Construction and properties of Bayesian nonparametric regression models

Isaac Newton Institute, Cambridge

Models beyond the Dirichlet process

Antonio Lijoi

University of Pavia

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Outline

- Discrete nonparametric priors
  - Neutral to the right priors and models for the cumulative hazard
  - Normalized random measures with independent increments
  - Poisson–Kingman models
  - Species sampling models & stick–breaking priors
- Priors for density estimation
  - Hierarchical mixture models
  - Random Bernstein polynomials
  - Pólya trees

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For any $n \geq 1$,

$$(X_1, \dots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \dots, X_{\pi(n)})$$

for any permutation π of the vector $(1, \dots, n)$

$X^{(\infty)} = (X_n)_{n \geq 1}$	sequence of $\mathbb{X}$–valued random variables
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de Finetti's representation theorem

The sequence $X^{(\infty)}$ is exchangeable if and only if

$$\mathbb{P}\left[X^{(\infty)} \in A\right] = \int_{\mathscr{P}(\mathbb{X})} p^{\infty}(A)\, Q(\mathrm{d}p) \qquad \forall A \in \mathscr{X}^{\infty}$$

where p^{∞} is the infinite product measure $p \times p \times \cdots$ on $(\mathbb{X}^{\infty}, \mathscr{X}^{\infty})$

Conditionally on a random probability measure $\tilde{p}$, the observations $X_1, \ldots, X_n$ are iid with common distribution $\tilde{p}$, *i.e.*

$$\Pr\left[X_1 \in A_1, \ldots, X_n \in A_n \,\middle|\, \tilde{p}\right] = \prod_{i=1}^{n} \tilde{p}(A_i)$$

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Moreover

$$\hat{p}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \;\Rightarrow\; \tilde{p} \qquad \text{a.s.--}\mathbb{P}$$

and $\Rightarrow$ denotes "weak convergence"

Dirichlet process

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Reasons for the use of the Dirichlet process

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- Simple system of predictive distributions which gives rise to simulation algorithms useful, *e.g.*, in mixture models

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$$\tilde{\mu} = \tilde{\mu}_c + \sum_{i=1}^{M} V_i \, \delta_{x_i}$$

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- the random measure $\tilde{\mu}_c$ has a Lévy measure ν on $\mathbb{X} \times \mathbb{R}^+$

The Lévy–Khintchine representation yields

$$\mathbb{E}\left[\mathrm{e}^{-\int_{\mathbb{X}} f(x)\,\tilde{\mu}_c(\mathrm{d}x)}\right] = \exp\left\{-\int_{\mathbb{X}\times\mathbb{R}^+}\left[1-\mathrm{e}^{-s\,f(x)}\right]\,\nu(\mathrm{d}x,\mathrm{d}s)\right\}$$

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Suppose there exist

- measure η on $\mathbb{X}$
- measure ρ_x on $\mathbb{R}^+$, for any $x \in \mathbb{X}$

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If $\rho_x = \rho$, then $\tilde{\mu}_c$ is said *homogeneous*

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Definition (Doksum, 1974)

A random distribution function F on $\mathbb{R}^+$ is neutral to the right (NTR) if, for any $0 < t_1 < t_2 < \cdots < t_k < \infty$ and $k \geq 1$, the random variables

$$F(t_1), \quad \frac{F(t_2) - F(t_1)}{1 - F(t_1)}, \quad \ldots \quad , \frac{F(t_k) - F(t_{k-1})}{1 - F(t_{k-1})}$$

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and $Y = \{Y_t : \ t \geq 0\}$ is an independent increments process

Connection with independent increments processes (Doksum, 1974)

F is NTR if and only if there exists a process with independent increments $Y = \{Y_t : \ t \geq 0\}$ such that $\mathbb{P}\left[\lim_{t\to\infty} Y_t = \infty\right] = 1$, $\mathbb{P}\left[Y_0 = 0\right] = 1$ and

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where we suppose $\tilde{\mu}$ has no fixed points of discontinuity

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Description of Y^* (Ferguson, 1974)

If $X_1^*, \ldots, X_k^*$ are the distinct observations among the X_i's,

$$Y_t^* = Y_t^c + \sum_{\{i\colon X_i^* \leq t\}} J_i$$

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$$\nu^*((0,t], \mathrm{d}s) = \int_{(0,t]} \mathrm{e}^{-\bar{N}(x)\, s}\, \eta(\mathrm{d}x)\, \rho_x(\mathrm{d}s)$$

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$$\rho_{x_i^*} = \rho \quad \Longrightarrow \quad \text{distribution of } J_i \text{ does not depend on } X_i^*$$

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$$\Delta_i = \mathbb{I}_{(0,c_i)}(X_i) = \begin{cases} 1 & \text{if } X_i \text{ is actually observed} \\ 0 & \text{if } X_i \text{ is censored} \end{cases}$$

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Ferguson & Phadia (1979): NTR are conjugate also with respect to right–censored data

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Susarla and van Ryzin (1976)

- $F|\text{Data} \neq$ Dirichlet
- Kaplan Meier estimator as a limit of the Bayes' estimate $\mathbb{E}[S(t)|\text{data}]$, with $\eta(\mathbb{R}^+) \to 0$

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Definition

F a *beta–Stacy* process with parameters c and M

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F a *beta–Stacy* process with parameters c and M if $F \sim$NTR(Y) and Y is an independent increments process with Lévy measure

