MATROIDS

E GROUND SET (FINITE).

$\mathcal{I} \subseteq 2^E$ s.t.

1. $\phi \in \mathcal{I}$
2. $J \in \mathcal{I}$, $I \subseteq J \Rightarrow I \in \mathcal{I}$
3. $I, J \in \mathcal{I}$ $|J| > |I|$
$\Rightarrow \exists\ x \in J - I$ s.t. $I \cup \{x\} \in \mathcal{I}$.

$\mathcal{I}$ = <u>independent</u> <u>sets</u>.

ALSO

BASES, RANK, CIRCUITS,
FLATS, CLOSURE,

$$
\begin{array}{ccccccc}
a & b & c & d & e & f & g
\end{array}
$$
$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

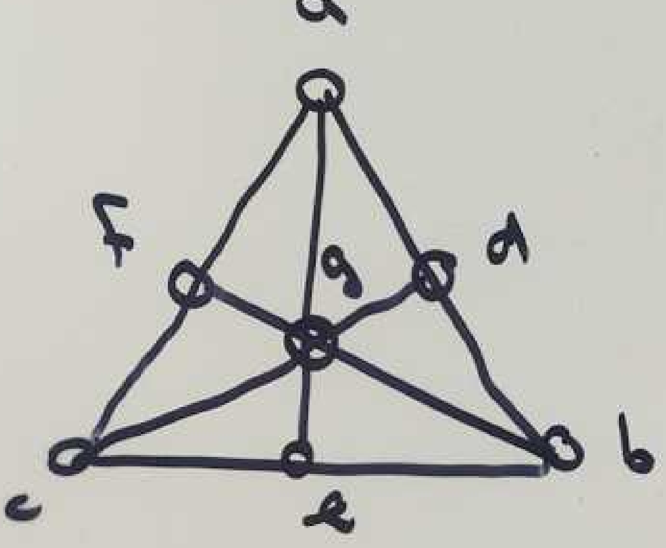
A <u>REPRESENTABLE</u> MATROID

BUT WHICH <u>FIELD</u>?

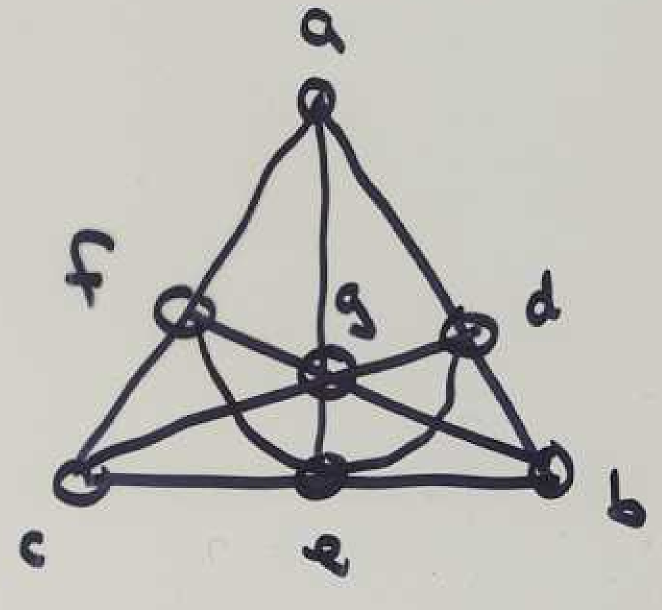
BINARY – REPRESENTABLE OVER GF(2)

TERNARY – REPRESENTABL OVER GF(3)

VISUALISING MATROIDS



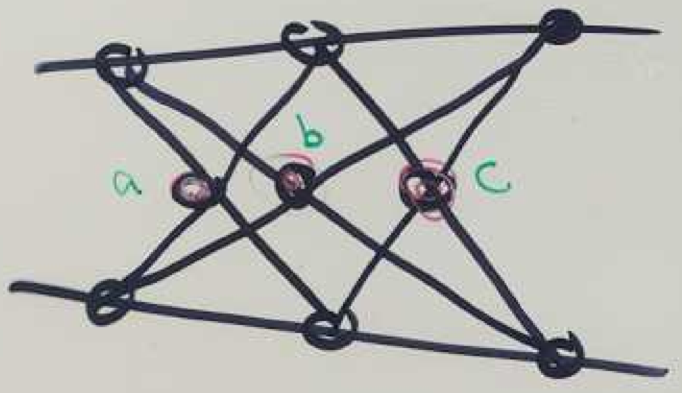
non - Fano.



