Conformal Field Theory and Combinatorics

Part III: Applications to Combinatorics

Jesper L. Jacobsen[1,2]

[1]Université Pierre et Marie Curie, Paris 6, France

[2]Institut de Physique Théorique, CEA/Saclay, France

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Summary

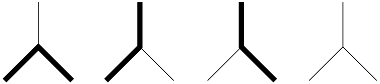
1. **Hamiltonian circuits and walks**
   - O(n) model revisited
   - FPL^2 model on the square lattice

2. **Coulomb Gas construction**
   - Compactification with respect to a lattice
   - Liouville field theory

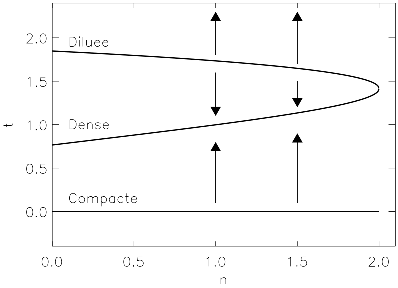
3. **Critical exponents**
   - Central charge and watermelon exponents
   - Standard exponents of polymer physics

Summary

O(n) MODEL ON THE HEXAGONAL LATTICE



- Low-T branch is RG attractive and $Z = Z(T^2)$
- Hence $T = 0$ is a line of fixed points



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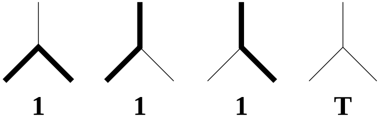

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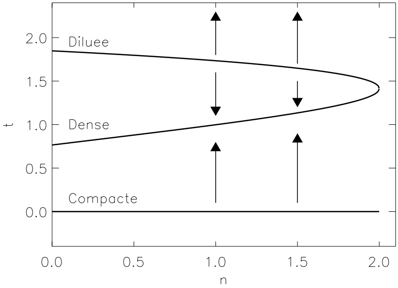

Right at $T = 0$ define partition function

$$Z = \sum_{\mathcal{G}} n^N$$

- $n \to 0$ describes Hamiltonian circuits
- Hamiltonian walks obtained by inserting defect pair

Mapping to $D = 2$ dimensional height model

- With $T = 0$, height differences satisfy $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$

$$\mathbf{A} = \left(\frac{1}{\sqrt{3}}, 0 \right), \quad \mathbf{B} = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2} \right), \quad \mathbf{C} = \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2} \right)$$

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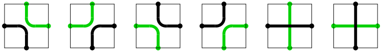
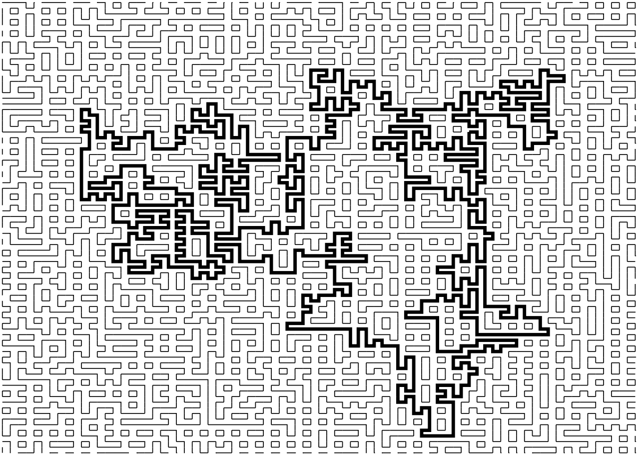
FPL^2 = Model of two types of fully-packed loops





Partition function of FPL^2 model

$$Z = \sum_{\mathcal{G}} n_{\rm b}^{N_{\rm b}} n_{\rm g}^{N_{\rm g}}$$

- Here $N_{\rm b}$ (resp. $N_{\rm g}$) is # black (resp. green) loops

Mapping to $D = 3$ dimensional height model

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = \mathbf{0}$$

$$\mathbf{A} = (-1, 1, 1) \qquad \mathbf{B} = (1, 1, -1)$$
$$\mathbf{C} = (-1, -1, -1) \qquad \mathbf{D} = (1, -1, 1)$$