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An independent increments process $Y = \{Y_t : \ t \geq 0\}$ with Lévy measure

$$\boldsymbol{\nu((0,t],ds) = \frac{ds}{1-e^{-s}} \int_{(0,t]} e^{-s\,[\beta(x)+\eta(\{x\})]}\, dF_\eta^c(x)}$$

is a log-beta process

Suitable choices of the measure η and of the function β of a log–beta process $Y = \{Y_t : t \geq 0\}$ lead to a beta–Stacy process $F = 1 - \mathrm{e}^{-Y}$ with parameters

$$M(x) = 1 - \mathrm{e}^{-\int_{(0,x]} \eta(\mathrm{d}v)/\beta(v)} \qquad c(x) = \frac{\beta(x)}{1 - M(x)}$$

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$$F \mid \text{ Data } \sim \text{ NTR}(Y^*)$$

where $Y^* = \{Y_t^* : t \geq 0\}$ is a log–beta process with parameters η^* and β^* such that

$$\mathbf{F}_{\eta^*}(\mathbf{x}) = \mathbf{F}_{\eta}(\mathbf{x}) + \sum_{\{\mathbf{i}:\, \boldsymbol{\Delta}_\mathbf{i}=1\}} \mathbb{I}(\mathbf{T}_\mathbf{i} \leq \mathbf{x})$$

$$\boldsymbol{\beta}^*(\mathbf{x}) = \boldsymbol{\beta}(\mathbf{x}) + \bar{\mathbf{N}}(\mathbf{x}) - \sum_{\{\mathbf{i}:\, \boldsymbol{\Delta}_\mathbf{i}=1\}} \mathbb{I}(\mathbf{T}_\mathbf{i} = \mathbf{x})$$

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Bayes estimate of F, under squared loss, is

$$\hat{F}(t) = \mathbb{E}[F(t) \,|\, \text{data}] = M^*(t)$$

and, as $c(\,\cdot\,) \to 0$, $\hat{F}$ converges to the Kaplan–Meier estimator of F

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- Computational algorithms: Ferguson & Klass (1972), Damien, Laud & Smith (1995), Walker & Damien (1998, 2000), Wolpert & Ickstadt (1998)
- Connections with some recent results on combinatorics. See Gnedin & Pitman (2005)

Models for cumulative hazards

As an alternative approach to inferential problems in survival analysis, instead of specifying a prior for F one could focus on the hazard rate

$$\frac{\mathrm{d}F(x)}{1 - F(x-)} = \text{Pr}[\text{death at } (x, x + \mathrm{d}x] \mid \text{ survival time } \geq x]$$

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Note that from the above relation one deduces

$$F(t) = 1 - \prod_{[0,t]} \{1 - \mathrm{d}A_x(F)\}$$

where $\prod_{[0,t]}$ is the product integral on $[0, t]$

The beta process (Hjort, 1990)

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Let

- A_0 be the baseline cumulative hazard with discontinuities at $x_1, x_2, \ldots$
- $\mathrm{d}A_{0,x_j}$ be the height of the jump of A_0 at x_j
- A_0^c continuous part of A_0
- $A_c = \{A_t^c : \ t \geq 0\}$ independent increments process with no fixed points of discontinuity and with Lévy measure

$$\nu(\mathrm{d}x, \mathrm{d}s) = \mathbf{c(x)\, s^{-1}\, (1 - s)^{c(x)-1}\, ds}\, \mathrm{d}A_{0,c}(x) = \boldsymbol{\rho_x(ds)}\, \eta(\mathrm{d}x)$$

  for any $x \geq 0$ and $0 < s < 1$ and for some function $c(\,\cdot\,)$
- $S_1, S_2, \ldots$ are mutually independent, independent from A_c and

$$S_j \sim \mathrm{Beta}(c(x_j)\mathrm{d}A_{0,x_j},\ c(x_j)[1 - \mathrm{d}A_{0,x_j}])$$

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See Hjort (1990) and Dey, Erickson & Ramamoorthi (2003)

Conjugacy of the beta process (Hjort, 1990)

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Posterior distribution

Let $A \sim \text{Beta}(c, A_0)$. Then

$$A|\,\text{Data} \sim \text{Beta}\left(c + \bar{N}, \int \frac{\mathrm{d}\bar{L} + c\,\mathrm{d}A_0}{c + \bar{N}}\right)$$

The previous statement implies that, if

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The Bayes estimators of A and F, under squared loss, are

$$\hat{A}_x = \int_{(0,x)} \frac{c\,\mathrm{d}A_0 + \mathrm{d}\bar{L}}{c + \bar{N}}$$

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Extensions of Hjort's work

- Discrete model with dependent hazard rates: Gamerman (1991), Gray (1994), Arias & Gasbarra (1994), Nieto–Barajas & Walker (2002)
- Inference on counting processes: Kim (1999)
- Spatial neutral to the right: James (2006)
- Connections with the Indian buffet process: Griffiths & Ghahramani (2006), Thibaux & Jordan (2007) and Teh, Görür & Ghahramani (2007). See Yee Whye's talk

