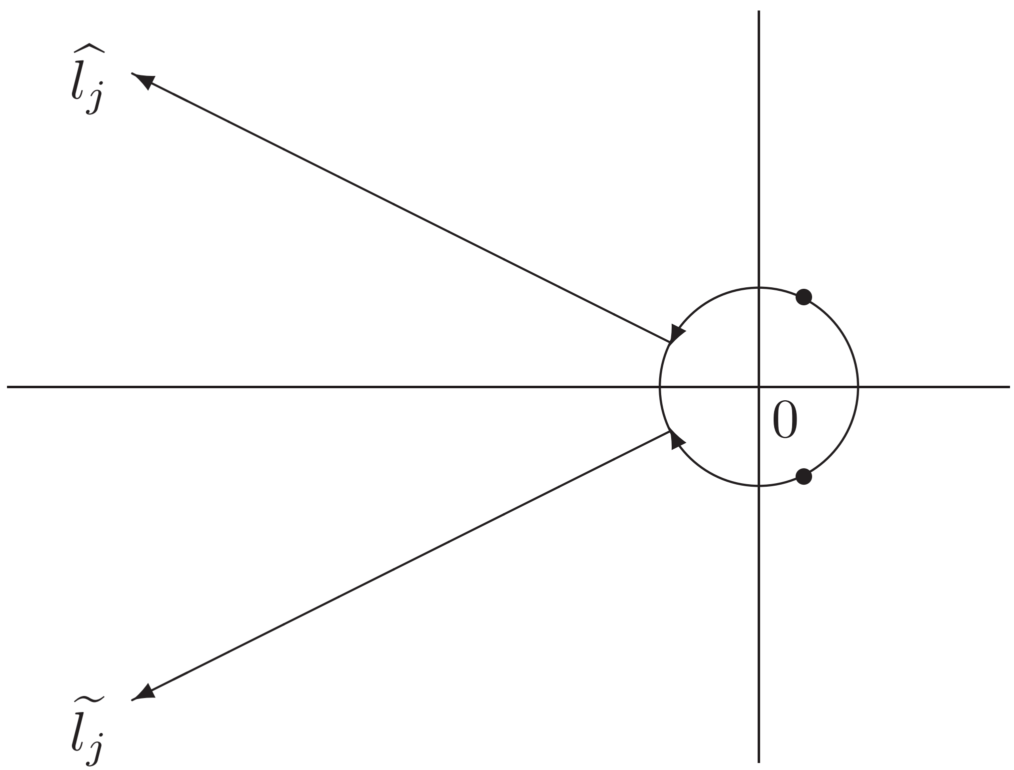
Again, we take $kh_j-\frac{\lambda}{k}\overline{h_j}=-l$. So

$$\cos\left(\phi+\alpha_j\right)=-\frac{l}{2\beta|h_j|}\in(-1,\,0)\quad\Rightarrow\quad\phi+\alpha_j\in\left(\frac{\pi}{2},\,\frac{3\pi}{2}\right).$$

We take $\phi\in\left(\frac{\pi}{2}-\alpha_j,\,\pi-\alpha_j\right)$.

For the second global relation, we take $k = \beta e^{i\phi}$ with

$$\cos(\phi - \alpha_j) = -\frac{l}{2\beta|h_j|} \in (-1,\, 0) \quad \text{and} \quad \phi \in \left(\pi + \alpha_j,\, \frac{3\pi}{2} + \alpha_j\right).$$



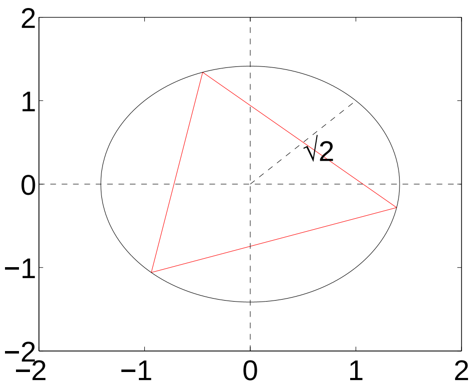
Numerical Results

We take $\lambda = -25$ and use the analytical solution

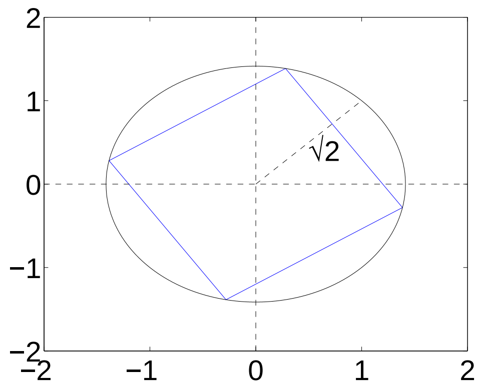
$$q(z, \bar{z}) = e^{4z - \frac{25}{4}\bar{z}}$$

to generate complex valued boundary data $\left\{ q^{(j)}(s),\, q_n^{(j)}(s) \right\}_{j=1}^{n}$.

