Isaac Newton Institute, **HOP 2007**:

Effective Computational Methods for Highly Oscillatory Solutions

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Open problems in the computational solution of Volterra functional equations with highly oscillatory kernels

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Volterra integral equations with HO kernels

Let

$$\mathbf{K}_{\omega,\alpha}(\mathbf{t},\mathbf{s}) := (\mathbf{t}-\mathbf{s})^{-\alpha}\mathbf{K_0}(\mathbf{t},\mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}(\mathbf{t}-\mathbf{s})},$$

with $\omega \gg \mathbf{1}$, $\mathbf{0} \le \alpha < \mathbf{1}$, and set

$$(\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}) := \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{K}_{\omega,\alpha}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds}\;.$$

Assume:

- $\mathbf{K_0}$ is smooth, **independent of** ω;
- $\mathbf{g}$ is smooth; $\mathbf{g}'(\mathbf{t}) \neq \mathbf{0}$ on $I := [0,T]$.

<u>*Volterra functional equations:*</u>

$$\mathbf{u}(\mathbf{t}) = \mathbf{f}(\mathbf{t}) + (\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}), \quad t \in I;$$

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}(\mathbf{t})\mathbf{u}(\mathbf{t}) + \mathbf{f}(\mathbf{t}) + (\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}), \quad t \in I;$$

$$(\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}) = \mathbf{f}(\mathbf{t}), \quad t \in I \quad (\mathbf{f}(\mathbf{0}) = \mathbf{0}).$$

<u>*Model problems:*</u> $\mathbf{g}(\mathbf{t}-\mathbf{s}) = \mathbf{t}-\mathbf{s}$.

More general oscillating kernels

Motivating example:

$$\int_0^{\mathbf{t}} \mathbf{J_0}(\omega(\mathbf{t}-\mathbf{s}))\mathbf{u}(\mathbf{s})\,\mathbf{ds} = \mathbf{f}(\mathbf{t}), \quad \omega \gg \mathbf{1} \quad (f(0)=0).$$

$\hookrightarrow$ Occurs as the (spatial) *Fourier transform* of a **retarded potential integral equation** (single layer potential for transient accoustic scattering from $2D$ surface).
(Details: **Davies** & **Duncan**, SINUM 42 (2004)).

For $f \in C^1(I)$ the (unique) **solution** is

$$\mathbf{u}(\mathbf{t}) = \mathbf{f}'(\mathbf{t}) + \omega \int_0^{\mathbf{t}} \underbrace{\mathbf{J_1}(\omega(\mathbf{t}-\mathbf{s}))}\frac{\mathbf{f}(\mathbf{s})}{\mathbf{s}}\,\mathbf{ds}, \quad t \in I;$$

$\mathbf{J_0}$ and $\mathbf{J_1}$ denote, respectively, the **Bessel** functions of orders zero and one.

Solution representations:

$$\mathbf{u(t) = f(t) + \int_0^t \underbrace{(t-s)^{-\alpha} K_0(t,s)}_{=:K_{0,\alpha}(t,s)} e^{i\omega(t-s)} u(s)\, ds} \ :$$

Let $\mathbf{K_0} \in \mathbf{C(D)}$ ($D := \{0 \leq s \leq t \leq T\}$) and $0 \leq \alpha < 1$. For $\mathbf{f} \in \mathbf{C(I)}$ the (unique) **solution** $\mathbf{u} \in \mathbf{C(I)}$ has the form

$$\mathbf{u(t) = f(t) + \int_0^t R_{\omega,\alpha}(t,s) f(s)\, ds}, \quad t \in I,$$

with

$$\mathbf{R_{\omega,\alpha}(t,s) := R_{0,\alpha}(t,s) e^{i\omega(t-s)}} \ .$$

Here,

$$\mathbf{R_{0,\alpha}(t,s) = (t-s)^{-\alpha} Q_{0,\alpha}(t,s)}$$

is the **resolvent kernel** associated with the **non-oscillatory** kernel

$$\mathbf{K_{0,\alpha}(t-s) := (t-s)^{-\alpha} K_0(t,s)} \ .$$

<u>**Illustration:**</u> $\mathbf{K_0(t,s)} = \lambda =$ const, $\alpha = \mathbf{0}$

$$\hookrightarrow \ \mathbf{u(t) = f(t)} + \lambda \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{i\omega(t-s)} u(s)\, ds}, \qquad t \in I.$$

