Chromatic Factorisation of Graphs

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CSM workshop - Zeros of graph polynomials, 2008

Joint work with Graham Farr

Overview

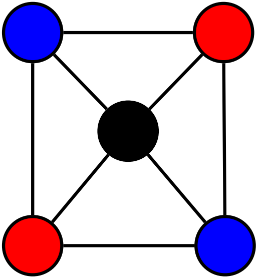
- Chromatic Polynomial
- Basic Properties
- Chromatic Factorisation
- Certificate of Factorisation
- Infinite Family of Graphs that have Chromatic Factorisations

The Chromatic Polynomial

$P(G, \lambda)$ gives the number of proper λ-colourings of a graph G

$$P(G, \lambda) = \lambda^5 - 8\lambda^4 + 24\lambda^3 - 31\lambda^2 + 14\lambda$$
$$= \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$$

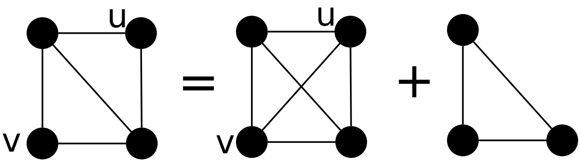
- $P(G, 0) = 0$
- $P(G, 1) = 0$
- $P(G, 2) = 0$
- $P(G, 3) = 6$
- $P(G, 4) = 72$
- $\chi(G) = 3$



The Chromatic Polynomial - Properties

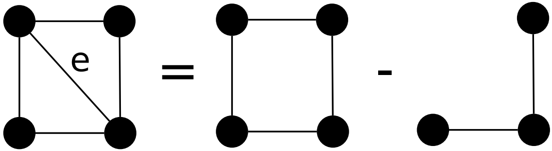
Addition-identification

$$P(G,\lambda) = P(G+uv,\lambda) + P(G/uv,\lambda)$$



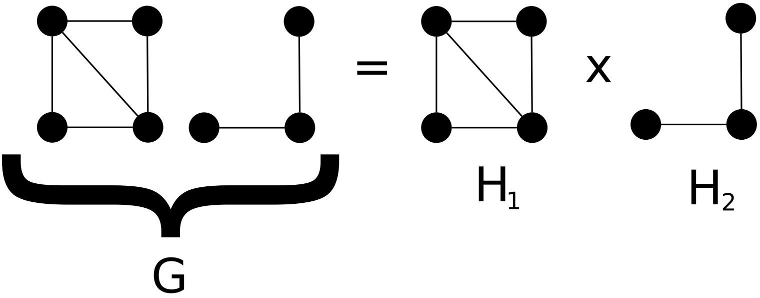
Deletion-contraction

$$P(G,\lambda) = P(G \setminus e, \lambda) - P(G/e, \lambda)$$



Chromatic polynomial of graph with more than a single component

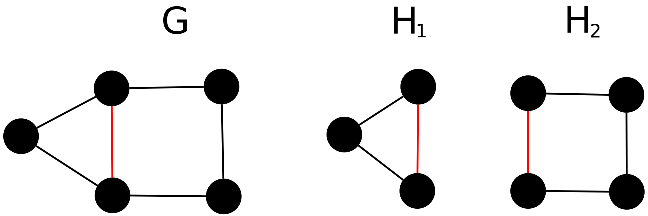
$$P(G, \lambda) = P(H_1, \lambda)P(H_2, \lambda)$$



The Chromatic Polynomial - Properties

Clique-gluings

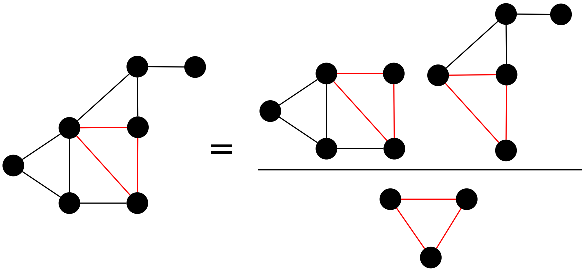
G is an *r-gluing*, or clique-gluing, of graphs H_1 and H_2, if G can be obtained by identifying an r-clique in H_1 with an r-clique in H_2.