FANO

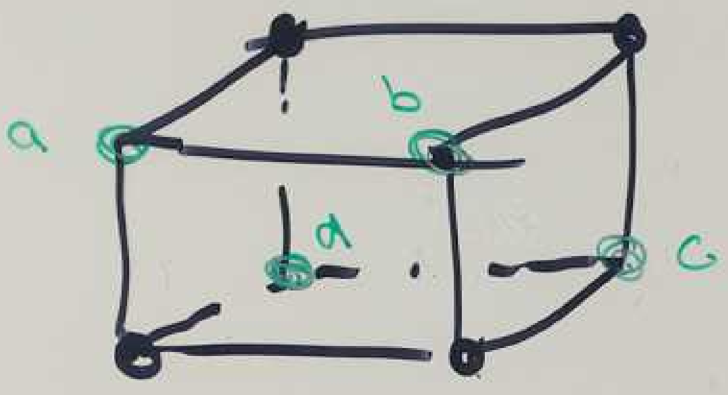
OK FOR LOW RANK.

NON-REPRESENTABLE MATROIDS



non-Pappus

a b c not collinear.



← Vamos matroid.

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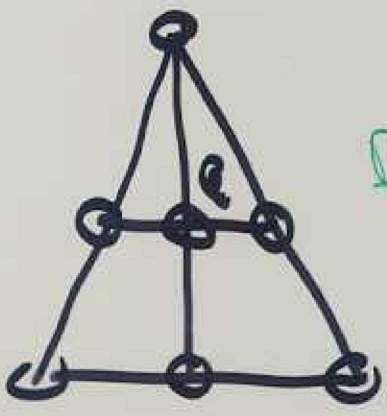
a b c d not coplanar

REMOVING ELEMENTS



DELETION

M $\qquad\qquad$ $M \setminus e$

CONTRACTION

M/e :

$$\mathcal{I}(M/e) = \{ I \subseteq E - \{e\}, \text{ s.t. } I \cup \{e\} \in \mathcal{I}(M) \}.$$

WHAT DOES THIS MEAN???

CONTRACTION IS PROJECTION FROM e

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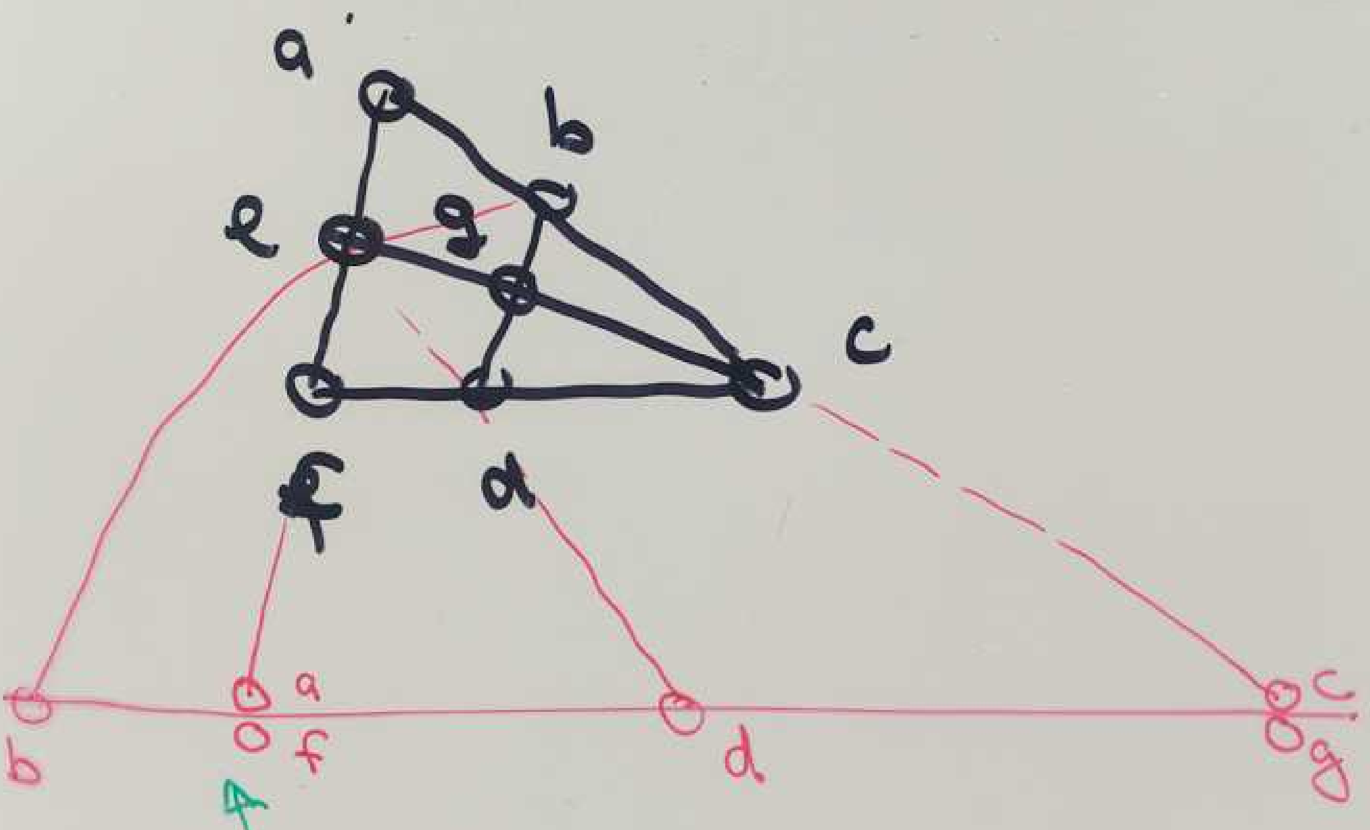