$\Rightarrow$ Resolvent kernel of $\mathbf{K_{\omega,0}(t,s)} = \lambda \mathbf{e^{i\omega(t-s)}}$ is

$$\mathbf{R_{\omega,0}(t,s)} = \lambda \mathbf{e}^{\lambda\mathbf{(t-s)}} \mathbf{e^{i\omega(t-s)}} \,.$$

Hence, the **solution** has the form

$$\mathbf{u(t) = f(t)} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{R_{\omega,0}(t,s) f(s)\, ds}, \qquad t \in [0, T],$$

or, if $\mathbf{f} \in \mathbf{C^1(I)}$,

$$\mathbf{u(t) = f(t)} + \frac{\lambda}{\lambda + \mathbf{i}\omega} \left(-\mathbf{f(t)} + \mathbf{e}^{(\lambda + \mathbf{i}\omega)\mathbf{t}} \mathbf{f(0)} \right)$$

$$+ \frac{\lambda}{\lambda + \mathbf{i}\omega} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}^{(\lambda + \mathbf{i}\omega)\mathbf{(t-s)}} \mathbf{f'(s)\, ds} \,.$$

<u>**Remark:**</u> Behaviour of solution $\mathbf{u(t)}$ as $\omega \to \infty$: Analogous result to the one by **Ursell** (1968) for **highly oscillatory Fredholm integral equation**,

$$\mathbf{u(x) = f(x)} + \mu \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{K_0(x,y) e^{i\omega|x-y|} u(y)\, dy} \,.$$

Volterra integro-differential equations

$$\mathbf{u}'(\mathbf{t}) = \mathbf{a}(\mathbf{t})\mathbf{u}(\mathbf{t}) + (\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}) + \mathbf{f}(\mathbf{t}), \quad u(0) = u_0,$$

with

$$(\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}) := \int_{\mathbf{0}}^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{-\alpha}\mathbf{K}_{\mathbf{0}}(\mathbf{t},\mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}(\mathbf{t}-\mathbf{s})}\mathbf{u}(\mathbf{s})\,\mathbf{ds}\ .$$

Equivalent Volterra integral equation:

$$\mathbf{u}(\mathbf{t}) = \mathbf{f}_{\mathbf{0}}(\mathbf{t}) + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{H}_{\omega,\alpha}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds},$$

Here,

$$\mathbf{H}_{\omega,\alpha}(\mathbf{t},\mathbf{s}) := \mathbf{a}(\mathbf{s}) + \int_{\mathbf{s}}^{\mathbf{t}} (\mathbf{v}-\mathbf{s})^{-\alpha}\mathbf{K}_{\mathbf{0}}(\mathbf{v},\mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}(\mathbf{v}-\mathbf{s})}\,\mathbf{dv}\ .$$

$\hookrightarrow$ *Special case:*

$$\mathbf{g}(\mathbf{t}-\mathbf{s}) = \mathbf{t}-\mathbf{s}, \ \ \alpha = \mathbf{0}, \ \ \mathbf{K}_{\mathbf{0}}(\mathbf{t},\mathbf{s}) = \lambda$$

$$\Rightarrow \ \ \mathbf{H}_{\omega,\mathbf{0}}(\mathbf{t},\mathbf{s}) = \mathbf{a}(\mathbf{s}) + \frac{\lambda}{i\omega}\left(\mathbf{e}^{\mathbf{i}\omega(\mathbf{t}-\mathbf{s})} - \mathbf{1}\right)\ .$$

