Conformal Field Theory and Combinatorics

Part II: Coulomb Gas construction and loop models

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Summary

1. Loop models and the Coulomb gas
   - General strategy
   - Potts model
   - O(n) model
2. Transformation to height models
   - The complex loop ensemble
   - Transformation to compactified height model
3. Liouville field theory
   - Background electric charge
   - Liouville term
   - Marginality requirement
4. Critical exponents
   - Watermelon exponents
   - Magnetic exponent in the Potts model

Summary

Coulomb Gas (CG) construction

- Exact solution of lattice models that renormalise to one or more free bosonic fields
- Involves complex Boltzmann weights
  - Hence difficult to make mathematically rigorous
  - But gives exact critical exponents & much more
- Especially well suited to loop models

Main ingredients

- Make the loop model local
- Transform it to a height model
- Identify corresponding *compactified boson* CFT

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Summary

Q-state Potts model on the square lattice $\mathcal{L}$

- Spin variables $\sigma_i = 1, 2, \ldots, Q$ for each $i \in \mathcal{L}$
- Nearest-neighbour interaction $-K\,\delta_{\sigma_i,\sigma_j}$
- Reduced coupling $J = K/k_{\rm B}T$

$$Z = \sum_{\{\sigma\}} \prod_{(ij)\in E} {\rm e}^{J\delta_{\sigma_i,\sigma_j}}$$

Random cluster model

- For each edge $(ij) \in E$: ${\rm e}^{J\delta_{\sigma_i,\sigma_j}} = 1 + ({\rm e}^J - 1)\delta_{\sigma_i,\sigma_j}$

$$Z = \sum_{E' \subseteq E} ({\rm e}^J - 1)^{|E'|} Q^{c(E')}$$

- E' runs over $2^{|E|}$ subsets of E
- $c(E') = \#$ connected components in graph induced by E'

Q-STATE POTTS MODEL ON THE SQUARE LATTICE $\mathcal{L}$

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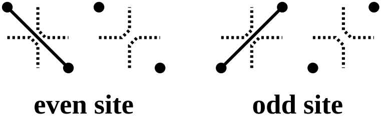
Loop model on surrounding lattice $S_{\mathcal{L}}$

- Vertices of $S_{\mathcal{L}}$ = midpoints of edges in $\mathcal{L}$
- On $S_{\mathcal{L}}$, define transition system for each $E' \subseteq E$



- Use Euler relation (planar lattice, N vertices)

$$Z = Q^{N/2} \sum_{E' \subseteq E} \left(\frac{e^J - 1}{\sqrt{Q}} \right)^{|E'|} Q^{l(T(E'))/2}$$

- Set $n = \sqrt{Q}$, take selfdual point (critical for $-2 \leq n \leq 2$)

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LOOP MODEL ON SURROUNDING LATTICE $S_{\mathcal{L}}$

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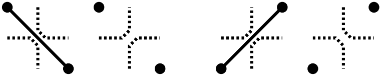

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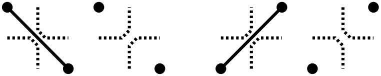

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O(n) model on the hexagonal lattice $\mathcal{L}$

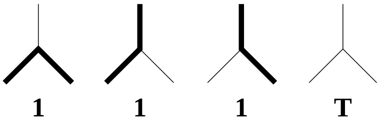
- Vector spins $\mathbf{S}_i \in \mathbb{R}^n$ for each $i \in \mathcal{L}$
- Nearest-neighbour interaction $-J\, \mathbf{S}_i \cdot \mathbf{S}_j$
- Integration measure $\int \mathrm{d}\mathbf{S}_i \mathrm{d}\mathbf{S}_j\, S_i^\alpha S_j^\beta = \delta_{\alpha,\beta}$

Expand out Boltzmann weights $\tilde{w}_{ij} = \exp(J\, \mathbf{S}_i \cdot \mathbf{S}_j / k_{\mathrm{B}} T)$

- Lattice prevents loops from crossing
- $w_{ij} \equiv 1 + K\, \mathbf{S}_i \cdot \mathbf{S}_j$ prevents multiple edge occupancy

$$Z = \sum_{\mathcal{G}} K^{|\mathcal{G}|} n^{l(\mathcal{G})}$$

- Critical temp. $T^2 = 1/K^2 = 2 \pm \sqrt{2-n}$ for $-2 \leq n \leq 2$



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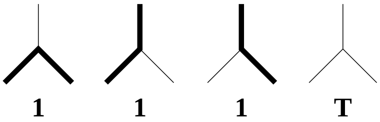

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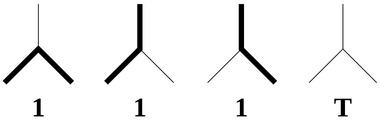
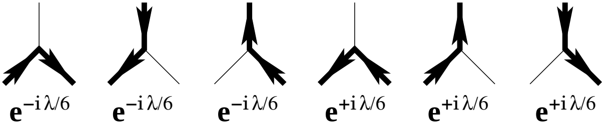


Summary

- When taking loop weight $n \in \mathbb{R}$, *locality* is lost...
- ...but may be recovered by transforming to height model