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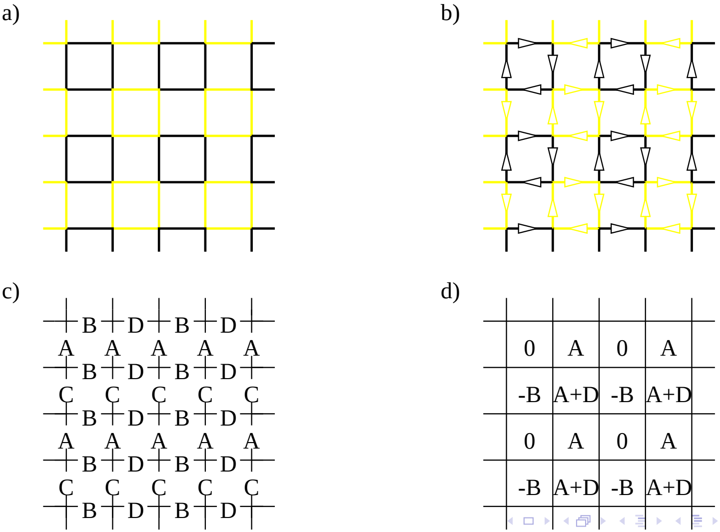
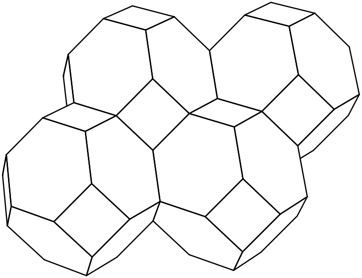
Summary

Ideal states = MACROSCOPICALLY FLAT STATES



CONSTRUCTION OF *ideal state graph* $\mathcal{I}$

- Transition between ideal states: transpose colour pair
- Associated change of average height $\in \mathbb{R}^D$
- In $\mathcal{I}$, vertex = ideal state, edge = transition



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- Defines *repeat lattice* $\mathcal{R}$
- Here FCC lattice with cubic cell of edge length 4

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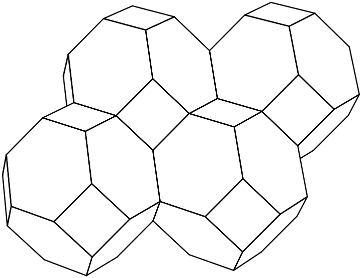

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Compactified height model

- $\mathbf{h}(\mathbf{x}) \in \mathbb{R}^D/\mathcal{R}$ with EM charges $\mathbf{e} \in \mathcal{R}^*$ and $\mathbf{m} \in \mathcal{R}$

Elastic term $S_{\rm E}$

- Constrained by rotational invariance...

$$S_{\rm E} = \frac{1}{2} \int {\rm d}^2\mathbf{x}\, K_{\alpha\beta}\, \partial h^\alpha \cdot \partial h^\beta$$

- ...and by loop reversal symmetries (**A** $\leftrightarrow$ **B** and **C** $\leftrightarrow$ **D**)

$$S_{\rm E} = \frac{1}{2} \int {\rm d}^2\mathbf{x} \left\{ K_{11}[(\partial h^1)^2 + (\partial h^3)^2] + 2K_{13}(\partial h^1 \cdot \partial h^3) + K_{22}(\partial h^2)^2 \right\}$$

- Diagonalised by a change of basis for $\mathbf{h}$

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BOUNDARY TERM $S_{\rm B}$

$$S_{\rm B} = \frac{\rm i}{4\pi} \int {\rm d}^2\mathbf{x}\, (\mathbf{e}_0 \cdot \mathbf{h}) \mathcal{R}(\mathbf{x})$$