Neutral delay VIDEs:

$$\frac{\mathbf{d}}{\mathbf{dt}}[\mathbf{u(t)} - (\mathcal{W}_{\omega,\alpha}\mathbf{u})(\mathbf{t})] = \mathbf{a(t)u(t)} + \mathbf{b(t)u}(\theta(\mathbf{t})), \quad t \geq 0,$$

with $\mathbf{u(t)} = \phi(\mathbf{t})$ if $t \leq 0$, and

$$(\mathcal{W}_{\omega,\alpha}\mathbf{u})(\mathbf{t}) := \int_{\mathbf{0}}^{\theta(\mathbf{t})} (\mathbf{t}-\mathbf{s})^{-\alpha}\mathbf{K_0}(\mathbf{t}-\mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}(\mathbf{t}-\mathbf{s})}\mathbf{u(s)}\,\mathbf{ds}\,.$$

Non-vanishing delay: $\theta(\mathbf{t}) := \mathbf{t} - \tau(\mathbf{t}),\ \tau(\mathbf{t}) \geq \tau_\mathbf{0} > \mathbf{0}$.

Let $\quad \mathbf{z(t)} := \mathbf{u(t)} - (\mathcal{W}_{\omega,\alpha}\mathbf{u})(\mathbf{t})$.

Since the delay function $\theta(\mathbf{t})$ induces **primary discontinuity points** $\{\xi_\mu\}$ by

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 1 \quad (\xi_0 := 0),$$

the given NVIDE is equivalent to the (local) ODEs with **highly oscillatory forcing terms**,

$$\mathbf{z}'(\mathbf{t}) = \mathbf{a(t)z(t)} + \underbrace{\mathbf{F}_{\omega,\alpha}(\mathbf{t})} + \mathbf{b(t)u}(\theta(\mathbf{t})), \quad t \in [\xi_\mu, \xi_{\mu+1}],$$

with known $\mathbf{z}(\xi_\mu)$ $(\mu = 0, 1, \ldots)$. Here,

$$\mathbf{F}_{\omega,\alpha}(\mathbf{t}) := \mathbf{a(t)} \int_{\mathbf{0}}^{\theta(\mathbf{t})} (\mathbf{t}-\mathbf{s})^{-\alpha}\mathbf{K_0}(\mathbf{t}-\mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}(\mathbf{t}-\mathbf{s})}\mathbf{u(s)}\,\mathbf{ds}\,.$$

For known $\mathbf{z(t)}$ on $[\xi_\mu, \xi_{\mu+1}]$ the solution on $[\xi_\mu, \xi_{m+1}]$ is

$$\mathbf{u(t)} = \mathbf{z(t)} + \underbrace{(\mathcal{W}_{\omega,\alpha}\mathbf{u})(\mathbf{t})}, \quad t \in [\xi_\mu, \xi_{\mu+1}]\,.$$

First-kind Volterra integral equations

$$(\mathcal{V}_{\omega,\alpha}\mathbf{u})(\mathbf{t}) = \mathbf{f}(\mathbf{t}), \quad t \in I \quad (\mathbf{f}(\mathbf{0}) = \mathbf{0}) \ :$$

Equivalent **regular** first-kind VIE:

$$\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{H}_{\omega,\alpha}(\mathbf{t},\mathbf{s})\mathbf{u}(\mathbf{s})\,\mathbf{ds} = \mathbf{f}_{\alpha}(\mathbf{t}),$$

with **smooth** kernel $\mathbf{H}_{\omega,\alpha}(\mathbf{t},\mathbf{s})$ given by

$$\int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1}-\mathbf{z})^{\alpha-\mathbf{1}}\mathbf{z}^{-\alpha}\,\underbrace{\mathbf{K_0}(\mathbf{s}+(\mathbf{t}-\mathbf{s})\mathbf{z},\mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}((\mathbf{t}-\mathbf{s})\mathbf{z})}}\,\mathbf{dz}$$

and **non-smooth** right-hand side

$$\mathbf{f}_{\alpha}(\mathbf{t}) := \int_{\mathbf{0}}^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\alpha-\mathbf{1}}\mathbf{f}'(\mathbf{s})\,\mathbf{ds}\ .$$

