Graph Theory I

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- G is **simple** if each edge in E is associated to a different pair of distinct vertices of V. In this case we can consider E to be a family of size two subsets of V.
- G is **finite** if V and E are both finite. All graphs considered in these talks are assumed to be finite.

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- G **contract** e is the graph G/e obtained by **deleting** e from E, deleting v, w from V and then adding a new vertex z which is incident to all edges in $E - e$ which were incident to v or w.

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- G **contract** e is the graph G/e obtained by **deleting** e from E, deleting v, w from V and then adding a new vertex z which is incident to all edges in $E - e$ which were incident to v or w.
- Note that if e is a loop then $G - e = G/e$.

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- A **contraction** of G is any graph which can be obtained by recursively contracting edges in G.
- A **minor** of G is any graph which can be obtained by recursively deleting or contracting edges and deleting isolated vertices from G.

Robertson-Seymour Theorem

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Corollary

Let $\mathcal{F}$ be a minor closed family of graphs. Then there exists a finite set of graphs $\mathcal{H}$ such that, for any graph G, $G \in \mathcal{F}$ if and only if G does not have a minor in $\mathcal{H}$.

Parallel and Series Extension/Reduction

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- If H can be obtained from G by contracting an edge incident to a vertex of degree two then H is a **series reduction** of G, and G is a **series extension** of H. If G can be obtained from H by a sequence of series extensions, then G is a **subdivision** of H.

Unions and Intersections

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- The **intersection** of H_1 and H_2 is the subgraph of G given by $H_1 \cap H_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

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Lemma

(a) G is a forest if and only if G does not have the cycle of length one as a minor.
(b) G is series parallel if and only if G is a 2-connected loopless graph which does not have the complete graph on four vertices K_4 as a minor.

Graphs on Surfaces

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Theorem

- G is outerplanar if and only if G does not have the complete graph K_4 or the complete bipartite graph $K_{2,3}$ as a minor.
- G is planar if and only if G does not have K_5 or $K_{3,3}$ as a minor. (Kuratowski)
- For each surface S, there exists a finite set of graphs $\mathcal{H}$ such that, for any graph G, G can be embedded in S if and only if G has no minor in $\mathcal{H}$. (Robertson and Seymour)

Let $G = (V, E)$ be a graph and $U \subseteq V$.

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Induced Subgraphs and Line Graphs

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Beineke's Theorem

There exists a set $\mathcal{F}$ of nine graphs on four, five, and six vertices such that a graph G is a line graph if and only if G does not have an induced subgraph in $\mathcal{F}$.

Claw-free Graphs

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Note

$K_{1,3}$ belongs to Beineke's list of nine excluded induced subgraphs for line graphs, so all line graphs are claw free.
Many properties of line graphs extend to the larger family of claw-free graphs.

Let $G = (V, E)$ be a connected graph, $U \subseteq V$, and $k \geq 1$ be an integer.

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CONNECTIVITY OF GRAPHS

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- G is k-**edge-connected** if every edge-cut of G contains at least k edges.
- U is a **vertex-cut** of G if $G - U$ is disconnected.
- G is k-**connected** if G has at least $k + 1$ vertices and every vertex-cut of G contains at least k vertices.

Menger's Theorem

Let G be a graph and u, v be vertices of G.

- The minimum size of an edge-cut which separates u and v in G is equal to the maximum number of pairwise edge-disjoint uv-paths in G.
- If u and v are not adjacent, then the minimum size of a vertex-cut which separates u and v in G is equal to the maximum number of pairwise openly-disjoint uv-paths in G.

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Corollary

- G is k-edge-connected if and only if every pair of vertices of G are joined by k pairwise edge-disjoint paths.
- G is k-connected if and only if G has at least $k + 1$ vertices and every pair of non-adjacent vertices of G are joined by k pairwise openly-disjoint paths.