a, f parallel.

SEQUENCE OF DELETIONS AND CONTRACTIONS GIVES A <u>MINOR</u>.

- MINOR CLOSED CLASS
  - eg $\mathbb{F}$ - REPRESENTABLE MATROIDS.

- EXCLUDED MINOR

$U_{2,4}$.

- 

$U_{2,4}$ NOT BINARY

- ALL PROPER MINORS BINARY

$\therefore$ $U_{2,4}$ IS EXCUDED MINOR FOR GF(2) REPRESENTATION.

THEOREM (TUTTE)

M BINARY IFF NO $U_{2,4}$ -MINOR.

- ROTA'S CONJECTURE
- GF(3) ✓ GF(4)? $q > 4$??
- INFINITE FIELDS X.

DUALITY:

$\mathcal{B}(M)$ = set of bases of M.

$\mathcal{B}(M^*) = \{ E - B : B$ a basis of $M \}$.

- $(M^*)^* = M$.
- $(M \backslash e)^* = M^* / e$.

IS DUALITY INTERESTING?

$G = (V, E)$ a graph

$M(G)$ CYCLE MATROID OF G

- GROUND SET OF $M(G)$ IS E
  (VERTICES DISAPPEAR!!)

- INDEPENDENT SETS OF
  $M(G)$ ARE FORESTS OF G.

- $M(G \setminus e) = M(G) \setminus e$

- $M(G/e) = M(G)/e$

- G PLANAR $M(G^*) = [M(G)]^*$.

(DUALITY IS INTERESTING!!)