Note that

$$\mathbf{H}_{\omega,\alpha}(\mathbf{t},\mathbf{t}) = \mathbf{K_0}(\mathbf{t},\mathbf{t})\,\underbrace{\int_{\mathbf{0}}^{\mathbf{1}} (\mathbf{1}-\mathbf{z})^{\alpha-\mathbf{1}}\mathbf{z}^{-\alpha}\,\mathbf{dz}}_{=\Gamma(\alpha,1-\alpha)}\ .$$

If $\mathbf{K_0}(\mathbf{t},\mathbf{t}) \neq \mathbf{0}$ on $\mathbf{I}$ then the given V1 is equivalent to a second-kind Volterra integral equation with regular (highly oscillatory) kernel. **BUT:**

Solution of V1 with highly oscillatory kernel is not necessarily highly oscillatory!

<u>Example:</u>

$$\int_0^{\mathbf{t}} \mathbf{e}^{\mathbf{i}\omega(\mathbf{t}-\mathbf{s})}\mathbf{u}(\mathbf{s})\,\mathbf{ds} = \mathbf{f}(\mathbf{t}), \quad t \in [0,T] \quad (\mathbf{f}(\mathbf{0}) = \mathbf{0}) \ :$$

$\hookrightarrow$ Equivalent **second-kind** Volterra equation is

$$\mathbf{u}(\mathbf{t}) = \mathbf{f}'(\mathbf{t}) + \int_0^{\mathbf{t}} \underbrace{(-\mathbf{i}\omega)\mathbf{e}^{\mathbf{i}\omega(\mathbf{t}-\mathbf{s})}}_{=\mathbf{K}_{\omega,\mathbf{0}}(\mathbf{t},\mathbf{s})}\mathbf{u}(\mathbf{s})\,\mathbf{ds} \ ,$$

where the kernel $\mathbf{K_0}(\mathbf{t},\mathbf{s}) = -\mathbf{i}\omega$ **depends on** ω. The resolvent of the kernel $\mathbf{K}_{\omega,\mathbf{0}}(\mathbf{t},\mathbf{s})$ is

$$\mathbf{R}_{\omega,\mathbf{0}}(\mathbf{t},\mathbf{s}) = -\mathbf{i}\omega \ ,$$

and hence the solution of **V1** is

$$\mathbf{u}(\mathbf{t}) = \mathbf{f}'(\mathbf{t}) - \mathbf{i}\omega \cdot \mathbf{f}(\mathbf{t}), \quad t \in [0,T] \ .$$

Collocation solutions

$$\mathbf{u(t) = f(t) + \int_0^t K_{\omega,\alpha}(t,s)u(s)\,ds},\ t \in [0,T]\ :$$

Let $\mathbf{I_h} := \{\mathbf{t_n} :\ \mathbf{0 = t_0 < t_1 < \cdots < t_N = T}\}$, with

$$\mathbf{e_n} := (\mathbf{t_n}, \mathbf{t_{n+1}}], \quad \mathbf{h_n} := \mathbf{t_{n+1} - t_n}, \quad \mathbf{h} := \max_{(\mathbf{n})}\{\mathbf{h_n}\}.$$

- **Collocation space:**

$$\mathbf{S^{(-1)}_{m-1}(I_h)} := \{\mathbf{v} :\ \mathbf{v}|_{\mathbf{e_n}} \in \pi_{\mathbf{m-1}}\ \ (0 \le n \le N)\}\ .$$

- **Collocation points:**

$$\mathbf{X_h} := \{\mathbf{t_n + c_k h_n} :\ 0 \le n \le N-1\}\ ,$$

with $\mathbf{0 \le c_1 < \cdots < c_m} \le 1$.

- **Collocation equation:** Find $\mathbf{u_h} \in \mathbf{S^{(-1)}_{m-1}(I_h)}$ so that

$$\mathbf{u_h(t) = f(t)} + (\mathcal{V}_{\omega,\alpha}\mathbf{u_h})(\mathbf{t})\ \text{ for }\ \underline{\mathbf{t \in X_h}}.$$

