Real Zeros of Chromatic and Flow Polynomials

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Real Zeros of Chromatic and Flow Polynomials

- Chromatic Polynomials of Graphs
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  - Planar Triangulations
  - Graphs of Bounded Maximum Degree
- Flow Polynomials of Graphs
  - Arbitrary Graphs
  - Cubic Graphs
  - Graphs of Bounded Diameter

Chromatic Polynomials: Arbitrary Graphs

Theorem

Let G be a loopless graph with n vertices, c components, and b blocks which are not isolated vertices. Then:

- $P_G(t)$ is non-zero with sign $(-1)^n$ for $t \in (-\infty, 0)$;
- $P_G(t)$ has a zero of multiplicity c at $t = 0$;
- $P_G(t)$ is non-zero with sign $(-1)^{n+c}$ for $t \in (0, 1)$;
- $P_G(t)$ has a zero of multiplicity b at $t = 1$, (Woodall);
- $P_G(t)$ is non-zero with sign $(-1)^{n+c+b}$ for $t \in (1, \frac{32}{27}] \approx (1, 1.185]$.

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Theorem (Thomassen)

The real chromatic roots of graphs are dense everywhere in $[\frac{32}{27}, \infty)$.

Chromatic Polynomials: 3-Connected Graphs

Conjecture

- If G is a loopless 3-connected graph, then G has no chromatic roots in $(1, \alpha)$ where $\alpha \approx 1.781$ is the chromatic root of $K_{3,4}$.
- For all $\epsilon > 0$, there are only finitely many 3-connected cubic graphs with a chromatic root in $(1, 2-\epsilon)$.

Chromatic Polynomials: Planar Graphs

Theorem

If G is a loopless planar graph, then $P_G(t) > 0$ for $t \in [5, \infty)$, (Birkhoff and Lewis).

Theorem

The real chromatic roots of planar graphs are dense everywhere in $[\frac{32}{27}, 3]$. (Thomassen)

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Conjecture

If G is a loopless planar graph, then $P_G(t) > 0$ for $t \in [4, \infty)$, (Birkhoff and Lewis).

Conjecture

The real chromatic roots of planar graphs are dense everywhere in $[\frac{32}{27}, 4]$. (Thomassen)

Chromatic Polynomials: Plane (Near) Triangulations

A **near triangulation** is a plane graph with at most one face of size other than three.

Theorem

Let G be a 3-connected plane near triangulation with n vertices. Then

- $P_G(t)$ is non-zero with sign $(-1)^n$ for $t \in (1,2)$, (Birkhoff and Lewis);
- $P_G(t)$ has a simple zero at $t = 2$, (Woodall);
- If G is a triangulation then $P_G(t)$ is non-zero with sign $(-1)^{n+1}$ for $t \in (2, \delta)$, where $\delta \approx 2.546$ is the chromatic root of the octahedron in $(2, 3)$, (Woodall).

Chromatic Polynomials: Plane Triangulations

Conjecture (Woodall)

- If G is a 4-connected plane triangulation then $P_G(t)$ has at most one zero in $(2, \tau^2)$, where $\tau = \frac{1+\sqrt{5}}{2}$ and $\tau^2 \approx 2.6180$.
- For all $\epsilon > 0$, there exist only finitely many 4-connected plane triangulations with a chromatic root in $(2, \tau^2 - \epsilon)$.

Chromatic Polynomials: Graphs of Bounded Maximum Degree

Theorem (Sokal)

Let G be a loopless graph of maximum degree Δ. Then $P(G, z) \neq 0$ for all complex z with $|z| \geq C\Delta$, where $C \approx 7.9639$.

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Let G be a loopless graph of maximum degree Δ. Then $P(G, t) > 0$ for all $t > \Delta$.

Theorem (Thomassen, Woodall)

Let G be a graph. Suppose that every simple minor of G has a vertex of degree at most d. Then $P(G, t) > 0$ for all $t > d$.

Let t be a positive integer and G be a graph.

Construct a digraph $\vec{G}$ by giving the edges of G an arbitrary orientation.

A **nowhere-zero $\mathbb{Z}_t$-flow** of G (with respect to $\vec{G}$) is an assignment of flow values $1, 2, 3, t-1$ to the arcs of $\vec{G}$ such that the total flow entering each vertex is congruent to the total flow leaving each vertex, modulo t.

Conjecture (Tutte)

Let G be a bridgeless graph.

- G has a nowhere zero 5-flow.
- If G has no edge-cuts of size three then G has a nowhere zero 3-flow.

Tutte's Conjectures and the Jaeger-Seymour Theorems

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Theorem

Let G be a bridgeless graph.

- G has a nowhere zero 6-flow. (Seymour)
- If G has no edge-cuts of size three then G has a nowhere zero 4-flow. (Jaeger)

The Flow Polynomial

Let $G = (V, E)$ be a graph. For each positive integer t, let $F_G(t)$ be the number of nowhere-zero $\mathbb{Z}_t$-flows of G. (By definition $F_G(t) \equiv 1$ if $E = \emptyset$.)

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Contraction-Deletion Lemma

Let G be a graph and e be an edge of G which is not a bridge. Then

$$F_G(t) = F_{G/e}(t) - F_{G-e}(t).$$

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This implies that $F_G(t)$ is a polynomial in t, the **flow polynomial** of G.