The Chromatic Polynomial - Properties

A graph is *clique-separable* if it is isomorphic to the graph obtained by an r-gluing of graphs H_1 and H_2.

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)}$$



Chromatic Equivalence

- G is chromatically equivalent to H if $P(G, \lambda) = P(H, \lambda)$
- $G \sim H$

Motivation

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- Large amount of research on roots of chromatic polynomials
- Little research into the algebraic theory of chromatic roots
- First step in finding roots of a polynomial is factorisation

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Chromatic Factorisation

If there exist graphs G, H_1, H_2 such that

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)}$$

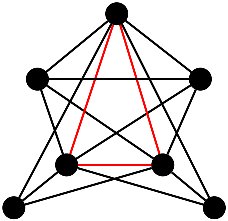
where $r \leq \min\{\chi(H_1), \chi(H_2)\}$, then $P(G,\lambda)$ has a *chromatic factorisation* with *chromatic factors* $P(H_1,\lambda)$ and $P(H_2,\lambda)$.

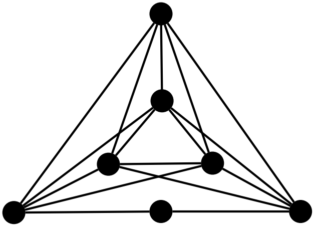
If either H_1 or H_2 is the complete graph K_s then $r < s$.

Chromatic Factorisation

$P(G, \lambda)$ has a chromatic factorisation

- G is an r-gluing of graphs H_1 and H_2,
- $G \sim H$, and H is an r-gluing of graphs H_1 and H_2
- Any others?





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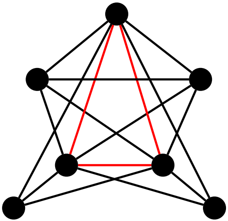

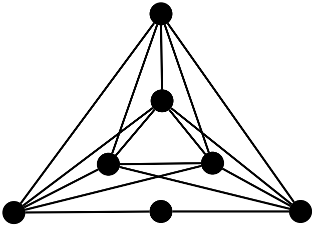

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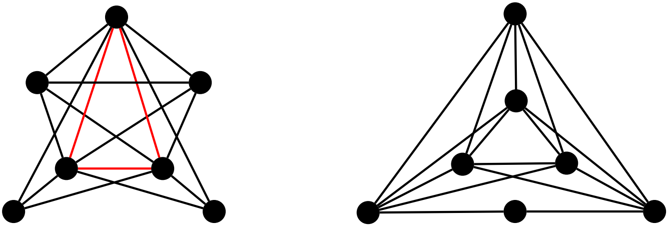

Does there exist $P(G, \lambda)$ that has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph?

Yes ...

n	Chromatic polynomials	Non-isomorphic graphs
8	2	3
9	25	97
10	485	3018
$n \leq 10$	512	3118

Certificates of Factorisation

Certificate of Factorisation

Sequence of steps $P_0, \ldots, P_i$

- P_0 is $P(G, \lambda)$
- P_i is $\frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)}$
- P_j is formed from P_{j-1} by a certification step

A Certificate of Factorisation: Certificate 1

$$
\begin{aligned}
P(G,\lambda) &= P(G+uv,\lambda) + P(G/uv,\lambda) && (1)\\
&= \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)} + \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_s,\lambda)} && (2)\\
&= P(H_1,\lambda)\left(\frac{P(H_3,\lambda)}{P(K_s,\lambda)} + \frac{P(H_4,\lambda)}{P(K_t,\lambda)}\right) && (3)\\
&= \frac{P(H_1,\lambda)}{P(K_r,\lambda)}\left(\frac{P(K_r,\lambda)P(H_3,\lambda)}{P(K_s,\lambda)} + \frac{P(K_r,\lambda)P(H_4,\lambda)}{P(K_t,\lambda)}\right) && (4)\\
&= \frac{P(H_1,\lambda)}{P(K_r,\lambda)}\left(P(H_5,\lambda) + P(H_6,\lambda)\right) && (5)\\
&= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} && (6)
\end{aligned}
$$

(1) add-ident., (2) clique-gluing, (3) common factor,
(4) multiply by $\frac{P(K_r,\lambda)}{P(K_r,\lambda)}$, (5) clique-gluing and (6) add-ident.