Models for hazard rates

Instead of defining a prior for A one can set a prior for the hazard rate

$$\alpha(s) = \frac{F'(s)}{1 - F(s)}$$

If $\tilde{\mu}$ is a completely random measure on a complete and separable metric space $\mathbb{Y}$ and $k(\,\cdot\,|\,\cdot\,)$ is some kernel such that

$$\alpha(s) = \int_{\mathbb{Y}} k(s|y)\,\tilde{\mu}(\mathrm{d}y)$$

and $\lim_{T\to\infty} \int_0^T \alpha(s)\,\mathrm{d}s = \infty$

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Multiplicative intensity model: data from counting processes

$$\text{Intensity rate at } s \;=\; Z_{i,s}\, \alpha(s)$$

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Multiplicative intensity model: data from counting processes

$$\text{Intensity rate at } s \;=\; Z_{i,s}\,\alpha(s)$$

The factor $Z_{i,s}$ characterizes the censoring mechanism

$$Z_{i,s} = \mathbb{I}(T_i \geq s) \qquad \text{right censored data}$$

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Models for hazard rates

Instead of defining a prior for A one can set a prior for the hazard rate

$$\alpha(s) = \frac{F'(s)}{1 - F(s)}$$

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Different choices of $\tilde{\mu}$, $Z_{i,s}$ and of the kernel k yield different models used in the literature. Dykstra & Laud (1981) and Lo & Weng (1989).

Multiplicative intensity models governed by any completely random measure $\tilde{\mu}$ with Lévy measure

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Full Bayesian analysis requires implementation of simulation scheme: Nieto–Barajas & Walker (2004), Ishwaran & James (2004) and James (2005)

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Definition

The random probability measures

$$\tilde{p} = \frac{\tilde{\mu}}{\tilde{\mu}(\mathbb{X})}$$

is a *normalized random measure with independent increments* (NRMI)

Use of NRMIs

- Ferguson (1973): Dirichlet process as a normalized gamma process
- Kingman (1975): normalized α–stable process for application to optimal storage
- Regazzini, L. & Prünster (2003): distributions of linear functionals of non–homogeneous NRMIs on $\mathbb{R}$
- James (2002): non–homogeneous NRMI on more general spaces
- James, L. & Prünster (2005): posterior analysis of non–homogeneous NRMIs
- Griffin (2007): use of NRMIs for constructing time dependent random probability measures

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Brix (1999). The random probability measure $\tilde{p} = \tilde{\mu}_{\alpha,\tau} / \tilde{\mu}_{\alpha,\tau}(\mathbb{X})$ is a *generalized gamma process prior*. See Pitman (2003), James (2002) and L., Mena & Prünster (2007)

Inference with NRMIs

Suppose

$$\begin{aligned} X_i \,|\, \tilde{p} &\stackrel{\text{iid}}{\sim} \tilde{p} \qquad i = 1, \ldots, n \\ \tilde{p} &= \tilde{\mu}/\tilde{\mu}(\mathbb{X}) \end{aligned}$$

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- The density functions q_0 and $q_{n_1, \ldots, n_k}$ expressed in terms of the Lévy measure $\rho_x(\mathrm{d}s)\eta(\mathrm{d}x)$

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Data $X_1, \ldots, X_n$ consist of k distinct values $X_1^*, \ldots, X_k^*$. Then

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(i) $\tilde{\mu}_u$ is a completely random measure with no fixed jumps and with Lévy measure

$$\nu^{(u)}(\mathrm{d}x, \mathrm{d}s) = \mathrm{e}^{-us} \, \rho_x(\mathrm{d}s) \, \eta(\mathrm{d}x)$$

(ii) jump $J_i^{(u)}$ has density $f_i(s)\mathrm{d}s \propto s^{n_i} \mathrm{e}^{-us} \rho_{X_i^*}(\mathrm{d}s)$

(iii) $\tilde{\mu}_u$ and $J_i^{(u)}$ $(i = 1, \ldots, k)$ are independent

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James, L. & Prünster (2005)

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Predictive distribution

Set $P_0 = \eta/\eta(\mathbb{X})$. Then

$$\Pr[X_{n+1} \in \mathrm{d}x_{n+1} | X_1, \ldots, X_n] = w^{(n)}\, P_0(\mathrm{d}x_{n+1}) + \frac{1}{n} \sum_{j=1}^{k} w_j^{(n)}\, \delta_{X_j^*}(\mathrm{d}x_{n+1})$$

where

$$w^{(n)} = \frac{1}{n} \int_0^{+\infty} u\, \tau_1(u|x_{n+1})\, q_{n_1,\ldots,n_k}(u)\, \mathrm{d}u$$

$$w_j^{(n)} = \int_0^{+\infty} u\, \frac{\tau_{n_j+1}(u|X_j^*)}{\tau_{n_j}(u|X_j^*)}\, q_{n_1,\ldots,n_k}(u)\, \mathrm{d}u$$

and $\tau_m(u|x) = \int_0^\infty s^m\, \mathrm{e}^{-us}\, \rho_x(\mathrm{d}s)$

Sampling conditional on U_n

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Predictive

$$\Pr[X_{i+1} \in \mathrm{d}x_{i+1} | X_1, \ldots, X_i, U_i = u] = m(\mathrm{d}x_{i+1} | x_1, \ldots, x_i, u)$$

$$\propto \kappa_1(u)\, \tau_1(u|x_{i+1})\, P_0(\mathrm{d}x_{i+1}) + \sum_{j=1}^{k} \frac{\tau_{n_{j,i}+1}(u|X^*_{j,i})}{\tau_{n_{j,i}}(u|X^*_{j,i})}\, \delta_{X^*_{j,i}}(\mathrm{d}x_{i+1})$$

