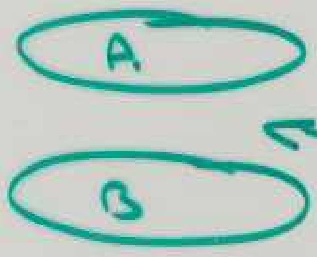
CONNECTIVITY:

(A, B) PARTITION OF E.

$$\lambda(A,B) = r(A) + r(B) - r(M).$$

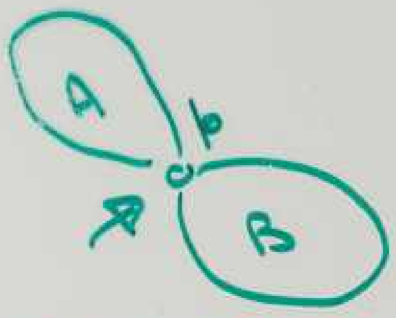
WHAT DOES THIS MEASURE?



SKEW

$\lambda(A,B) = 0$

A SEPARATION

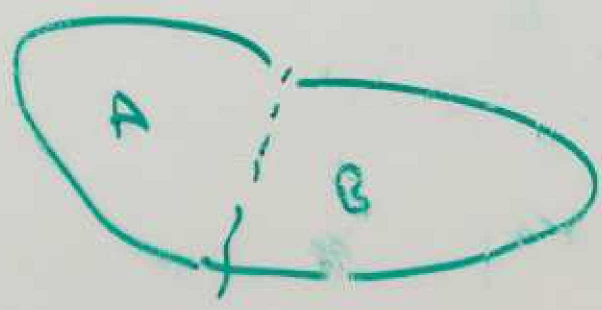


p NOT NECESSARILY IN E.

$\lambda(A,B) = 1$

A 2-SEPARATION

ODD?



$\lambda(A,B) = 2$

A 3-SEPARATION.

MATROID CONNECTIVITY GENERALISES VERTEX CONNECTIVITY

$G = (V, E)$, (A, B) PARTITION OF E.

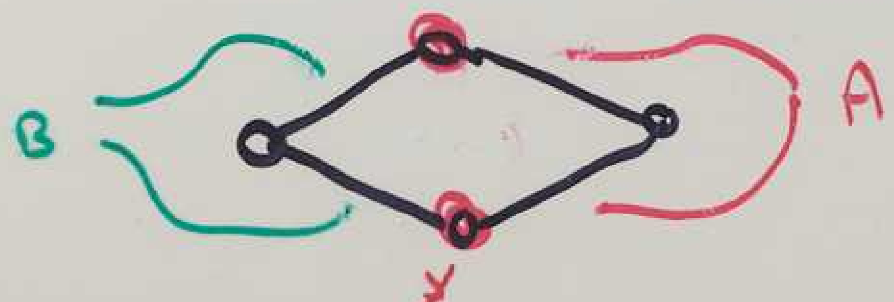
- ASSUME A, B INDUCE CONNECTED SUBGRAPHS

