Graph Theory II

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- U is a **vertex cover** of G if every edge of G is incident to a vertex in U.
- Note that if M is a matching in G and U is a vertex cover of G then $|M| \leq |U|$.

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Theorem

Let G be a bipartite graph with bipartition $\{X, Y\}$.

- G has a matching which saturates X if and only if all $S \subseteq X$ have at least $|S|$ neighbours in Y. (Hall)
- The maximum size of a matching in G is equal to the minimum size of a vertex cover of G. (König)

Given a graph G, let $k^{odd}(G)$ denote the number of connected components of G with an odd number of vertices.

Matchings in General Graphs: The Tutte-Berge Theorems

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Theorem

Let $G = (V, E)$ be a graph.

- G has a perfect matching if and only if $k^{odd}(G - S) \leq |S|$ for all $S \subset V$. (Tutte)
- The maximum size of a matching in G is equal to $\min\{(|V| - k^{odd}(G - S) + |S|)/2)\}$ over all $S \subset V$. (Berge)

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Note

Edmonds gave a polynomial time algorithm for determining a maximum size matching in a graph.

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- We can determine a maximum size independent set of vertices in a line graph since this is equivalent to determining a maximum size matching in the original graph (which we can do by Edmond's algorithm).
- Minty showed that Edmond's algorithm could also be used to give a polynomial time algorithm for finding a maximum size independent set of vertices for the more general family of claw-free graphs. (An error in Minty's algorithm was subsequently corrected by Nakamura and Tamura.)

GRAPH COLOURING: Vertex-Colouring

Let $G = (V, E)$ be a graph and $k \geq 1$ be an integer.

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Theorem (Brooks)

Let G be a connected simple graph of maximum degree Δ. Then $\chi(G) \leq \Delta + 1$ with equality if and only if $G = K_{\Delta+1}$, or $\Delta = 2$ and G is an odd cycle.

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- If G can be embedded in a surface of Euler characteristic $c \neq 2$ then $\chi(G) \leq \lfloor (7 + \sqrt{49 - 24c})/2 \rfloor$. (Heawood) This bound is best possible for all surfaces except the Klein bottle. (Ringel and Youngs)

Perfect Graphs

Let $G = (V, E)$ be a simple graph and $U \subseteq V$.

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Theorem (Chudnovsky, Robertson, Seymour and Thomas)

Let G be a simple graph. Then G is perfect if and only if G does not contain an odd cycle of length at least five or its complement as an induced subgraph.

Edge-Colouring

Let $G = (V, E)$ be a graph of maximum degree Δ and $k \geq 1$ be an integer.

- A **proper** k**-edge-colouring** of G is an assignment of k colours to the edges of G such that no pair of adjacent edges receive the same colour. (Equivalently, it is a partition of E into k matchings.)

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- If G is simple and H is the line graph of G then $\chi'(G) = \chi(H)$.
- For each $k \geq 3$, it is NP-complete to decide if a graph has a proper k-edge-colouring. (Holyer)

Theorem

Let G be a graph of maximum degree Δ.

- $\chi'(G) \leq 3\Delta/2$. (Shannon)
- $\chi'(G) \leq \Delta + \mu(G)$ where $\mu(G)$ denotes the maximum multiplicity of an edge of G. (Vizing)
- If G is bipartite then $\chi'(G) = \Delta$. (König)

NOWHERE-ZERO FLOWS IN GRAPHS:
Group valued flows

Let $G = (V, E)$ be a graph and Γ be an additive abelian group.

- Construct a digraph $\vec{G}$ by giving the edges of G an arbitrary orientation. For $U \subseteq V$ and $\bar{U} = V - U$, let $E^+(U)$ be the set of arcs from U to $\bar{U}$ in $\vec{G}$ and $E^-(U) = E^+(\bar{U})$.

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- Let $f : E(\vec{G}) \to \Gamma$ and put $f^+(U) = \sum_{e \in E^+(U)} f(e)$ and $f^-(U) = \sum_{e \in E^-(U)} f(e)$.

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- f is a Γ-**flow** for G, with respect to $\vec{G}$, if $f^+(v) = f^-(v)$ for all $v \in V(G)$.

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- f is a Γ-**flow** for G, with respect to $\vec{G}$, if $f^+(v) = f^-(v)$ for all $v \in V(G)$.
- If, in addition, $f(e) \neq 0$ for all $e \in E(G)$, then f is a **nowhere-zero** Γ-**flow** for G.

Group valued flows, continued

- The condition $f^+(v) = f^-(v)$ for all $v \in V(G)$ is equivalent to the apparently stronger condition that $f^+(U) = f^-(U)$ for all $U \subseteq V(G)$.

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- This implies that, if G has a nowhere-zero Γ-flow, then G is bridgeless. (A **bridge** in G is an edge-cut of size one.)
- Since reversing the orientation on an edge e of $\vec{G}$ is equivalent to replacing $f(e)$ by $-f(e)$, the number of distinct nowhere-zero Γ-flows for G is independent of the chosen orientation $\vec{G}$ of G.

Integer valued flows

Let $G = (V, E)$ be a graph and $k \geq 1$ be an integer.