Certification Steps

Certification Steps:

- Addition-identification
- Deletion-contraction
- Chromatic equivalence
- Clique-gluings
- Basic algebra

Certificates of Factorisation

- Simple
  - Graph is clique-separable
  - Graph is chromatically equivalent to a clique-separable graph
- Identified cases where $P(G, \lambda)$ has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph
  - Certificates of factorisation, $n \leq 9$
- Identified an infinite family of graphs that have a chromatic factorisation
  - Not clique-separable
  - Certificate of factorisation for this family

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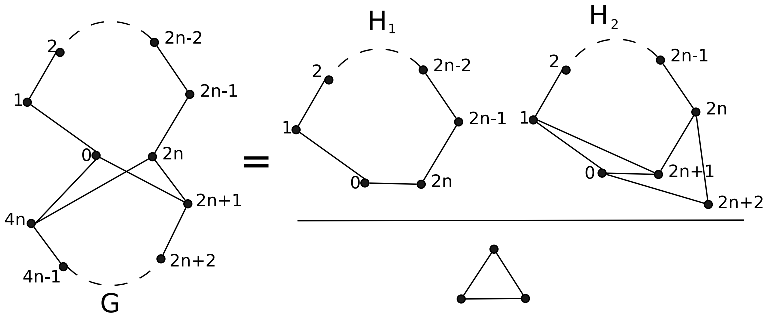
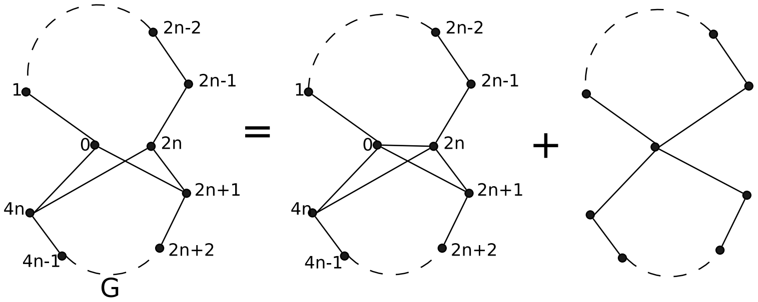
A Certificate of Factorisation

- G is $C_{4n+1} + (0, 2n+1) + (2n, 4n)$, $n \geq 2$
- H_1 is C_{2n+1}



A Certificate of Factorisation

$$P(G, \lambda) = P(G + (0, 2n), \lambda) + P(G/(0, 2n), \lambda)$$

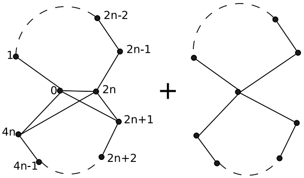


A Certificate of Factorisation

$$= P(G + (0,2n), \lambda) + P(G/(0,2n), \lambda)$$

$$= \frac{P(C_{2n+1}, \lambda) P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n+1}, \lambda) P(C_{2n}, \lambda)}{P(K_1, \lambda)}$$

=



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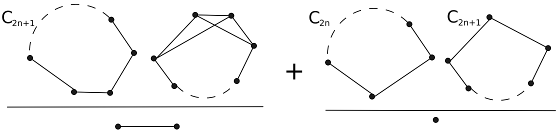
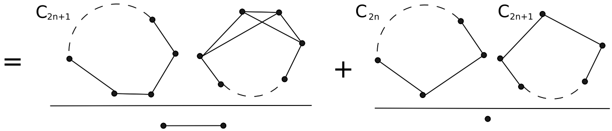
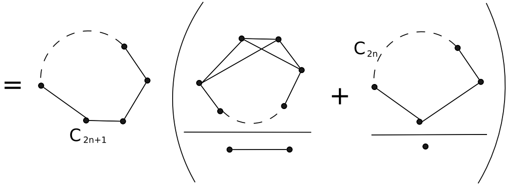