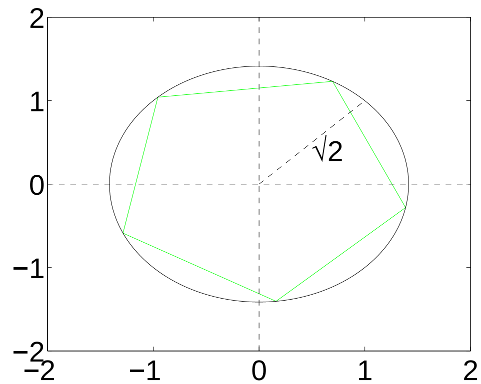
Regular Polygons



$n = 3$



$n = 4$



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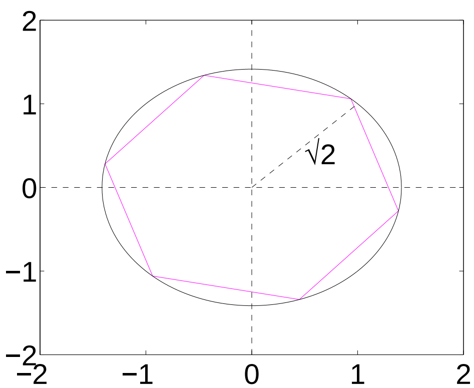

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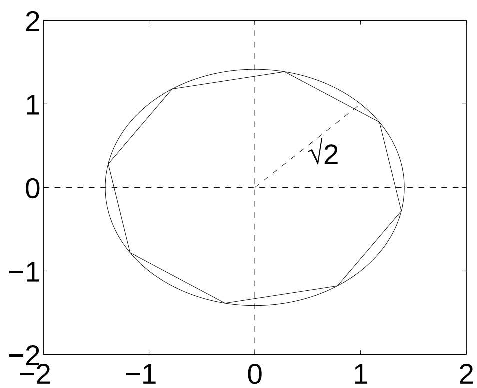
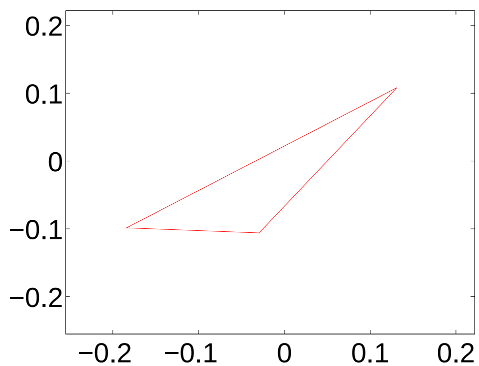


$n = 8$

Irregular Polygons



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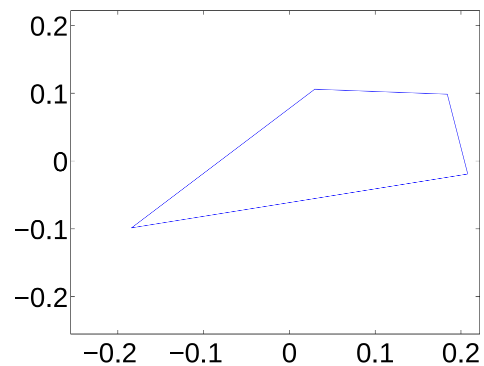

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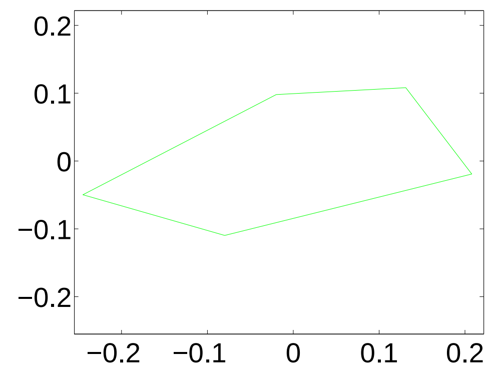

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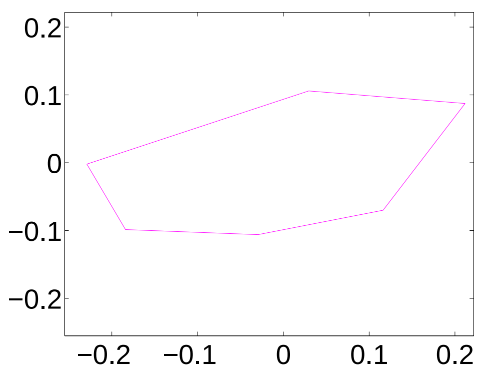