Gomory-Hu Trees

Theorem (Gomory and Hu)

Let $G = (V, E)$ be a connected graph. Then there exists a tree $T = (V, F)$ and a map $c : F \to \mathbb{N}$ such that, for all $u, v \in V$, the minimum size of an edge-cut which separates u and v in G is equal to the $\min\{c(e)\}$ over all edges e in the unique uv-path in T. Furthermore, if e is an edge of T for which this minimum is attained, and $\{U, V - U\}$ is the partition of V given by the connected components of $T - e$, then $E_G(U, V - U)$ is an edge-cut in G of size $c(e)$.

Someone asked during my talk if it is always possible to find a Gomory-Hu tree T for G such that T is a spanning tree of G. Rob Waters pointed that $G = K_{2,3}$ is a counterexample: we cannot even find a spanning tree $T = (V, F)$ of $K_{2,3}$ and a map $c : F \to \mathbb{N}$ such that, for all $u, v \in V$, the minimum size of an edge-cut which separates u and v in $K_{2,3}$ is equal to the $\min\{c(e)\}$ over all edges e in the unique uv-path in T.

Decomposing Connected Graphs: The Block/Cut-Vertex Tree

Let $G = (V, E)$ be a connected graph.

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Theorem

Let G be a connected graph. Let B be the set of blocks of G and X be the set of cut vertices of G. Let $T = (U, F)$ be the simple graph such that $U = B \cup X$, and $bx \in F$ if and only if $b \in B$, $x \in X$, and x is a vertex of b. Then T is a tree.

Decomposing 2-Connected Graphs: Hinges

- Let G be a 2-connected graph.
- Let x and y be vertices of G such that $G - \{x, y\}$ has components $G_1, G_2, \ldots, G_r$, with $r \geq 2$ if xy is not a multiple edge of G.

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- An $\{x, y\}$-**component** of G is a subgraph of G induced by $V(G_i) \cup \{x, y\}$, but with any edges joining x and y deleted. In addition, if $xy \in E(G)$, then the subgraph induced by $\{x, y\}$ is a **trivial** $\{x, y\}$-**component** of G.

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- Let H be an $\{x, y\}$-component of G and put $H' = G - (H - \{x, y\})$. H is **excisable** if H is not trivial and either H or H' is a 2-connected graph or a multiple edge.

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- If an excisable $\{x, y\}$-component H of G exists, we say that $\{x, y\}$ is a **hinge** of G and H is a **hinge component** of G.
- A hinge $\{x, y\}$ of G is **Type I** if G has exactly two $\{x, y\}$-components and **Type II** otherwise.

- Construct the **augmented graph** G^{aug} from G by adding a new edge incident to x and y for each hinge $\{x, y\}$ and each excisable $\{x, y\}$-component of G. These new edges are called **virtual edges**. Two distinct hinge components of G give rise to the same virtual edge if and only if they are the two $\{x, y\}$-components of the same hinge $\{x, y\}$ of Type I.

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- If H is an excisable $\{x, y\}$-component of G, the two graphs D_1 and D_2 derived from H and H' by adjoining to each of H and H' the virtual edge e associated with H are called the **cleavage graphs** of G at e.

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- If H is an excisable $\{x, y\}$-component of G, the two graphs D_1 and D_2 derived from H and H' by adjoining to each of H and H' the virtual edge e associated with H are called the **cleavage graphs** of G at e.
- The **cleavage units** of G are the minimal cleavage graphs obtained by recursively constructing cleavage graphs from cleavage graphs. (No cleavage unit of G can have a hinge, and each virtual edge of G belongs to exactly two cleavage units.)

Decomposing 2-Connected Graphs: The Cleavage-Unit Tree

- The *cleavage unit tree* T of G is the graph whose vertices and edges are the cleavage units and virtual edges, respectively, of G, in which a cleavage unit D and a virtual edge e are incident in T if and only if e is an edge of D.

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- Tutte showed that the cleavage unit tree of a 2-connected graph G is indeed a tree and that each cleavage unit of G is either a 3-connected simple graph or a cycle of length at least three, or a multiple edge with at least three edges.