A Certificate of Factorisation

$$= \frac{P(C_{2n+1},\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(C_{2n+1},\lambda)P(C_{2n},\lambda)}{P(K_1,\lambda)}$$

$$= P(C_{2n+1},\lambda)\left(\frac{P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(C_{2n},\lambda)}{P(K_1,\lambda)}\right)$$

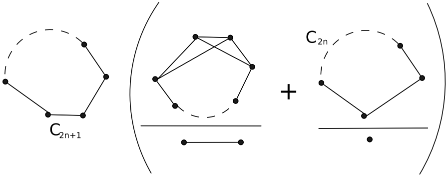


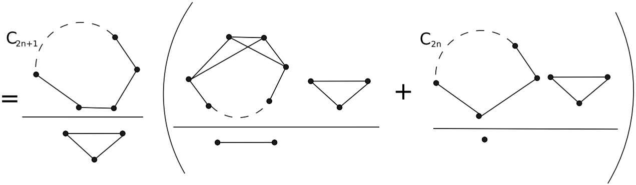


A Certificate of Factorisation

$$= P(C_{2n+1}, \lambda) \left(\frac{P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)}{P(K_1, \lambda)} \right)$$

$$= \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)} \left(\frac{P(H_3, \lambda) P(K_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda) P(K_3, \lambda)}{P(K_1, \lambda)} \right)$$





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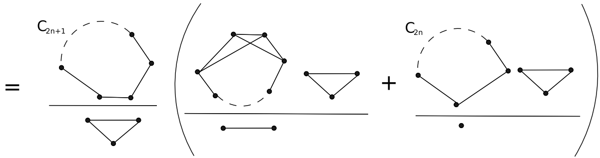

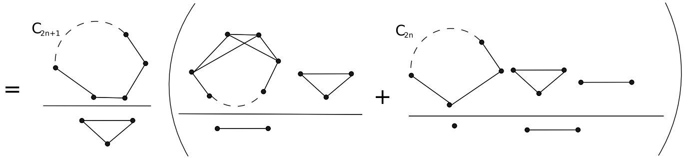

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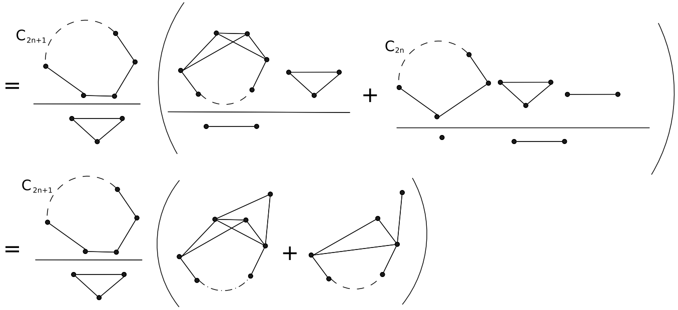
$$= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(P(H_5,\lambda) + P(H_6,\lambda)\right)$$



$$= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(P(H_5,\lambda)+P(H_6,\lambda)\right)$$

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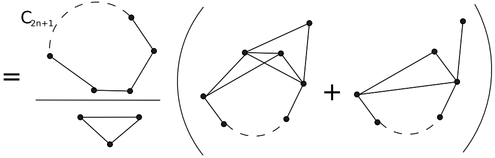

