Some good news about nonparametric priors in density estimation

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joint work with I. Prünster

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- Models based on mixtures

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- Tools for the sensitivity Analysis

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Introduction

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- Mixture models:

$$f(x) = \int k(x, y)\mathrm{P}(\mathrm{d}y)$$

where $k(x, y)$ is a parametric kernel and P a mixing distribution

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  - Classical: Take P the e.d.f. $\Rightarrow$ popular kernel density estimator (Silverman, 1986)
  - Bayesian: Take P as a NP prior $\Rightarrow$ $f(x)$ is a random density function and the Bayes estimate becomes

$$\hat{f}_n(x) = \mathrm{E}\left[\int k(x,y)\mathrm{P}(\mathrm{d}y)\Big| X_1,\ldots,X_n\right]$$

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- Ferguson (1983): Formulated the MDP as a hierarchical model

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- Escobar and West (1995): Developed MCMC algorithms $\Rightarrow$ MDP to be exploited in several ways

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- James (2006): P $\sim$ Spatial neutral to the right

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- Lijoi, Mena and Prünster (2005, 2007): Studied special tractable cases of mixtures of NRMI and highlighted their different cluster behaviour with respect to the MDP's.
- Dunson and Park (2006): Introduced dependent stick-breaking priors for epidemiologic and bioassay problems.

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  - $\bar{k}$ is a non-negative weighting function
  - $A(\cdot)$ in an IAP

$$\Leftrightarrow A(y) \text{ is weighted with } \bar{k} \text{ and normalized}$$

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- Perturb $\equiv$ external modification that cannot be achieved by a simple modification of the hyper-parameters

$$\Rightarrow \text{ sensitivity } + \text{ robustness}$$

The perturbed mixture model

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- The perturbed model can be expressed in a hierarchical way: Let $(X_i)_{i\geq 1}$ be a sequence of observable r.v.'s and let $(Y_i)_{i\geq 1}$ be a sequence of latent r.v.'s, therefore

$$\begin{aligned} X_i \mid Y_i &\overset{\text{ind}}{\sim} k(\,\cdot\,, Y_i) \\ Y_i \mid \mathrm{P} &\overset{\text{iid}}{\sim} \mathrm{P} \\ \mathrm{P} &\sim \mathscr{P} \end{aligned}$$

where $\mathscr{P}$ is the distribution of a normalized weighted IAP

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- The role of $\bar{k}$ is to amplify or squeeze the jump heights of the IAP A

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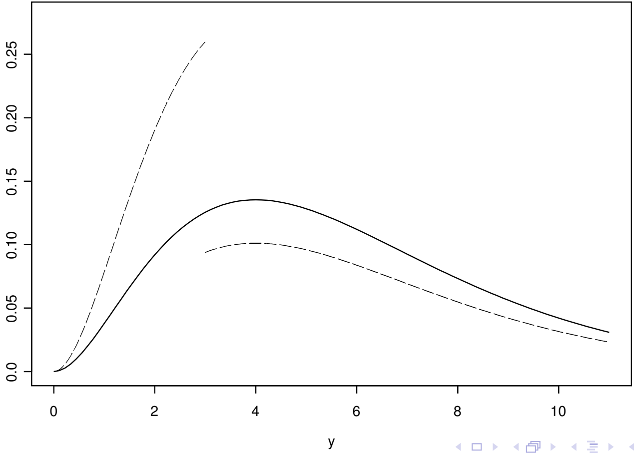

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    1. Marginal method: Integrate out P and resort to the predictive distribution
    2. Conditional method: Characterize P conditionally (usually on latent variables)
- The predictive distributions of the perturbed NRMI are intractable so we will follow the conditional method

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$$(Y_1,\ldots,Y_n) \to \left\{ \begin{array}{l} (Y_1^*,\ldots,Y_r^*) \\ (n_1,\ldots,n_r) \end{array} \right. , \text{ con } \sum_{j=1}^{r} n_j = n$$

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- Characterize the posterior law through conditional distributions
$[A_k(\cdot)|X,U,Y]$, $[U|X,Y,A_k]$ and $[Y|X,U,A_k]$

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- This result can also be obtained as an extension of James et al. (2005)