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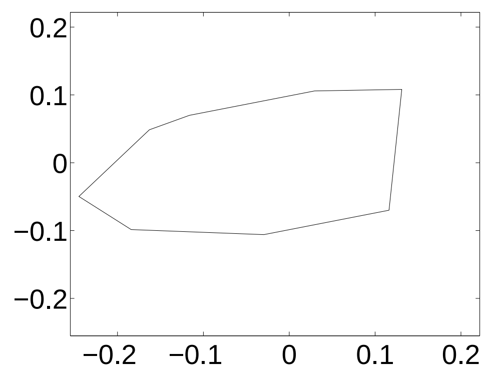
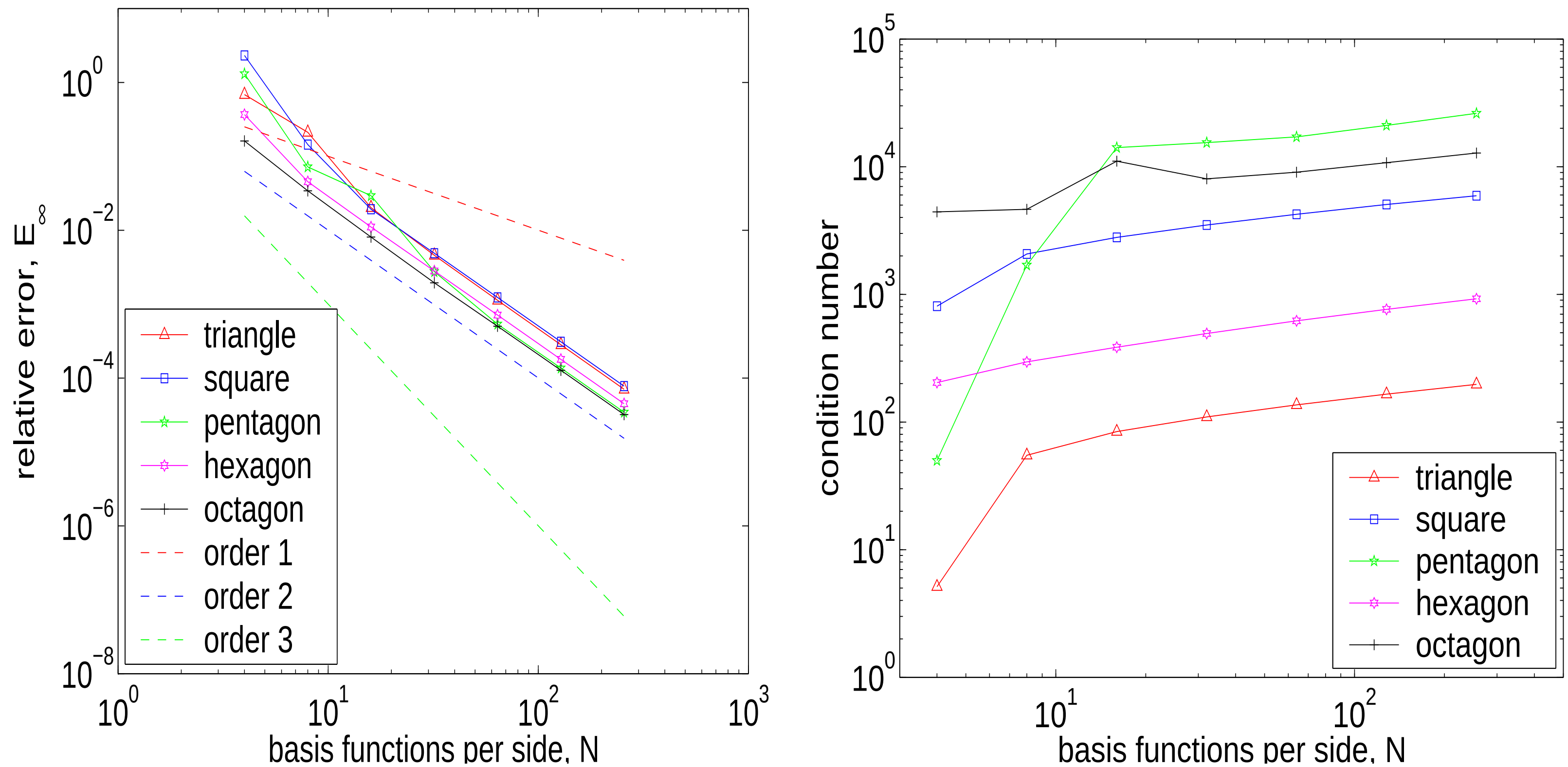
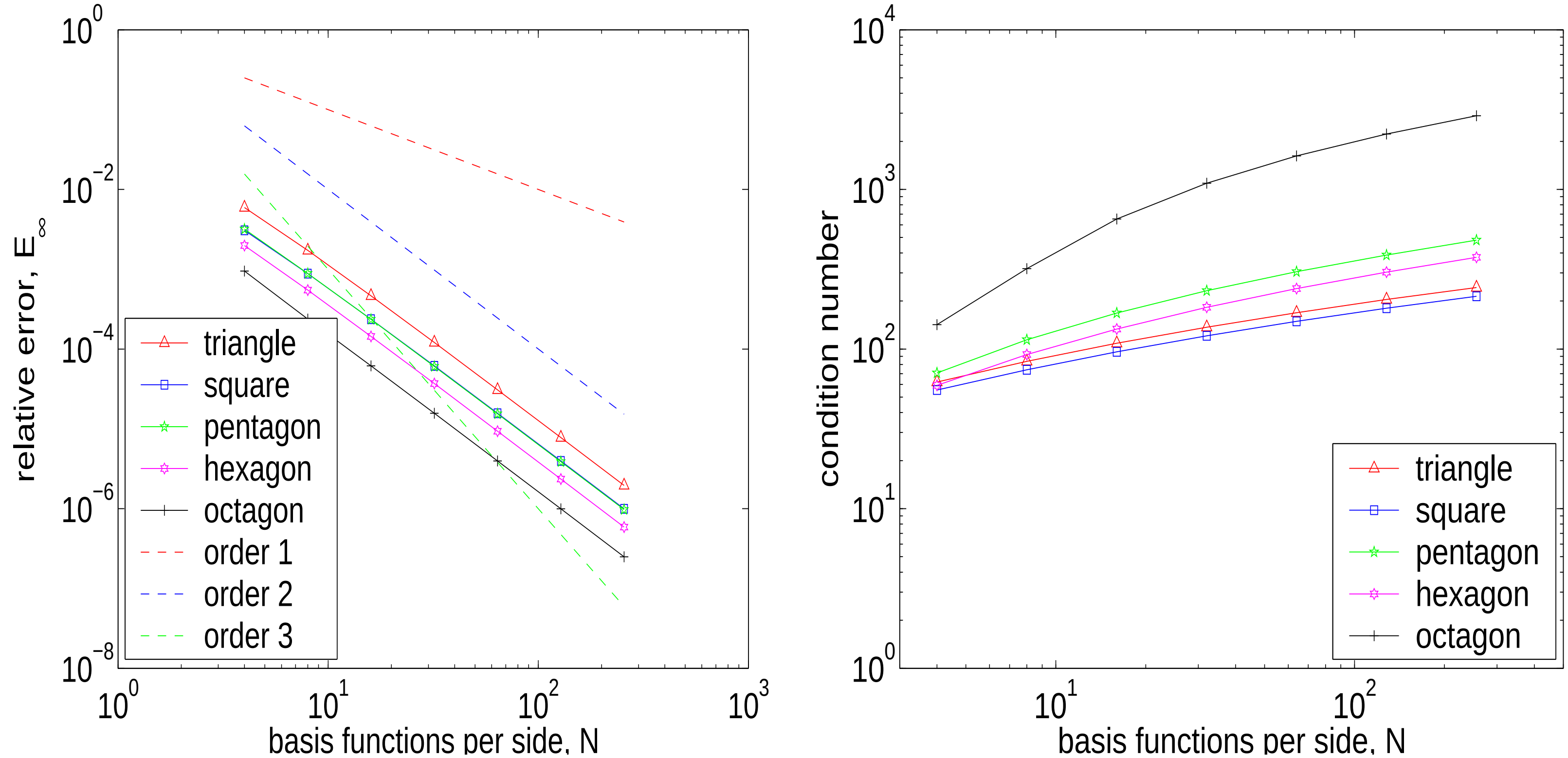


$n = 8$

Regular Polygons, Sine Basis Functions



Irregular Polygons, Sine Basis Functions



Current/Future Work

- Direct approximation of $\left\{ q_n^{(j)}(s) \right\}_{j=1}^{n}$ using functions from $C[-\pi, \pi]$;
- computation of the Neumann to Dirichlet map for the Helmholtz and modified Helmholtz equations;
- analytical proof of convergence.

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