$r_{M(G)}(A) = |V(A)| - 1$

$r_{M(G)}(B) = |V(B)| - 1$

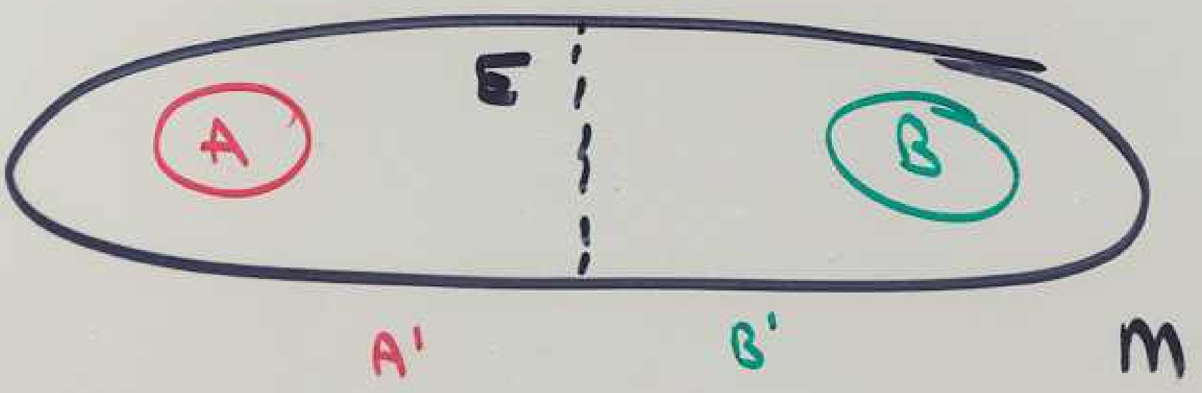
$r(M(G)) = |V| - 1$

$$\therefore \lambda_{M(G)}(A, B) = \underbrace{|V(A) \cap V(B)|}_{\text{SIZE OF A VERTEX CUT.}} - 1$$



$|V(A) \cap V(B)| = 2.$

MENGER'S THEOREM FOR MATROIDS



$$\kappa(A,B) = \min\{\lambda(A',B') : A' \supseteq A,\ B' \supseteq B\}$$

$\kappa(A,B)$ IS AN UPPER BOUND FOR AMOUNT OF COMMUNICATION BETWEEN A AND B.

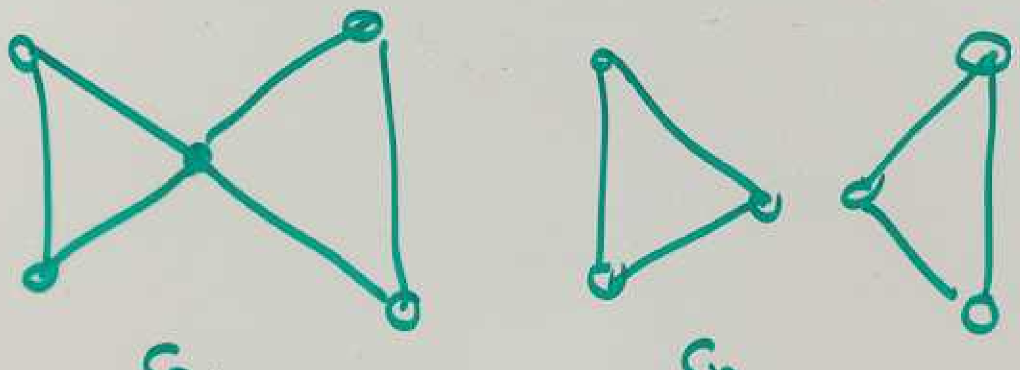
TUTTE'S LINKING LEMMA:

THERE EXISTS PARTITION X, Y OF $E - (A \cup B)$ s.t.

$$\lambda_{M \backslash X / Y}(A,B) = \kappa_M(A,B).$$

$\boxed{\text{EASY PROOF}}$

M k-CONNECTED IF NO NON-TRIVIAL k-SEPARATION.



G_1 G_2

$M(G_1)$, $M(G_2)$ NOT CONNECTED.

- G 2-CONNECTED $\iff$ $M(G)$ 2-CONNECTED.

- G 3-CONNECTED $\iff$ $M(G)$ 3-CONNECTED.

ALMOST - UP TO PARALLEL ~~ELEMENTS~~. EDGES.

BILL'S DECOMPOSITION OF 2-CONNECTED GRAPHS INTO 3-CONNECTED PIECES EXTENDS TO MATROIDS.

$\Rightarrow$ OFTEN SUFFICES TO SOLVE PROBLEMS FOR 3-CONNECTED MATROIDS.

REGULAR MATROIDS

MATRIX OVER $\mathbb{Q}$ IS <u>TOTALLY UNIMODULAR</u> IF ALL SUBDETERMINANTS ARE IN $\{0, \pm 1\}$.

MATROID M IS <u>REGULAR</u> IF IT CAN BE REPRESENTED BY A TOTALLY UNIMODULAR MATRIX.

EXAMPLES:

- GRAPHIC MATROIDS
- COGRAPHIC MATROIDS

R_{10}

$$\begin{bmatrix} 1 & 0 & & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & \ddots & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

THEOREM (TUTTE)

THE FOLLOWING ARE EQUIVALENT.