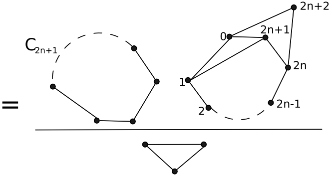

Certificate 1 with $H_1 \cong C_{2n+1}$

$$\begin{aligned}
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&= P(C_{2n+1},\lambda)\left(\frac{P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_4,\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(\frac{P(H_3,\lambda)P(K_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_4,\lambda)P(K_3,\lambda)P(K_2,\lambda)}{P(K_1,\lambda)P(K_2,\lambda)}\right) \\
&= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(P(H_5,\lambda) + P(H_6,\lambda)\right) \\
&= \frac{P(C_{2n+1},\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}
\end{aligned}$$

Theorem

There exists an infinite family of non-clique-separable graphs $\mathcal{G}$ such that for all $G \in \mathcal{G}$, $P(G, \lambda)$ has a chromatic factorisation.

Note: Every $G \in \mathcal{G}$ satisfies Certificate 1 with $H_1 \cong C_{2n+1}$, $n \geq 2$.

Theorem

Graphs in $\mathcal{G}$ are the only graphs that have a chromatic factorisation in the form of Certificate 1 with $r = 3$ and $H_1 \cong C_{2n+1}$.

$P(G,\lambda)$ has a chromatic factorisation but is not the chromatic polynomial of any clique-separable graph.

Let t_G, t_1, t_2 be the number of triangles in G, H_1 and H_2 respectively. Then

- $t_G = t_1 + t_2 - \binom{r}{3}$
- If $r = 3$ then
  - $t_1 = 0$
  - $t_G = t_2 - 1$
  - In the case where $P(G,\lambda)$ has a chromatic factorisation in the form of Certificate 1:
    - $t_G = 0$
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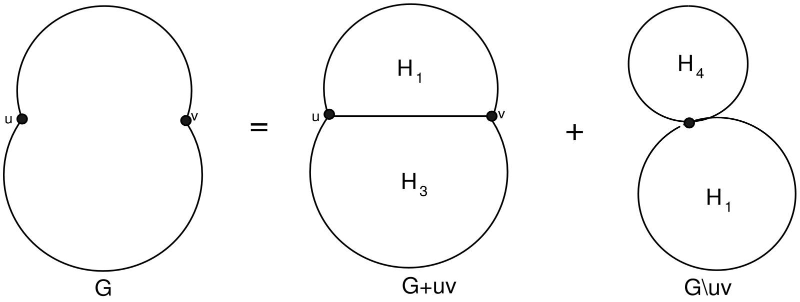
Proof (idea)
Some observations

- $H_1 \cong H_3/uv$
- $H_4 \cong H_1/uv \cong C_{2n}$



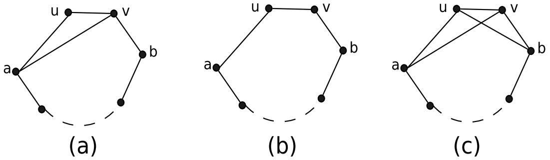
Proof (idea)

Now

- $H_1 \cong H_3/uv \cong C_{2n+1}$

Three options for H_3

- $C_{2n+2} + av$
- C_{2n+2}
- $C_{2n+2} + av + bu$



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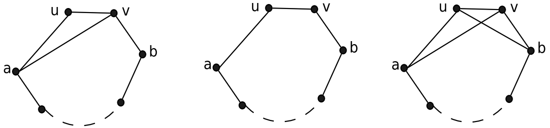


(a) (b) (c)

Certificate 1 requires:

- H_5 isomorphic to a K_2-gluing of H_3 and K_3
- $H_5 \setminus e$ isomorphic to H_2
- $P(H_5/e, \lambda)$ isomorphic to $P(H_6, \lambda) = \frac{P(C_{2n},\lambda)P(K_3,\lambda)P(K_2,\lambda)}{P(K_2,\lambda)P(K_1,\lambda)}$

H_3 is option (c)