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$$f_{U_n|Y}(u) \propto u^{n-1} \exp\{-\psi_{\bar{k}}(u)\} \prod_{j=1}^{r} \tau_{n_j}(u|y_j^*),$$

where $\psi_{\bar{k}}(u) = \int_{\mathbb{R}\times\mathbb{R}^+} (1 - e^{-u\bar{k}(s)v})\rho(\mathrm{d}v|s)\alpha(\mathrm{d}s)$
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- $[Y|X,U,A_k] = [Y|X,A_k]$ are n univariate discrete densities

$$f(y_i|x_i,A) \propto k(x_i,y_i)\bar{k}(y_i)A\{\mathrm{d}y_i\}$$

for $i = 1,\ldots,n$

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$$A_k(y) = \sum_i \bar{k}(\tau_i) J_i I(\tau_i \leq y),$$
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5. Compute a path of the random density

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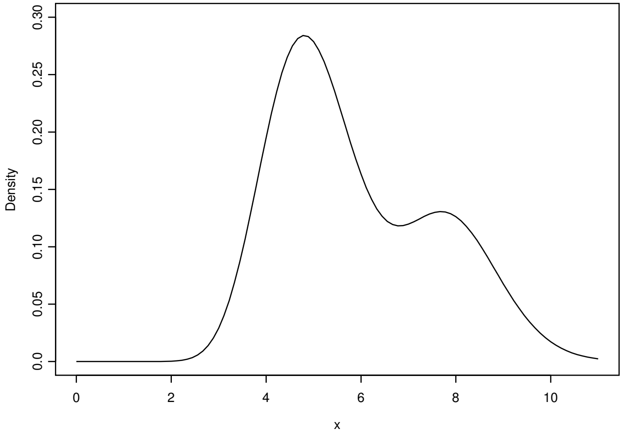
A simulated data example



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$$\bar{k}(y) = \begin{cases} 10 & \text{if } y \in [0,3] \\ 1 & \text{if } y > 3 \end{cases}$$

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- In any case, the prior density estimates are:

$$\hat{f}(x) = E[f(x)] = \int_{\mathbb{R}^+} (\sqrt{2\pi})^{-1} \mathrm{e}^{-\frac{1}{2}(x-y)^2} \times \mathrm{P}_0(\mathrm{d}y)$$

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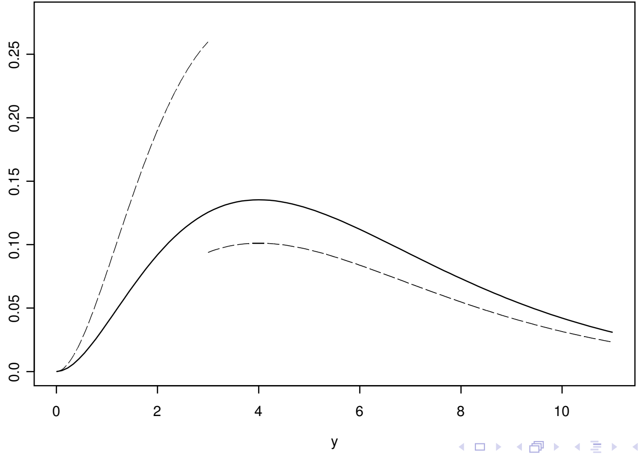

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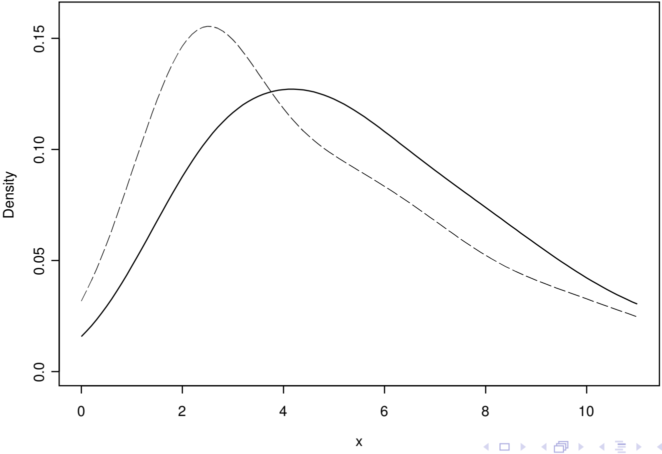

A simulated data example

- What will be the influence of perturbing the prior and force it to select more means (locations) in the interval $[0, 3]$??