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Moreover

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(i) Generalization of Pólya urn scheme: one can generate a sample $X_1, \ldots, X_n$ from $\tilde{p}$

- Generate U_0 from q_0
- Generate X_1 from $m(\mathrm{d}x|U_0)$
- Generate U_1 from q_1
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(ii) Sample from the posterior random measure, given U_n and the data,

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- Sample $J_i^{(U_n)}$ from $f_i(s)\mathrm{d}s \propto s^{n_i}\,\mathrm{e}^{-U_n s}\,\rho_{X_i^*}(\mathrm{d}s)$
- Simulate $\tilde{\mu}_{U_n}$ via the Ferguson & Klass algorithm

See Luis' talk.

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The random probability measure

$$\tilde{\mathrm{p}}_{(\alpha,\theta)} = \frac{\tilde{\mu}_{(\alpha,\theta)}}{\tilde{\mu}_{(\alpha,\theta)}(\mathbb{X})}$$

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- For any I such that $\rho(I) < \infty$

$$\mathbb{P}[\tilde{\Pi}(I) = k] = \frac{\mathrm{e}^{-\rho(I)}\, \rho(I)^k}{k!} \qquad k = 0, 1, \ldots$$

It is assumed that the Lévy measure ρ satisfies suitable conditions that guarantee

$$\mathbb{P}\left[\sum_{i\geq 1} J_i < \infty\right] = 1$$

and we set

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It is also assumed that the distribution of T has density function f_T

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Definition

Let

$$p(\rho|t) \; = \; \text{conditional distribution of } (P_i)_{i\geq 1} \text{ on } S^*, \text{ given } T = t$$

$$\gamma \; = \; \text{probability distribution on } \mathbb{R}^+$$

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$\mathrm{PK}(\rho, \gamma)$ distribution = discrete nonparametric prior

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If $\alpha \to 0$, the $\mathrm{PD}(\alpha,\theta)$ process reduces to the one–parameter $\mathrm{PD}(\theta)$ process introduced by Kingman (1975)

Since a PK(ρ, γ) distribution selects (almost surely) discrete probability distributions, if

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then

$$\mathbb{P}[X_i = X_j] > 0 \qquad \forall i \neq j$$

and the n data $X_1, \ldots, X_n$ can be partitioned into

- $K_n = k$ classes such that two observations belong to the same class if and only if they coincide
- the K_n classes have frequencies $N_{1,n}, \ldots, N_{K_n,n}$ such that $\sum_{i=1}^{K_n} N_{i,n} = n$

For the PD(α, θ) process one has

$$\Pr(\text{data in } k \text{ classes with frequencies } n_1, \ldots, n_k)$$

$$= \Pr[K_n = k, N_{1,n} = n_1, \ldots, N_{k,n} = n_k]$$

$$= \frac{\prod_{i=1}^{k-1}(\theta + i\alpha)}{(\theta+1)_{n-1}} \prod_{j=1}^{k} (1-\alpha)_{n_j - 1}$$

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A sequence $(X_i)_{i\geq 1}$ such that

$$\begin{aligned} X_i \mid \tilde{p} \quad &\sim \quad \tilde{p} \qquad\qquad i = 1, 2, \ldots \\ \tilde{p} \quad &= \quad \text{species sampling model} \end{aligned}$$

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such that

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Posterior distribution of a PD(α, θ) model (Pitman, 1996)

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- Ishwaran & James (2003): species sampling models with stick–breaking construction
- James, L. & Prünster (2005): species sampling models with weights obtained by normalizing jumps of a completely random measure

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Nonparametric hierarchical model

$$
\begin{aligned}
Y_i | X_i &\overset{\text{ind}}{\sim} f(Y_i | X_i) \qquad i = 1, \ldots, n \\
X_i | \tilde{p} &\overset{\text{iid}}{\sim} \tilde{p} \qquad i = 1, \ldots, n \\
\tilde{p} &= \text{discrete random probability measure}
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Equivalently

$$\begin{aligned} Y_i | \tilde{f} &\overset{\text{iid}}{\sim} \tilde{f} \qquad i = 1, \dots, n \\ \tilde{f}(\,\cdot\,) &= \int_{\mathbb{X}} f(\,\cdot\,|x)\, \tilde{p}(\mathrm{d}x) \end{aligned}$$

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Recall that $X_1^*, \ldots, X_k^*$ denote the k distinct values among the $X_1, \ldots, X_n$ and

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Number of distinct X_i's

$\Downarrow$

Number K_n of clusters for the observations Y_i

- Evaluate $\mathbb{E}[\tilde{f} \mid Y_1, \ldots, Y_n]$
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Let

$\mathscr{P}_{n,k}$ set of all partitions $\mathbf{p}$ of $\{1, \ldots, n\}$ into k sets

$\mathbf{Y}^{(n)}$ vector $(Y_1, \ldots, Y_n)$

and

$$\mathscr{L}(\mathrm{d}x_1^*, \ldots, \mathrm{d}x_k^* \mid \mathbf{Y}^{(n)}, \mathbf{p}) = \text{ posterior distribution of the latent}$$