$\hookrightarrow$ **<u>Time-stepping:</u>** At $\mathbf{t = t_n + c_j h_n \in e_n}$,

$$\mathbf{u_h(t) = f(t) + \Phi_n(t)}$$

$$\mathbf{+h_n \int_0^{c_j} K_{\omega,\alpha}(t, t_n + sh_n)u_h(t_n + sh_n)\,ds},$$

with *memory term*

$$\mathbf{\Phi_n(t)} := \mathbf{\int_0^{t_n} K_{\omega,\alpha}(t,s)u_h(s)\,ds}\ .$$

For local (Lagrange) basis $\{\phi_{\mathbf{n,k}}\}$ of $\mathbf{S}^{(-1)}_{\mathbf{m-1}}(\mathbf{I_h})$, need to compute the **local oscillatory integrals**

$$\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{K}_{\omega,\alpha}(\mathbf{t}, \mathbf{t}_\ell + \mathbf{s}\mathbf{h}_\ell)\phi_{\ell,\mathbf{k}}(\mathbf{s})\,\mathbf{ds} \quad (\ell < n)$$

and

$$\int_{\mathbf{0}}^{\mathbf{c_j}} \mathbf{K}_{\omega,\alpha}(\mathbf{t}, \mathbf{t_n} + \mathbf{s}\mathbf{h_n})\phi_{\mathbf{n,k}}(\mathbf{s})\,\mathbf{ds} \;\; (j = 1, \ldots, m),$$

where

$$\mathbf{K}_{\omega,\alpha}(\mathbf{t}, \mathbf{s}) = (\mathbf{t} - \mathbf{s})^{-\alpha}\mathbf{K_0}(\mathbf{t}, \mathbf{s})\mathbf{e}^{\mathbf{i}\omega\mathbf{g}(\mathbf{t}-\mathbf{s})} \,.$$

Case I: ω *small* :

$\hookrightarrow$ Collocation with $\{\mathbf{c_k}\}$ given by **Radau II** points ($\mathbf{c_m} = \mathbf{1}$) or **Lobatto** points ($\mathbf{c_1} = \mathbf{0}$, $\mathbf{c_m} = \mathbf{1}$) yields the classical (optimal) estimates (for $\alpha = \mathbf{0}$),

$$\max\{|\mathbf{u}(\mathbf{t}) - \mathbf{u_h}(\mathbf{t})| : \; \mathbf{t} \in \mathbf{I_h}\} \le \mathbf{Q}_\kappa(\omega)\mathbf{h}^{\mathbf{2m}-\kappa},$$

with $\kappa = 1$ (Radau II) and $\kappa = 2$ (Lobatto), and '**small**' constant $\mathbf{Q}_\kappa(\omega)$.
(These estimates also hold for *fully discretised* collocation, provided the quadrature abscissas are the collocation points.)

Case II: $\omega \gg 1$: $\mathbf{u_h \in S_h(I_h)} \hookrightarrow$ **defect** $\delta_{\mathbf{h}}$ given by

$$\delta_{\mathbf{h}}(\mathbf{t}) = -\mathbf{u_h}(\mathbf{t}) + \mathbf{f}(\mathbf{t}) + (\mathcal{V}_{\omega,\alpha}\mathbf{u_h})(\mathbf{t}), \quad \underline{\mathbf{t} \in \mathbf{I}},$$

with

$$\underline{\delta_{\mathbf{h}}(\mathbf{t}) = \mathbf{0}} \quad \text{for} \quad \underline{\mathbf{t} \in \mathbf{X_h}} \ .$$

Collocation error $\mathbf{e_h} := \mathbf{u} - \mathbf{u_h}$:

$$\mathbf{e_h}(\mathbf{t}) = \delta_{\mathbf{h}}(\mathbf{t}) + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{R}_{\omega,\alpha}(\mathbf{t},\mathbf{s})\delta_{\mathbf{h}}(\mathbf{s})\,\mathbf{ds}. \quad t \in I \ ,$$

with

$$\mathbf{R}\omega, \alpha(\mathbf{t},\mathbf{s}) = \mathbf{R}_{\mathbf{0},\alpha}(\mathbf{t},\mathbf{s})\mathbf{e}^{\mathbf{i}\omega(\mathbf{t}-\mathbf{s})} \ .$$