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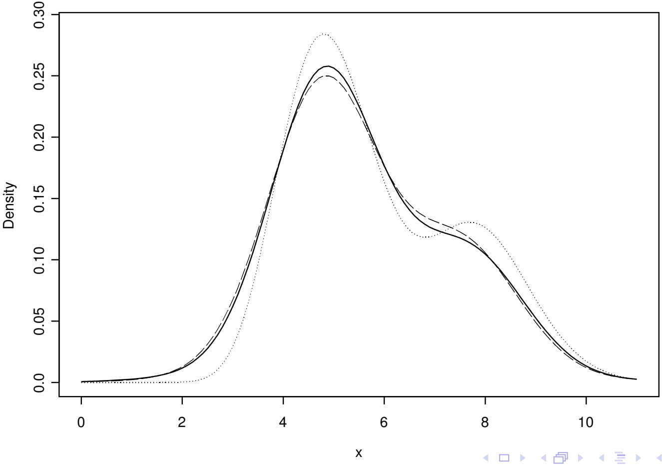

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- How is this difference going to be neutralized a-posteriori?
- The answer depends on the latent U and the configuration of the latents $Y^{(30)}$.
- Consider four cases: 0, 10, 20 and 30 latents Y's fall in $[0,3]$ $\Rightarrow$ a different shape of the density of $U|Y^{(30)}$ in the perturbed-Dirichlet

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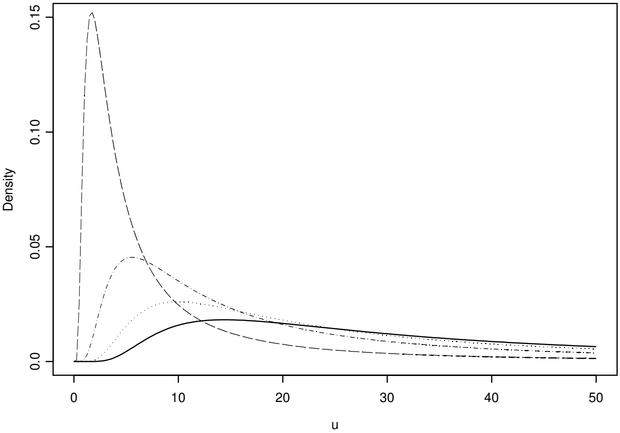

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- The distribution of $U|Y^{(30)} \Rightarrow$ infinite mean. So we take the medians as average representatives.
- We computed the posterior probability of selecting a mean (location) in $[0,3]$ conditional on $U|Y^{(30)}$ and $Y^{(30)}$

$$\mathrm{E}\left[P\left([0,3]\right)\middle| U;\, Y^{(30)}\right] = \mathrm{E}\left[\frac{\int_{[0,3]} \bar{k}(y)A^{u,y^*}(\mathrm{d}y) + \sum_{y_i^* \in [0,3]} k(y_i^*)J_i^{u,y^*}}{\int_{\mathbb{R}^+} \bar{k}(y)A^{u,y^*}(\mathrm{d}y) + \sum_{y_i^* \in \mathbb{R}^+} k(y_i^*)J_i^{u,y^*}}\right]$$

for the Dirichlet and perturbed-Dirichlet

A simulated data example

# latents in $[0, 3]$	Dirichlet (Indep. of U)	Perturbed Dirichlet $u = 1$	$u = 20$	$u = 50$	(Median of U)
30	0.974	0.984	0.970	0.968	(5.10) 0.971
20	0.651	0.767	0.655	0.648	(16.9) 0.657
10	0.328	0.465	0.335	0.325	(29.7) 0.329
0	0.006	0.011	0.005	0.003	(42.6) 0.004

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- What we observed from the Gibbs sampler for the distribution of $U|Y^{(30)}$ was a median of 28.5 $\Rightarrow$ approximately 10 latents fall in $[0,3]$

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$$\bar{k}(y) = \begin{cases} 20 & \text{if } y \in (11,16] \\ 1 & \text{if } y \in [0,11] \cup (16,\infty) \end{cases}$$