A posterior estimate of $f(y)$

$$\mathbb{E}\left[\tilde{f}(y) \mid \mathbf{Y}^{(n)}\right] = \sum_{k=1}^{n} \sum_{\mathbf{p} \in \mathscr{P}_{n,k}} \int_{\mathbb{X}^k} \left\{ \int_{\mathbb{X}} f(y|x)\, \mathbb{E}\left[\tilde{P}(\mathrm{d}x) | x_1^*, \ldots, x_k^*\right] \right\}$$

$$\times \; \mathscr{L}(\mathrm{d}x_1^*, \ldots, \mathrm{d}x_k^* \mid \mathbf{Y}^{(n)}, \mathbf{p})$$

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$\mathbb{E}\left[\tilde{P}(\mathrm{d}x)|x_1^*,\ldots,x_k^*\right]$ predictive distribution

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The posterior distribution of K_n cannot be evaluated exactly, as well

MCMC algorithm

Suppose

$$\tilde{p} = \frac{\tilde{\mu}}{\tilde{\mu}(\mathbb{X})}$$

where $\tilde{\mu}$ has intensity $\nu(\mathrm{d}x, \mathrm{d}s) = \rho(\mathrm{d}s)\,\eta(\mathrm{d}x)$.

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In order $X_1, \ldots, X_n$ from the posterior $\mathscr{L}(\,\cdot\,|\mathbf{Y}^{(n)})$ one makes use of

$$\Pr\left[X_i \in \mathrm{d}x_i \,\middle|\, \mathbf{X}_{-i}^{(n)}, \mathbf{Y}^{(n)}\right] = q_{i,0}^*\, P_0(\mathrm{d}x_i)\, f(Y_i|x_i) + \sum_{j=1}^{k_{i,n-1}} q_{i,j}^*\, \delta_{X_{i,j}^*}(\mathrm{d}x_i)$$

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where

- $\mathbf{X}_{-i}^{(n)} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$
- $k_{i,n-1}$ = number of distinct observations, $X_{i,j}^*$, in $\mathbf{X}_{-i}^{(n)}$

Weights

$$q^*_{i,0} \propto w_i^{(n-1)} \int_{\mathbb{X}} f(Y_i|x)\, P_0(\mathrm{d}x) \qquad q^*_{i,j} \propto w_{i,j}^{(n-1)}\, f(Y_i|X^*_{i,j})$$

such that $\sum_{j=0}^{k_{i,n-1}} q^*_{i,j} = 1$. Moreover,

$$w_{i,0}^{(n-1)} = \frac{1}{n-1} \int_0^\infty u\, \tau_1(u)\, q_{n_1,\dots,n_{k_{i,n-1}}}(u)\, \mathrm{d}u$$

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Non–homogeneous case: the weights depend on the $X^*_{i,j}$ as well

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$$
\begin{aligned}
X_{1,t} &\sim \Pr\left[X_1 \,\middle|\, X_{2,t-1}, \ldots, X_{n,t-1}, \mathbf{Y}^{(n)}\right] \\
X_{2,t} &\sim \Pr\left[X_2 \,\middle|\, X_{1,t}, X_{3,t-1}, \ldots, X_{n,t-1}, \mathbf{Y}^{(n)}\right] \\
\vdots \quad & \quad \vdots \\
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$$\vdots \qquad \vdots$$

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Output of the algorithm is used to evaluate

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(I) A posterior estimate of f

$$\hat{f}(y) = \frac{1}{N} \sum_{t=1}^{N} \int_{\mathbb{X}} f(y|x)\, \mathbb{E}\left[\tilde{P}(\mathrm{d}x) \,|\, X^*_{1,t}, \dots, X^*_{k_t,t}\right]$$

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(II) Posterior of the number of clusters K_n

$$\mathbb{P}\left[K_n = k \,|\, \mathbf{Y}^{(n)}\right] \approx \frac{1}{N} \sum_{t=1}^{N} \mathbb{I}\{k^{(t)} = k\}$$

where $k^{(t)}$ is the number of distinct values among $X_{1,t}, \ldots, X_{n,t}$ generated at iteration t.

MCMC algorithm inspired by the work of Escobar (1988,1994) and Escobar & West (1995) for the MDP.

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$$\pi(x_1, \ldots, x_{k^{(t)}} \,|\, \mathbf{Y}, k^{(t)}) \propto \prod_{j=1}^{k^{(t)}} P_0(\mathrm{d}x_j) \prod_{i \in I_j} f(y_i | x_j)$$

See MacEachern (1994,1998) and Ishwaran & James (2001).

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- Algorithm for stick–breaking priors: Ishwaran & James (2001,2003), Papaspiliopoulos & Roberts (2004) and Walker (2006)

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Mixture of the Dirichlet process: $\tilde{p}$ a Dirichlet process with baseline measure $\theta\, P_0$

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Mixture of more general discrete priors: one can take $\tilde{p}$ not to be the Dirichlet process. For example

$$\tilde{p} = \text{ normalized random measure}$$

$$\tilde{p} = \text{ stick–breaking prior}$$

Why using a mixing measure different from the Dirichlet process?