Oriented loops with complex Boltzmann weights

- Weight $e^{i\alpha\gamma/2\pi}$ for left turn through angle α
- Total weight $n = 2\cos\gamma \in [-2, 2]$ with $\gamma \in [0, \pi]$



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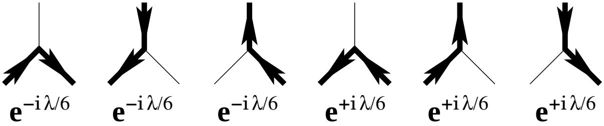
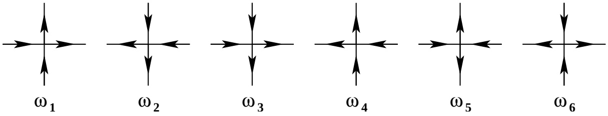


From Potts model to the six-vertex model

- Sum loop weights over oriented transition systems compatible with edge orientations



$$\begin{aligned} \omega_1, \ldots, \omega_6 &= 1, 1, x, x, e^{i\gamma/2} + x e^{-i\gamma/2}, e^{-i\gamma/2} + x e^{i\gamma/2} \\ \omega'_1, \ldots, \omega'_6 &= x, x, 1, 1, e^{-i\gamma/2} + x e^{i\gamma/2}, e^{i\gamma/2} + x e^{-i\gamma/2} \end{aligned}$$

- We have set $x = (e^J - 1)/\sqrt{Q}$
- Anisotropy parameter $\Delta \equiv \frac{\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6}{2\sqrt{\omega_1\omega_2\omega_3\omega_4}} = -\cos\gamma$

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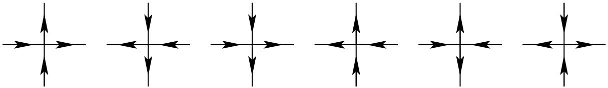

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From vertex model to height model

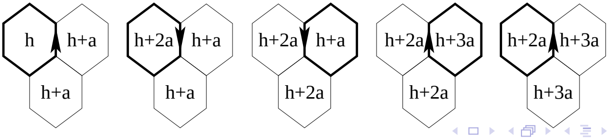
- Heights defined on dual lattice
- Arrows define height steps (in arbitrary units *a*)

From heights to *compactified boson* CFT

- Discrete heights converge to field $\phi(\mathbf{x})$ as lattice mesh tends to zero
- Expect fluctuations to be controlled by elastic action

$$S_{\mathrm{E}} = \frac{g}{4\pi} \int \mathrm{d}^2\mathbf{x}\, (\nabla\phi)^2$$

- Geometrical proof of compactification $\phi(\mathbf{x}) \in \mathbb{R}/(2a\mathbb{Z})$



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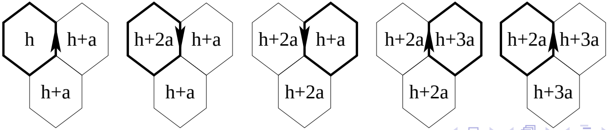

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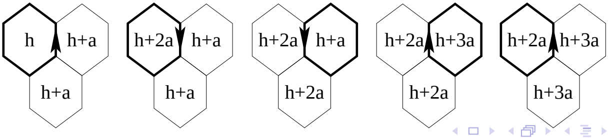

Summary

Proper treatment of loops of non-trivial homotopy

- Convenient to define theory on a cylinder
- Weight of winding loop would be $2 = 1 + 1$, not $n = 2\cos\gamma$
- Cure: Give extra weight $\mathrm{e}^{\pm i\gamma}$ to oriented loop traversing the periodic boundary condition

Corresponding *boundary term* in the action

$$S_{\mathrm{B}} = \frac{i\gamma}{4\pi a} \int \mathrm{d}^2\mathbf{x}\, \phi(\mathbf{x}) \mathcal{R}(\mathbf{x})$$

- $\mathcal{R}$ is scalar curvature of space $\mathbf{x}$ (cf. Gauss-Bonnet)
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Summary of Liouville field theory this far

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\begin{aligned}
Z &= \int \mathcal{D}\phi(\mathbf{x}) \exp\left(-S[\phi(\mathbf{x})]\right) \\
S[\phi(\mathbf{x})] &= S_{\rm E} + S_{\rm B} + S_{\rm L} \\
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Identification with compactified boson CFT

- Electric charge e is that of vertex operator $e^{ie\phi}$
- Magnetic charge m is a height dislocation $\Delta\phi = m$
- e and m must indeed live on dual lattices
- Set $a = \pi$ and $\gamma = \pi e_0$ from now on

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Critical exponents of compactified boson

- Must correct for presence of *background charge* e_0
- e_0 conjugate to height difference between cylinder ends
  - Can gauge away constant height tilt
  - Easy to see its effect on elastic free energy
  - In turn linked to shift in c and $\Delta_{e,m}$

$$c = 1 - \frac{6e_0^2}{g}$$

$$\Delta_{e,m} = \frac{1}{2}\left[\frac{e(e-2e_0)}{g} + gm^2\right]$$

Remaining problems

- We haven't said a word about bulk loop weights (or $S_{\rm L}$)
- We must link the coupling g to the loop weight $n = 2\cos(\pi e_0)$