Lemma (Tutte)

If G is a connected plane graph and G^* is its planar dual then

$$F_G(t) = t^{-1} P_{G^*}(t).$$

Flow Polynomials: Arbitrary Graphs

Theorem (Wakelin)

Let G be a bridgeless graph with n vertices, m edges, b blocks, and no isolated vertices. Then:

- $F_G(t)$ is non-zero with sign $(-1)^{m-n+1}$ for $t \in (-\infty, 1)$;
- $F_G(t)$ has a zero of multiplicity b at $t = 1$;
- $F_G(t)$ is non-zero with sign $(-1)^{m-n+b+1}$ for $t \in (1, \frac{32}{27}] \approx (1, 1.185]$.

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Theorem (Thomassen)

The real flow roots of graphs are dense everywhere in $[\frac{32}{27}, 3)$.

Flow Polynomials: Arbitrary Graphs

Conjecture (Thomassen)

The flow roots of graphs are dense in $[\frac{32}{27}, 4]$.

Conjecture (Welsh)

If G is a bridgeless graph, then $F_G(t) > 0$ for all $t > 4$.

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Theorem

Let G be a bridgeless graph. If every every 3-edge-connected minor of G has a cycle of length at most d, then $F_G(t) > 0$ for all $t > d$.

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Theorem

Let G be a bridgeless graph. If every every 3-edge-connected minor of G has a cycle of length at most d, then $F_G(t) > 0$ for all $t > d$.

Corollary

Let G be a bridgeless graph with n vertices. Then $F_G(t) > 0$ for all $t > 2 \log_2 n$.

Flow Polynomials: (Near) Cubic Graphs

A graph G is **near cubic** if it has at most one vertex of degree other than three.

Theorem

Let G be a 3-connected near cubic with n vertices and m edges. Then:

- $F_G(t)$ is non-zero with sign $(-1)^{m-n}$ for $t \in (1,2)$;
- $F_G(t)$ has a simple zero at $t = 2$;
- $F_G(t)$ is non-zero with sign $(-1)^{m-n+1}$ for $t \in (2,\rho]$, where $\gamma \approx 2.225$ is the root of $t^4 - 8t^3 + 22t^2 - 28t + 17$ in $(2,3)$;
- If G is cubic then $F_G(t)$ has no zeros in $(2,\delta]$, where $\delta \approx 2.546$ is the flow root of the cube in $(2,3)$.

Conjecture

- If G is a cyclically-4-connected cubic graph then $F_G(t)$ has at most one zero in $(2, \tau^2)$, where $\tau = \frac{1+\sqrt{5}}{2}$ and $\tau^2 \approx 2.6180$.
- For all $\epsilon > 0$, there exist only finitely many cyclically 4-connected cubic graphs with a flow root in $(2, \tau^2 - \epsilon)$.

Let $G = (V, E)$ be a graph and $Z \subset E$. Then Z is a $\mathbb{Z}_2$-**cycle** of G if each vertex of G is incident to an even number of edges of Z. The set of $\mathbb{Z}_2$-cycles of G is closed under symmetric difference and hence forms a vector space over $\mathbb{Z}_2$, the **cycle space**, $\mathcal{Z}(G)$, of G. For each basis B of $\mathcal{Z}(G)$, let $\Lambda(B)$ be the maximum size of a $\mathbb{Z}_2$-cycle in B. Let $\Lambda(G)$ be the minimum of $\Lambda(B)$ over all bases B for $\mathcal{Z}(G)$.

Flow Polynomials: Graphs of Bounded Diameter

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Conjecture (Sokal and BJ)

There exists a constant $c > 0$ such that, for all bridgeless graphs G, and all complex z with $|z| > c\Lambda(G)$, we have $F_G(z) \neq 0$.

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There exists a constant $c > 0$ such that, for all bridgeless graphs G of diameter D, and all complex z with $|z| > cD$, we have $F_G(z) \neq 0$.

The **multivariate Tutte polynomial** of a graph $G = (V, E)$ is the $(|E| + 1)$-variable polynomial given by

$$Z_G(q, \mathbf{w}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} w_e,$$

where $\mathbf{w} = (w_e)_{e \in E}$ is a vector of indeterminates.
We have $P_G(q) = Z_G(q, -\mathbf{1})$.

The Multivariate Tutte Polynomial

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We have $P_G(q) = Z_G(q, -\mathbf{1})$.

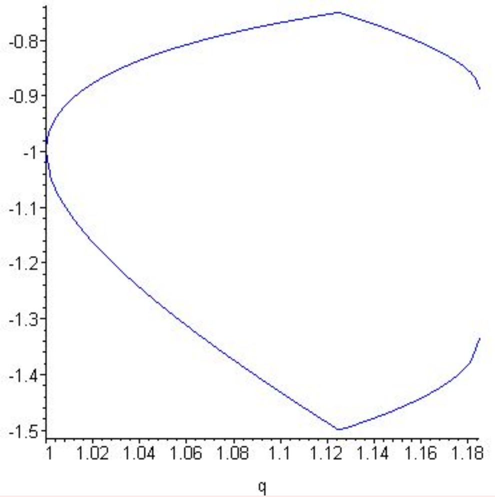
Theorem (Sokal and BJ)

Suppose $G = (V, E)$ is a loopless 2-connected graph and $q \in (1, 32/27]$. Let $w_\diamond$ be the root of $w^3 - 2wq - q^2$ which lies in $(-1, 0)$. Put

$$L_q = \begin{cases} (-q - \sqrt{q^2 - q}, -q + \sqrt{q^2 - q}) & \text{if } q \in (1, 9/8] \\ (q/w_\diamond, w_\diamond) & \text{if } q \in (9/8, 32/27] \end{cases}$$

If $w_e \in L_q$ for all $e \in E$, then $(-1)^{|V|} Z_G(q, \mathbf{w}) > 0$.

The Interval L_q for $1 < q < 32/27$



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