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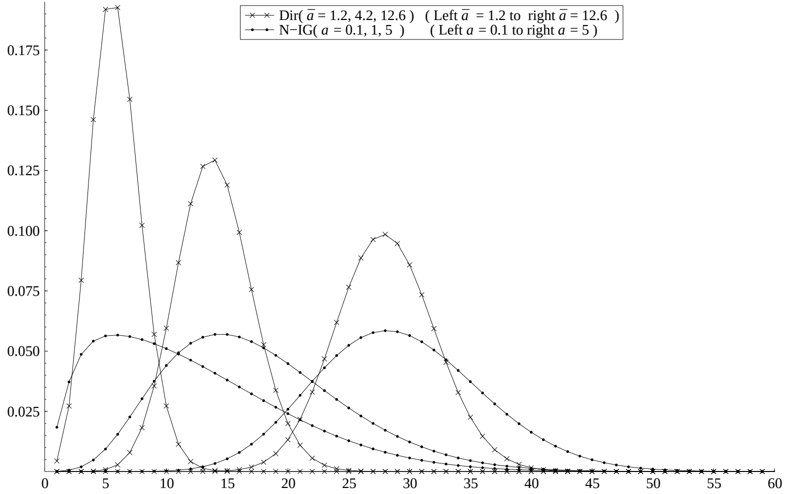
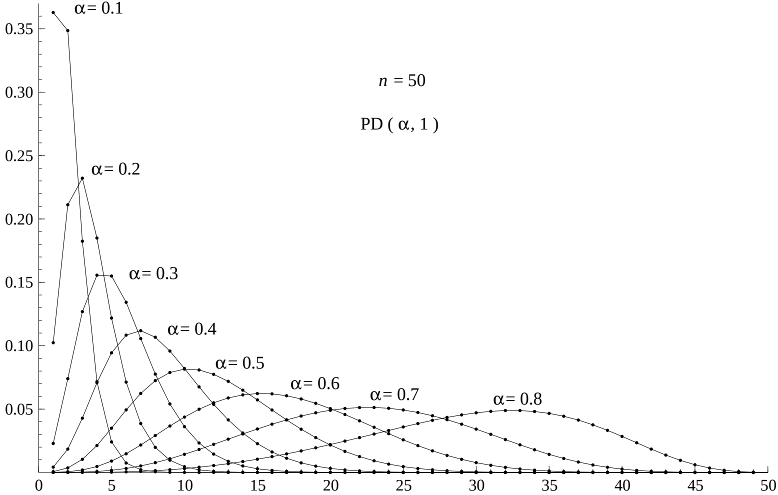
For mixtures with more general mixing random probabilities see Ishwaran & James (2001, 2003), L., Mena & Prünster (2005, 2007) and, for an application to AFT models, see Argiento, Guglielmi & Pievatolo (2007).



Dir($\bar{a}$ = 1.2, 4.2, 12.6)   (Left $\bar{a}$ = 1.2 to right $\bar{a}$ = 12.6)
N−IG(a = 0.1, 1, 5)   (Left a = 0.1 to right a = 5)
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0
5
10
15
20
25
30
35
40
45
50
55
60



α= 0.1
α= 0.2
α= 0.3
α= 0.4
α= 0.5
α= 0.6
α= 0.7
α= 0.8
n = 50
PD (α, 1)
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0
5
10
15
20
25
30
35
40
45
50

Density estimation: random Bernstein polynomial (Petrone, 1999)

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Take $[a, b] = [0, 1]$ and define

$$B(x; k, F) = \sum_{j=1}^{k} F(j/k) \binom{k}{j} x^j (1 - x)^{k-j} \qquad x \in [0, 1]$$

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Case where

- $F : [0, 1] \to \mathbb{R}$ is a cumulative distribution function with density f
- $w_{j,k} = F(j/k) - F((j-1)/k)$
- $\mathbf{w}_k = (w_{1,k}, \ldots, w_{k,k})$

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$$\tilde{f}_k(x; \mathbf{w}_k) = \sum_{j=1}^{k} w_{j,k} \text{Beta}(x; j, k-j+1)$$

$$\lim_{k\to\infty} \tilde{f}_k(x;\mathbf{w}_k) = f(x) \qquad \forall x \in [0,1]$$

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Prior on $(K, \mathbf{W}_k)$

$$(W_{1,K}, \ldots, W_{K,K}) | K \sim \mathrm{Dir}(\alpha_{1,K}, \ldots, \alpha_{K,K})$$
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Posterior of $(K, \mathbf{W}_K)$

$$\pi(k, \mathbf{w}_k|X_1, \ldots, X_n) \propto \pi(k)\pi(\mathbf{w}_k|k) \prod_{i=1}^{n} \tilde{f}_k(x_i; \mathbf{w}_k)$$

and density estimate

$$\hat{f}(x) = \sum_{k\geq 1} \tilde{f}_K(x; \mathbb{E}[\mathbf{W}_K|X_1, \ldots, X_n, K])\, \pi(K|X_1, \ldots, X_n)$$

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mixture of random Bernstein polynomials

MCMC for random Bernstein polynomials

Introduce auxiliary random variables Y_i such that

$$
\begin{aligned}
X_i | Y_i, \tilde{F}, k &\overset{\text{ind}}{\sim} \text{Beta}(j, k + j - 1) \qquad \text{if } Y_i \in \left(\frac{j-1}{k}, \frac{j}{k}\right] \\
Y_i | \tilde{F}, k &\overset{\text{iid}}{\sim} \tilde{F} \\
\tilde{F} | k &\sim \text{Dir}(\theta\, F_0) \\
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Algorithm