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- $+$ a randomization of σ as in Richardson and Green (1997): $\sigma \sim \mathrm{Ga}(1,2) \Rightarrow$ extend the Gibbs sampler to include

$$\pi(\sigma|x,u,y,A_k) \propto \pi(\sigma)\prod_{i=1}^{n} k(x_i|y_i,\sigma)$$

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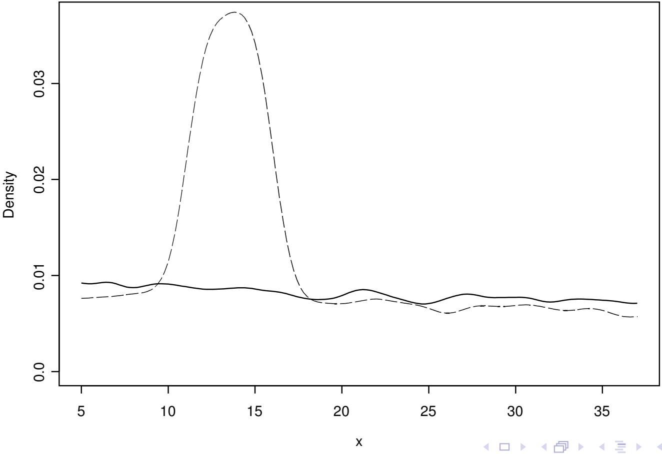

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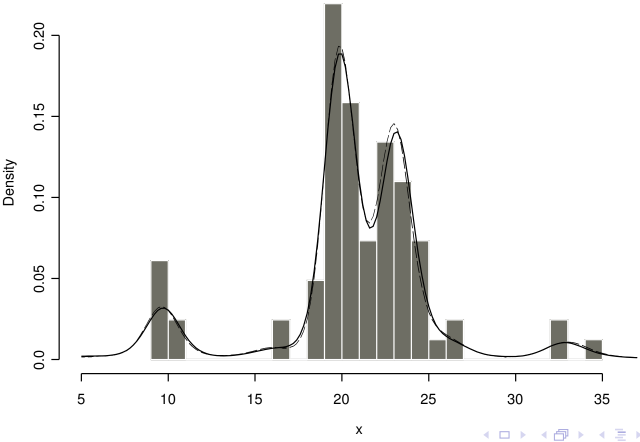

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$$\bar{k}(y) = \begin{cases} 0.1 & \text{if } y \in (4, 5] \\ 1 & \text{if } y \in [0, 4] \cup (5, \infty) \end{cases}$$

Note that the perturbing function squeezes by a factor of 10 the jumps of the process in $(4, 5]$

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- $+\ \sigma \sim \mathrm{Ga}(1, 10)$

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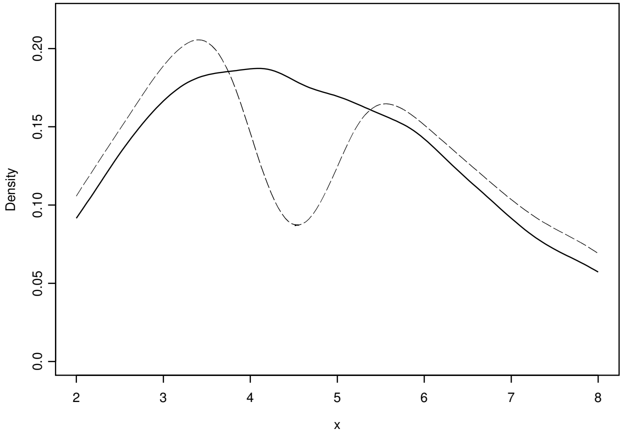

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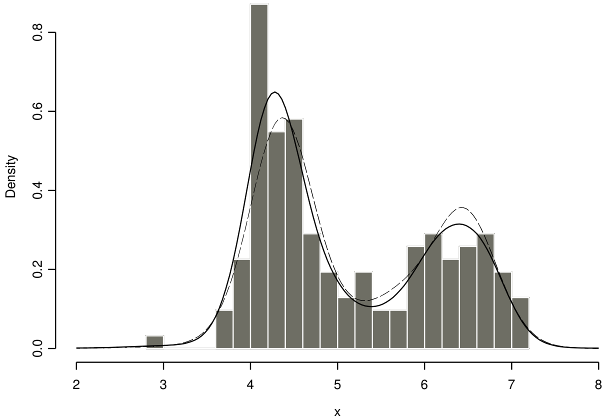

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