(1) M IS REGULAR

(2) M IS REPRESENTABLE OVER ALL FIELDS

(3) M IS REPRESENTABLE OVER $GF(2)$ AND $\mathbb{F}$ (NOT CHARACTERISTIC 2).

(3) M HAS NO MINOR ISOMORPHIC TO $U_{2,4}$, F_7, F_7^*.

BEAUTIFUL!

BUT OF LIMITED USE ALGORITHMICALLY.

DIRECT SUMS

2-SUMS

3-SUMS

PRESERVE REGULARITY.

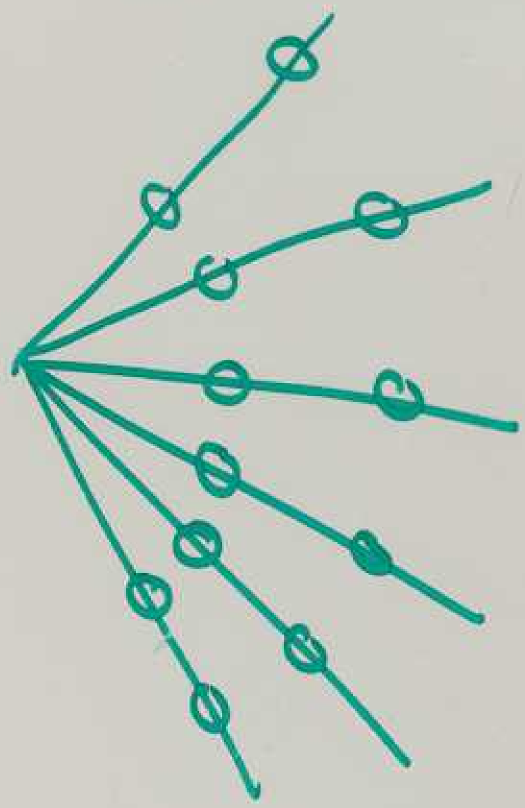
THEOREM (SEYMOUR):

M REGULAR IFF CAN BE BUILT BY DIRECT SUMS, 2-SUMS AND 3-SUMS FROM GRAPHIC MATROIDS, COGRAPHIC MATROIDS AND COPIES OF R_{10}.

LEADS TO POLYNOMIAL-TIME RECOGNITION ALGORITHM

ORACLE COMPLEXITY

DETERMINING IF M IS BINARY IS PROVABLY EXPONENTIAL.



THE BINARY SPIKE.

TERNARY CLASSES:

DYADIC – MATRICES OVER $\mathbb{Q}$, SUBDETERMINANTS $\{0, \pm 2^k\}$.

DYADIC

$= GF(3) \cap GF(5)$

$= GF(3) \cap \mathbb{Q}$

EXCLUDED MINORS?

STRUCTURE?

$\sqrt[6]{1}$ - MATROIDS

(SHOULD BE COMPLEX UNIMODULAR)

MATRICES OVER $\mathbb{C}$ SUBDETERMINANTS MODULUS 1.

$\sqrt[6]{1}$ - MATROIDS

$= GF(3) \cap GF(4)$

EXCLUDED MINORS ✓

STRUCTURE ?

(24)

GOLDEN MEAN.

α, β ROOTS OF $x^2 - x - 1 = 0$.

MATRICES OVER $\mathbb{R}$, SUBDETERMINANTS

$\{0, \pm \alpha^i \beta^j\}$

VERTIGAN (UNPUBLISHED)

GOLDEN MEAN

$= GF(4) \cap GF(5)$.

CHROMATIC POLYNOMIALS OF MATROIDS $P(M;\lambda)$

G A GRAPH $P(M(G);\lambda) = P(G;\lambda)$

(ESSENTIALLY)

$P(M^*(G);\lambda)$ = FLOW POLYNOMIAL OF G

THEOREM: M, GF(q) - REPRESENTED BY A. THEN,

$\max \{k$ s.t. $\exists$ A SUBSPACE W OF $PG(r-1,q)$ s.t. $W \cap A = \phi$ AND $r(W) = r-k\}$

$= \min \{k$ s.t. $P(M;q^k) > 0\}$.

CRAPO & ROTA

THE CRITICAL PROBLEM