- A *nowhere-zero k-flow* for G is a nowhere-zero $\mathbb{Z}$-flow, f, such that $|f(e)| \leq k - 1$ for all $e \in E(G)$.

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- The number of nowhere-zero $\mathbb{Z}_k$-flows for G may differ from the number of nowhere-zero k-flows for G.
- A connected plane graph G has a proper k-vertex-colouring if and only if its planar dual G^* has a nowhere-zero k-flow. (Tutte)

Conjecture (Tutte)

Let G be a bridgeless graph.

- G has a nowhere zero 5-flow.
- If G has no edge-cuts of size three then G has a nowhere zero 3-flow.

Tutte's Conjectures and the Jaeger-Seymour Theorems

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Theorem

Let G be a bridgeless graph.

- G has a nowhere zero 6-flow. (Seymour)
- If G has no edge-cuts of size three then G has a nowhere zero 4-flow. (Jaeger)

Let $G = (V, E)$ be a graph. For each positive integer t, let $P_G(t)$ be the number of proper t-vertex-colourings of G. (By definition $P_G(t) \equiv 1$ if $E = \emptyset$, and $P_G(t) \equiv 0$ if G has a loop.)

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Deletion-Contraction Lemma

Let G be a graph and e be an edge of G which is not a loop. Then

$$P_G(t) = P_{G-e}(t) - P_{G/e}(t).$$

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This implies that $P_G(t)$ is a polynomial in t, the **chromatic polynomial** of G.

The Flow Polynomial

Let $G = (V, E)$ be a graph. For each positive integer t, let $F_G(t)$ be the number of nowhere-zero $\mathbb{Z}_t$-flows of G. (By definition $F_G(t) \equiv 1$ if $E = \emptyset$.)

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Lemma (Tutte)

If G is a connected plane graph and G^* is its planar dual then

$$tF_G(t) = P_{G^*}(t).$$

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Deletion-Contraction Lemma

Let G be a graph and e be an edge of G. Then

- $T_G(x, y) = T_{G-e}(x, y) + T_{G/e}(x, y)$ if e is neither a loop not a bridge,
- $T_G(x, y) = xT_{G-e}(x, y)$ if e is a bridge,
- $T_G(x, y) = yT_{G-e}(x, y)$ if e is a loop.

Let $G = (V, E)$ be a graph.

- If G is embedded in the plane and G^* is its planar dual then $T_G(x, y) = T_{G^*}(y, x)$.

Properties and Specialisations of the Tutte Polynomial

Let $G = (V, E)$ be a graph.

- If G is embedded in the plane and G^* is its planar dual then $T_G(x, y) = T_{G^*}(y, x)$.
- If $G = G_1 \cup G_2$ and $G_1 \cap G_2$ has at most one vertex and no edges then $T_G(x, y) = T_{G_1}(x, y)T_{G_2}(x, y)$.

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- $T_G(1, 1)$ is the number of spanning trees of G.

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- $T_G(1, 1)$ is the number of spanning trees of G.
- $T_G(2, 0)$ is the number of acyclic orientations of G.

Let $G = (V, E)$ be a graph.

- If G is embedded in the plane and G^* is its planar dual then $T_G(x, y) = T_{G^*}(y, x)$.
- If $G = G_1 \cup G_2$ and $G_1 \cap G_2$ has at most one vertex and no edges then $T_G(x, y) = T_{G_1}(x, y)T_{G_2}(x, y)$.
- $P_G(t) = (-1)^{r(E)} t^{k(E)} T_G(1 - t, 0)$.
- $F_G(t) = (-1)^{|V|} T_G(0, 1 - t)$.
- $T_G(1, 1)$ is the number of spanning trees of G.
- $T_G(2, 0)$ is the number of acyclic orientations of G.
- $T_G(0, 2)$ is the number of totally-cyclic orientations of G.

The Partition Function

The **Pott's model partition function**, or **multivariate Tutte polynomial**, of a graph $G = (V, E)$ is the $(|E| + 1)$-variable polynomial given by

$$Z_G(q, \mathbf{w}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} w_e,$$

where $\mathbf{w} = (w_e)_{e \in E}$ is a vector of indeterminates.

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Deletion-Contraction Lemma

Let G be a graph and e be an edge of G. Then

$$Z_G(q, \mathbf{w}) = Z_{G-e}(q, \mathbf{w}|_{E-e}) + w_e Z_{G/e}(q, \mathbf{w}|_{E-e}).$$

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Lemma

$$T_G(x, y) = (x-1)^{-k(E)}(y-1)^{-|V|} Z_G\left((x-1)(y-1), (y-1)\mathbf{1}\right).$$

The Fortuin-Kasteleyn Expansion of the Partition Function

Theorem (Fortuin-Kasteleyn)

Let G be a graph and q be a positive integer. Let $S = \{1, 2, \ldots, q\}$. Then

$$Z_G(q, \mathbf{w}) = \sum_{\sigma : V \to S} \prod_{e \in E} (1 + w_e \delta_e),$$

where $\delta_e = 1$ if σ maps the end-vertices of e onto the same element of S, and $\delta_e = 0$ otherwise.