- Generate $(k^{(t)}, \mathbf{y}^{(t)})$ from $\pi(K, Y_1, \ldots, Y_n | X_1, \ldots, X_n)$
- Generate $\mathbf{w}_k^{(t)}$ from $\pi(\mathbf{W}_{k^{(t)}} | k^{(t)}, \mathbf{y}^{(t)}, X_1, \ldots, X_n)$ which is

$$\text{Dir}(\alpha^*_{1,k}, \ldots, \alpha^*_{k,k})$$

In order to sample $(k^{(t)}, \mathbf{y}^{(t)})$ one makes use of the full conditionals

- $\pi(K|\mathbf{Y}, \mathbf{X})$ is proportional to

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where $\zeta_k(Y_i) = j$ if $Y_i \in (\frac{j-1}{k}, \frac{j}{k}]$

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- $\pi(Y_i|k, \mathbf{Y}_{-i}, \mathbf{X})$ is proportional to

$$q(x_i, k) f_0(Y_i) \text{Beta}(x_i; \zeta_k(Y_i),\ k - \zeta_k(Y_i) + 1)\, \mathrm{d}Y_i + \sum_{j \neq i} q^*_{x_j,k}\, \delta_{Y_j}(\mathrm{d}Y_i)$$

  where

$$q(x_i, k) \propto \theta\, \text{Beta}(x_i; k, F_0) \qquad q^*_{x_j,k} \propto \text{Beta}(x_j; \zeta_k(Y_j),\ k - \zeta_k(Y_j) + 1)$$

  are such that $q(x_i, k) + \sum_{j \neq i} q^*_{x_j,k} = 1$

Density estimation: Pólya trees

Consider a sequence of nested partitions of $\mathbb{X} \subset \mathbb{R}$

- $\Pi_1 = \{B_0, B_1\}$
- $\Pi_2 = \{B_{00}, B_{01}, B_{10}, B_{11}\}$ where

$$B_{00} \cup B_{01} = B_0 \qquad B_{10} \cup B_{11} = B_1$$

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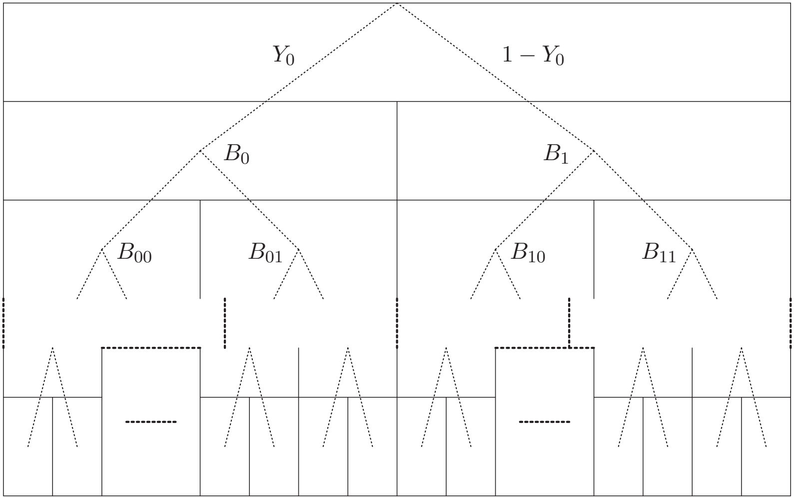

Y_0
$1 - Y_0$
B_0
B_1
B_{00}
B_{01}
B_{10}
B_{11}

Definition

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- $\{\tilde{p}(B_{\varepsilon 0} | B_{\varepsilon}) : \varepsilon \in E^*\}$ is a collection of independent random variables
- $\tilde{p}(B_{\varepsilon 0} | B_{\varepsilon}) \sim \text{Beta}(\alpha_{\varepsilon 0}, \alpha_{\varepsilon 1})$

In symbols, $\tilde{\mathbf{p}} \sim \mathbf{PT}(\mathscr{A}, \mathbf{\Pi})$

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See Ferguson (1974), Mauldin, Sudderth & Williams (1992), Lavine (1992, 1994)

Definition

Let $E^* = \cup_{k\geq 1} E^k$ and $\mathscr{A} = \{\alpha_{\varepsilon} : \varepsilon \in E^*\}$ is a set of non negative real numbers. A random probability measure $\tilde{p}$ is a Pólya tree process with respect to $\Pi = \{\Pi_k : k \geq 1\}$ and $\mathscr{A}$ if

- $\{\tilde{p}(B_{\varepsilon 0}|B_{\varepsilon}) : \varepsilon \in E^*\}$ is a collection of independent random variables
- $\tilde{p}(B_{\varepsilon 0}|B_{\varepsilon}) \sim \text{Beta}(\alpha_{\varepsilon 0}, \alpha_{\varepsilon 1})$

In symbols, $\tilde{\mathbf{p}} \sim \mathbf{PT}(\mathscr{A}, \boldsymbol{\Pi})$

See Ferguson (1974), Mauldin, Sudderth & Williams (1992), Lavine (1992, 1994)