At $\underline{\mathbf{t} = \mathbf{t_n} + \mathbf{v}\mathbf{h_n}}$ $(v \in (0,1])$:

$$\delta_{\mathbf{h}}(\mathbf{t}) = -\mathbf{u_h}(\mathbf{t}) + \sum_{\ell=0}^{\mathbf{n}-1} \mathbf{h}_\ell \int_{\mathbf{0}}^{\mathbf{1}} \underbrace{\mathbf{K}_{\omega,\alpha}(\mathbf{t},\mathbf{t}_\ell + \mathbf{s}\mathbf{h}_\ell)} \mathbf{u_h}(\cdots)\,\mathbf{ds}$$

$$+\mathbf{h_n} \int_{\mathbf{0}}^{\mathbf{v}} \mathbf{K}_{\omega,\alpha}(\mathbf{t},\mathbf{t_n} + \mathbf{s}\mathbf{h_n})\mathbf{u_h}(\mathbf{t_n} + \mathbf{s}\mathbf{h_n})\,\mathbf{ds} + \mathbf{f}(\mathbf{t}) \ ,$$

and

$$\mathbf{e_h}(\mathbf{t}) = \delta_{\mathbf{h}}(\mathbf{t}) + \sum_{\ell=0}^{\mathbf{n}-1} \mathbf{h}_\ell \int_{\mathbf{0}}^{\mathbf{1}} \underbrace{\mathbf{R}_{\omega,\alpha}(\mathbf{t},\mathbf{t}_\ell + \mathbf{s}\mathbf{h}_\ell)} \delta_{\mathbf{h}}(\cdots)\,\mathbf{ds}$$

$$+\mathbf{h_n} \int_{\mathbf{0}}^{\mathbf{v}} \mathbf{R}_{\omega,\alpha}(\mathbf{t},\mathbf{t_n} + \mathbf{s}\mathbf{h_n})\delta_{\mathbf{h}}(\mathbf{t_n} + \mathbf{s}\mathbf{h_n})\,\mathbf{ds} \ .$$

Questions:

- **Choice of collocation points X_h ?**

  $\hookrightarrow$ **Lobatto** points:

$$X_h = \{t_n + c_k h_n : \underline{0 = c_1} < \cdots < \underline{c_m = 1}\}$$

(possibly with *multiplicities* greater than one, depending on complexity of kernel $K_{0,\alpha}(t,s)$).

- **Evaluation of highly oscillatory integrals ?**

  $\hookrightarrow$ *Filon-type methods:*

Iserles and **Nørsett** (2005); **S. Olver** (2006). The latter method (extending **D. Levin** (1997)) does not require computation of moments.

For integrands given by **Bessel functions**, *e.g.* in collocation equations for

$$\int_0^t J_0(\omega(t-s))u(s)\,ds = f(t) \; ,$$

use approach of **D. Levin** (1997).

- **Choice of basis functions ?**

Some open problems

- **Pantograph-type integral equations**:

$$\mathbf{u(t) = f(t) + \int_0^{qt} (t-s)^{-\alpha} K_0(t,s) e^{i\omega g(t-s)} u(s)\, ds},$$

with $\underline{\mathbf{0 < q < 1}}$, $0 \le \alpha < 1$, and $\omega \gg 1$.
↪ *Proportional delay* vanishing at $\mathbf{t = 0}$.

- Irregular oscillators $\mathbf{g(t-s)}$ with **stationary points** ?

(See *e.g.*: **D. Levin** (1997), **Iserles & Nørsett** (2004), **Iserles** (2005), **S. Olver** (2006), **Huybrechs and Vandewalle** (2006)).

- Semi-discretised **discontinuous Galerkin** methods for

$$\mathbf{\int_0^t J_0(\omega(t-s))u(s)\, ds = f(t) \quad (\omega \gg 1)}$$

(**B., Davies & Duncan** (2007)).
Choice of quadrature (for inner products) / connection with **collocation solution** in $\mathbf{S^{(-1)}_{m-1}(I_h)}$?
(**D. Levin** (1997))

↪ **Stability** and **convergence properties** of *semi-discretised* dG solution depend strongly on the **quadrature method** !