- $\mathcal{R}(\mathbf{x})$ is the scalar curvature
- Parametrise weights: $n_{\rm b} = 2\cos(\pi e_{\rm b})$ and $n_{\rm g} = 2\cos(\pi e_{\rm g})$
- Constraints:
  $\mathbf{e}_0 \cdot \mathbf{A} = \pi e_{\rm b}, \quad \mathbf{e}_0 \cdot \mathbf{B} = -\pi e_{\rm b}, \quad \mathbf{e}_0 \cdot \mathbf{C} = \pi e_{\rm g}, \quad \mathbf{e}_0 \cdot \mathbf{D} = -\pi e_{\rm g}$
- Solution: $\mathbf{e}_0 = -\pi(e_{\rm b}, 0, e_{\rm g})$

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Liouville term $S_{\rm L}$

$$S_{\rm L} = \int {\rm d}^2\mathbf{x}\, w[\mathbf{h}(\mathbf{x})]$$

- $\exp(-w[\mathbf{h}(\mathbf{x})])$ is scaling limit of vertex weights in oriented loop representation
- Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be colours around a vertex. Then $w(\mathbf{h}) = \frac{\rm i}{16}\, \mathbf{e}_0 \cdot [(\sigma_1 - \sigma_3) \times (\sigma_2 - \sigma_4)]$

Fourier analysis

$$w(\mathbf{h}) = \sum_{\mathbf{e} \in \mathcal{R}_w^*} \tilde{w}_{\mathbf{e}} \exp({\rm i}\mathbf{e} \cdot \mathbf{h}) \,.$$

- $\mathcal{R}_w^*$ spanned by next-shortest vectors in $\mathcal{R}^*$

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DIMENSIONS OF ELECTROMAGNETIC OPERATORS

- In basis where coupling constant tensor g^α is diagonal:

$$\Delta_{\mathbf{e},\mathbf{m}} = \frac{1}{4\pi}\left[\frac{1}{g_\alpha} e_\alpha (e_\alpha - 2e_{0\alpha}) + g_\alpha (m^\alpha)^2\right]$$

MARGINALITY REQUIREMENT

- Most revelant **e**-charges in $\mathcal{R}_w^*$ are marginal

$$g_1 = \frac{\pi}{2}(1 - e_{\rm b}), \qquad \frac{1}{g_2} = \frac{1}{g_1} + \frac{1}{g_3}, \qquad g_3 = \frac{\pi}{2}(1 - e_{\rm g})$$

Dimensions of electromagnetic operators

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Summary

Central charge $c = D + 12\Delta_{\mathbf{e}_0,\mathbf{0}}$

$$c = \begin{cases} 2 - \frac{6e_0^2}{1-e_0} & \text{for the FPL model} \\ 3 - 6\left(\frac{e_\mathrm{b}^2}{1-e_\mathrm{b}} + \frac{e_\mathrm{g}^2}{1-e_\mathrm{g}}\right) & \text{for the FPL}^2\text{ model} \end{cases}$$

Watermelon exponents (from vector defects $\mathbf{m} \in \mathbb{R}^D$)

- For FPL model on honeycomb lattice

$$\Delta_\ell = \begin{cases} \frac{1}{8}g\ell^2 - \frac{(1-g)^2}{2g} & \text{for } \ell \text{ even} \\ \frac{1}{8}g\ell^2 - \frac{(1-g)^2}{2g} + \frac{3g}{8} & \text{for } \ell \text{ odd} \end{cases}$$

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Summary

Number of selfavoiding walks and polygons

- Length l, with one monomer attached to a fixed point

$$\mathcal{N}_{\rm SAW} \sim \mu^l \, l^{\gamma-1} \qquad \mathcal{N}_{\rm SAP} \sim \mu^l \, l^{-\nu d}$$

- Standard scaling relations

$$\Delta_1 = 1 - \frac{\gamma}{2\nu} \qquad \Delta_2 = 2 - \frac{1}{\nu}$$

Results for conformational exponents

Model	ν	γ
Dilute O($n \to 0$)	$\frac{3}{4}$	$\frac{43}{32}$
Dense O($n \to 0$)	$\frac{1}{2}$	$\frac{19}{16}$
FPL honeycomb	$\frac{1}{2}$	1
FPL square	$\frac{1}{2}$	$\frac{117}{112}$

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