If

$$\alpha_{\varepsilon} = \alpha_{\varepsilon 0} + \alpha_{\varepsilon 1}$$

then $\tilde{p}$ is the Dirichlet process

Let $Y_{\boldsymbol{\varepsilon}} = P(B_{\boldsymbol{\varepsilon}0} | B_{\boldsymbol{\varepsilon}})$. For any $\boldsymbol{\varepsilon} \in E^k$

$$\tilde{p}(B_{\boldsymbol{\varepsilon}}) = \left\{ \prod_{j=1,\, \varepsilon_j = 0}^{k} Y_{\varepsilon_1 \cdots \varepsilon_{j-1}} \right\} \left\{ \prod_{j=1,\, \varepsilon_j = 1}^{k} (1 - Y_{\varepsilon_1 \cdots \varepsilon_{j-1}}) \right\}$$

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Absolute continuity

Suppose $\mathbb{X} = [0, 1]$ and Π is a sequence of partitions of $[0, 1]$ into dyadic intervals: when $\boldsymbol{\varepsilon} \in E^k$ a set $B_{\boldsymbol{\varepsilon}} \in \Pi_k$ is of the form

$$\left(\sum_{j=1}^{k} \varepsilon_j \, 2^{-j}, \; \sum_{j=1}^{k} \varepsilon_j \, 2^{-j} + 2^{-k} \right]$$

If $\tilde{p} \sim \mathrm{PT}(\mathscr{A}, \Pi)$ and $\alpha_{\varepsilon_1 \cdots \varepsilon_k} = k^2$ then $\tilde{p}$ is (almost surely) a probability distribution absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$

Marginal distribution: Using independence of the beta random variables one has

$$\mathbb{P}[X_1 \in B_{\boldsymbol{\varepsilon}}] = \mathbb{E}[\tilde{p}(B_{\boldsymbol{\varepsilon}})] = \prod_{i=1}^{k} \frac{\alpha_{\varepsilon_1 \cdots \varepsilon_i}}{\alpha_{\varepsilon_1 \cdots \varepsilon_{i-1} 0} + \alpha_{\varepsilon_1 \cdots \varepsilon_{i-1} 1}}$$

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Posterior distribution: If

$$\begin{aligned} X_i \,|\, \tilde{p} \quad &\sim \quad \tilde{p} \qquad\qquad i = 1, \ldots, n \\ \tilde{p} \quad &\sim \quad \mathrm{PT}(\mathscr{A}, \Pi) \end{aligned}$$

then

$$\tilde{p} \,|\, (X_1, \ldots, X_n) \sim \mathrm{PT}(\mathscr{A}^*, \Pi)$$

where $\mathscr{A}^* = \{\alpha_{\boldsymbol{\varepsilon}}(\mathbf{X}) : \boldsymbol{\varepsilon} \in E^*\}$ and

$$\alpha_{\boldsymbol{\varepsilon}}(\mathbf{X}) = \alpha_{\boldsymbol{\varepsilon}} + \sum_{i=1}^{n} \mathbb{I}_{B_{\boldsymbol{\varepsilon}}}(X_i)$$

Using the representations of the marginal and of the posterior distribution one obtains

The predictive distribution: For any $\boldsymbol{\varepsilon} \in E^k$

$$\Pr[X_{n+1} \in B_{\boldsymbol{\varepsilon}} | X_1, \ldots, X_n] = \frac{\alpha_{\varepsilon_1} + n_{\varepsilon_1}}{\alpha_0 + \alpha_1 + n} \times$$

$$\times \; \frac{\alpha_{\varepsilon_1 \varepsilon_2} + n_{\varepsilon_1 \varepsilon_2}}{\alpha_{\varepsilon_1 0} + \alpha_{\varepsilon_1 1} + n_{\varepsilon_1}} \cdots \frac{\alpha_{\varepsilon_1 \cdots \varepsilon_k} + n_{\varepsilon_1 \cdots \varepsilon_{k-1}}}{\alpha_{\varepsilon_1 \cdots \varepsilon_{k-1} 0} + \alpha_{\varepsilon_1 \cdots \varepsilon_{k-1} 1} + n_{\varepsilon_1 \cdots \varepsilon_k}}$$

where $n_{\varepsilon_1 \cdots \varepsilon_i} =$ number of X_j's in $B_{\varepsilon_1 \cdots \varepsilon_i}$ $(j = 1, \ldots, n)$

Centering of $\tilde{p}$

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Then, for any $\boldsymbol{\varepsilon} \in E^k$ and $k \geq 1$ and

$$\mathbb{E}\left[\tilde{p}(B_{\boldsymbol{\varepsilon}})\right] = \prod_{i=1}^{k} \frac{\alpha_{\varepsilon_1 \cdots \varepsilon_i}}{\alpha_{\varepsilon_1 \cdots \varepsilon_{i-1} 0} + \alpha_{\varepsilon_1 \cdots \varepsilon_{i-1} 1}} = 2^{-k} = P_0(B_{\boldsymbol{\varepsilon}})$$

- Drawback: posterior inferences depend on the partition Π
  Lavine (1992): mixture of Pólya trees

$$\begin{aligned}\tilde{p} \,|\, \theta, \xi &\sim \mathrm{PT}(\mathscr{A}^{\theta}, \Pi^{\xi}) \\ \theta &\sim \pi_1(\theta) \\ \xi &\sim \pi_2(\theta)\end{aligned}$$

  See also Hanson & Johnson (2002)