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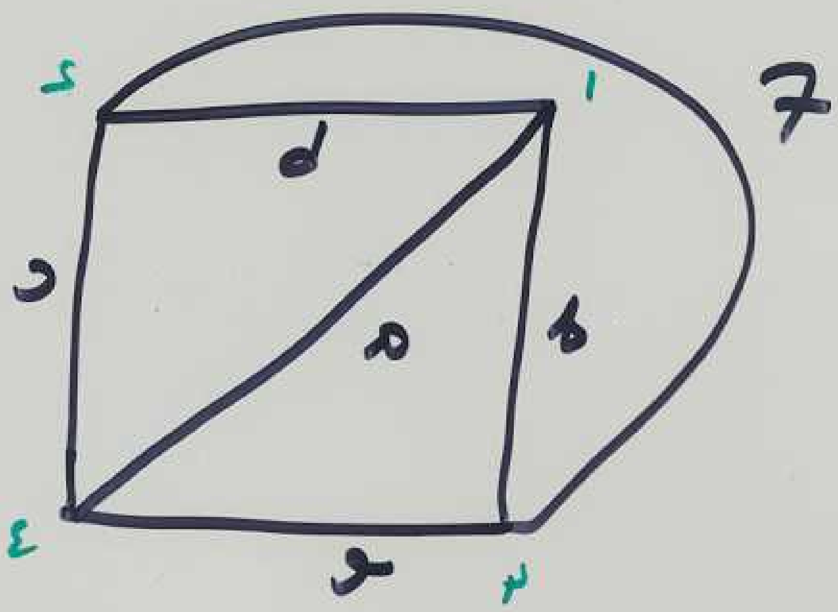
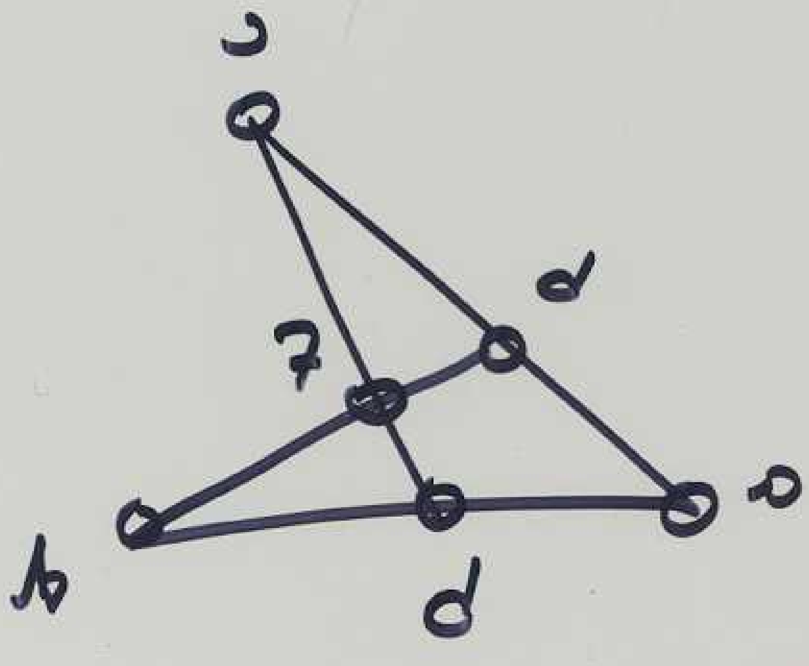
G



m(G)



$$
\begin{array}{c|cccccc}
 & a & b & c & d & e & f \\
\hline
1 & 1 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & -1 & 1 & 0 & 0 & 1 \\
3 & -1 & 0 & -1 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & -1 & -1 & -1
\end{array}
$$

HOW MUCH INFORMATION ABOUT G DO YOU LOSE IN M(G) ??

- M(G) DOES NOT DISTINGUISH FORESTS

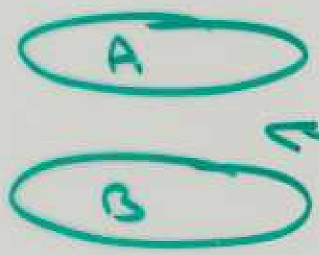
- G 3-CONNECTED, THEN M(G) DETERMINES G (<u>ESSENTIALLY</u>!)

CONNECTIVITY:

(A, B) PARTITION OF E.

$$\lambda(A, B) = r(A) + r(B) - r(M).$$

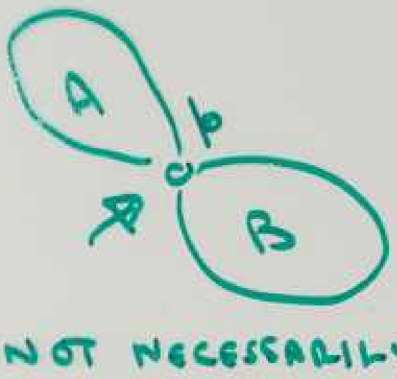
WHAT DOES THIS MEASURE?



SKEW

$\lambda(A, B) = 0$

A <u>SEPARATION</u>



p NOT NECESSARILY IN E.

$\lambda(A, B) = 1$

A 2-<u>SEPARATION</u>

ODD?

$\lambda(A, B) = 2$

A 3-<u>SEPARATION</u>.