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Summary

Continuum limit of bulk loop weights

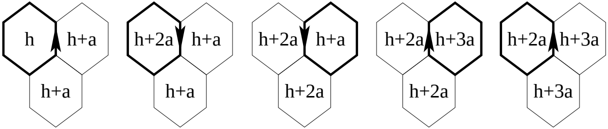
- Microscopic vertex weights w_i become $\mathrm{e}^{-w[\phi(\mathbf{x})]}$

$$S_{\rm L} = \int \mathrm{d}^2\mathbf{x}\, w[\phi(\mathbf{x})]$$

- $S_{\rm L}[\phi]$ is a periodic functional of ϕ; hence Fourier analysis

$$w[\phi] = \sum_{e \in \mathcal{L}_w} \tilde{w}_e\, \mathrm{e}^{ie\phi}$$

- $\mathcal{L}_w$ is a sublattice of the trivial $\mathcal{L}_0 \equiv \mathbb{Z}$



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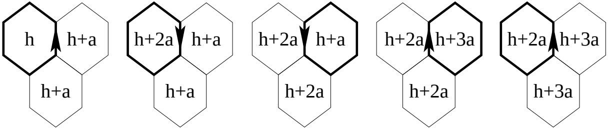

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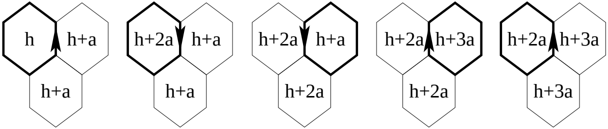

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LIOUVILLE TERM MUST BE RG MARGINAL

- Argument: loops of any size have weight n
- Most relevant vertex operator $e_w = 2$ must have $\Delta_{e_w,0} = 2$
- This fixes $g = 1 - e_0$

BUT THE AMPLITUDE MIGHT VANISH, $\tilde{w}_{e_w} = 0$

- ...by tuning the temperature of the O(n) model
- ...or by adding non-magnetic impurities in the Potts model
- Then take $\Delta_{-e_w,0} = 2$
- Both cases covered by

$$n = \pm\sqrt{Q} = -2\cos(\pi g)$$

- with $0 < g \leq 1$ (resp. $1 \leq g \leq 2$) in former (latter) case

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Liouville term must be RG marginal

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But the amplitude might vanish, $\tilde{w}_{e_w} = 0$

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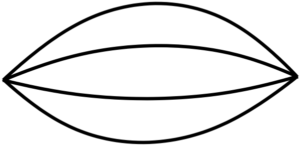
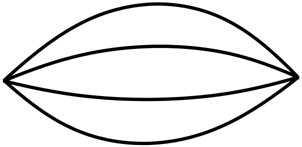
Watermelon (or fuseau, or ℓ-leg) exponents



- Orient legs from left to right
- Magnetic charges $\pm m_\ell = \pm \frac{\ell a}{2\pi} = \pm \frac{\ell}{2}$ at cylinder ends
- $e^{\pm i e_0 \phi}$ needed to cancel spurious phase factors

$$\Delta_\ell = \Delta_{e_0, m_\ell} = \frac{1}{8} g \ell^2 - \frac{(1-g)^2}{2g}$$

WATERMELON (OR FUSEAU, OR ℓ-LEG) EXPONENTS



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Fits nicely into Kac table of m'th minimal model

$$m = \begin{cases} \frac{g}{1-g} & \text{for the dense O}(n)\text{ model, or the critical Potts model} \\ \frac{1}{g-1} & \text{for the dilute O}(n)\text{ model} \end{cases}$$

$$\Delta_\ell = \begin{cases} 2h_{0,\ell/2} & \text{for the dense O}(n)\text{ model} \\ 2h_{\ell/2,0} & \text{for the dilute O}(n)\text{ model} \end{cases}$$

Summary

1. Loop models and the Coulomb gas
   - General strategy
   - Potts model
   - O(n) model
2. Transformation to height models
   - The complex loop ensemble
   - Transformation to compactified height model
3. Liouville field theory
   - Background electric charge
   - Liouville term
   - Marginality requirement
4. Critical exponents
   - Watermelon exponents
   - Magnetic exponent in the Potts model

Magnetic exponent in the Potts model

- $P(\sigma_{\mathbf{x}_1} = \sigma_{\mathbf{x}_2}) \propto P(\mathbf{x}_1, \mathbf{x}_2) \in$ same cluster
- Hence $\mathbf{x}_1$ and $\mathbf{x}_2$ not separated by winding loop
- Hence give weight $\bar{n} = 0$ to such loops
- Inserting $\mathrm{e}^{\pm i\pi e_1}$ gives $\bar{n} = 2\cos(\pi e_1)$; here $e_1 = \frac{1}{2}$
- Compare free energy to ground state $e_1 = e_0$:

$$\Delta_{\rm m} = \Delta_{\frac{1}{2},0} - \Delta_{e_0,0} = \frac{1 - 4(1-g)^2}{8g}$$

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