So

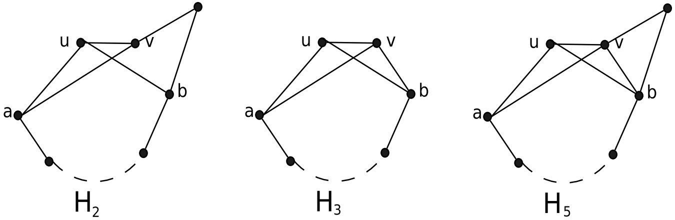
- H_3 is isomorphic to $C_{2n+2} + av + bu$
- H_5 is isomorphic to K_2-gluing of H_3 and K_3 on edge bv

Thus

- H_2 is isomorphic to $H_5 \setminus bv$

and

- $G + uv$ is a K_2-gluing of H_3 and C_{2n+1} on the edge uv
- G belongs to the family of graphs



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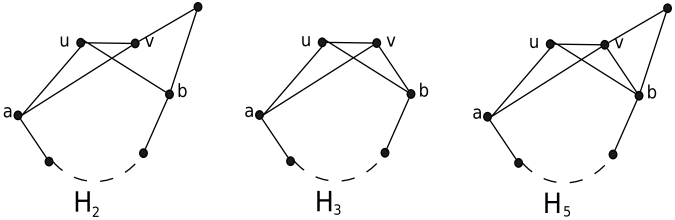

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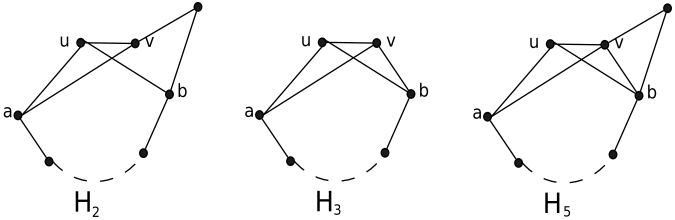

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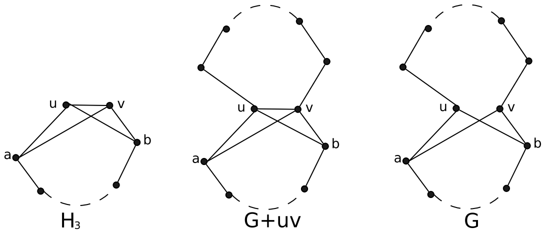

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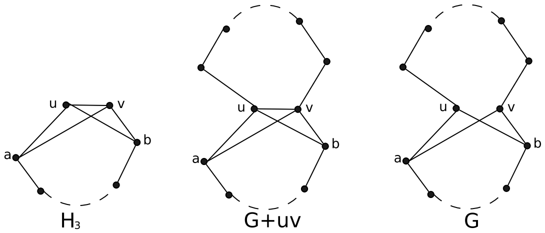
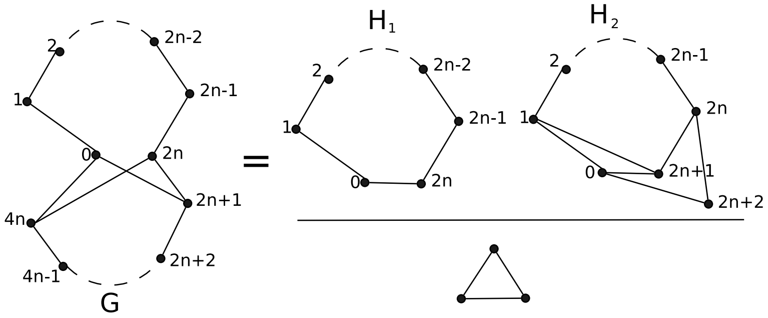
- H_2 is isomorphic to $H_5 \setminus bv$

and

- $G + uv$ is a K_2-gluing of H_3 and C_{2n+1} on the edge uv
- G belongs to the family of graphs



Chromatic Factorisation of $G \in \mathcal{G}$.



Further Work

- Identify some properties of non-clique-separable graphs that have chromatic factorisations
- Identify other infinite families of non-clique-separable graphs that have a chromatic factorisation
- Devise certificates to explain different types of chromatic factorisation
- Study chromatic roots using algebraic means