# Constructive resolution of two conjectures on real chromatic roots

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# Outline

# CHROMATIC FUNCTION

In 1912 Birkhoff introduced the function $P_G(k)$ such that for a graph $G$ and positive integer $k$,

$$P_G(k) = \text{the number of proper } k\text{-colourings of } G.$$

> *Type* 2. Here the emphasis is quantitative. Moreover no restriction is made on the number of colors considered. This point of view leads inevitably to certain polynomials each one of which gives the exact number of ways an associated map may be colored in any number of colors. The theory of these -called "chromatic" polynomials was initiated by Birkhoff in 1912 (Birkhoff

This was an attempt to develop *quantitative* tools to *count* the number of colourings of a planar graph, rather than the alternative *qualitative* approach ("Type 1") of just proving the existence of a 4-colouring.

# Chromatic Polynomials

Well known that $P_G(k)$ is a monic polynomial with alternating coefficients:

- For the complete graph $K_n$ we have

$$P_{K_n}(k) = k(k-1)(k-2)\dots(k-n+1) = \langle x \rangle_k$$

- For any tree $T$ on $n$ vertices we have

$$P_T(k) = k(k-1)^{n-1}$$

- For the Petersen graph $P$ we have $P_P(k)$ equal to

$$k^{10} - 15\,k^9 + 105\,k^8 - 455\,k^7 + 1353\,k^6 - 2861\,k^5 + 4275\,k^4 - 4305\,k^3 + 2606\,k^2 - 704\,k$$

# Chromatic Roots

Birkhoff & Lewis generalized the Five Colour Theorem:

- Five-Colour Theorem (Heawood 1890)
  If $G$ is planar then $P_G(5) > 0$.
- Birkhoff-Lewis Theorem (1946)
  If $G$ is planar and $x \geq 5$, then $P_G(x) > 0$.
- Birkhoff-Lewis Conjecture [still unsolved]
  If $G$ is planar and $x \geq 4$ then $P_G(x) > 0$.

The hope was that studying the *real chromatic roots* of a graph $G$ — namely the real numbers where $P_G(x) = 0$ — might tell us where $P_G(x) \neq 0$.

# FUNDAMENTAL RESULTS

We have already heard some of the many results regarding real and complex chromatic roots, including

- Chromatic root free intervals $(0, 1)$ and $(1, 32/27)$ and extensions of the latter interval for special classes of graphs.
- Complex chromatic roots arbitrarily close to any complex number, even for planar graphs.[1]

[1] perhaps not $|z - 1| < 1$

# Two conjectures on real chromatic roots

This talk considers two conjectures on the location of real chromatic roots.

- Jackson's Conjecture
  A 3-connected graph that is not bipartite of odd order has no chromatic roots in the interval $(1, 2)$.
  On to Jackson's Conjecture
- Beraha's Conjecture
  There are planar graphs with *real* chromatic roots arbitrarily close to $x = 4$.
  On to Beraha's Conjecture

# Jackson's conjecture

For chromatic roots in $(1, 2)$ we know

- No chromatic roots in $(1, \frac{32}{27}) \approx (1, 1.185)$ (Bill Jackson).
- Chromatic roots dense in $[\frac{32}{27}, \infty)$ (Carsten Thomassen).

The extremal graphs have many 2-cuts and so perhaps requiring $3$-connectivity might push the root-free interval up, perhaps even to $(1, 2)$.

**Conjecture (not Jackson)**

A 3-connected graph has no chromatic roots in the interval $(1, 2)$.

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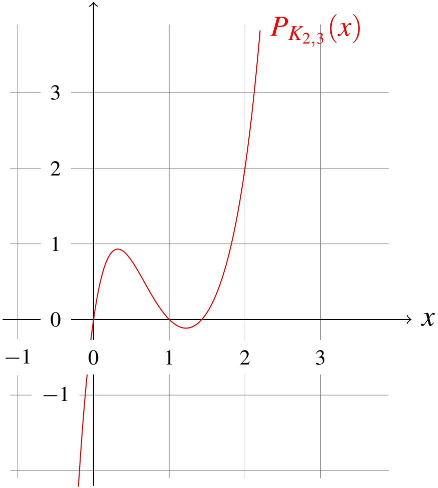
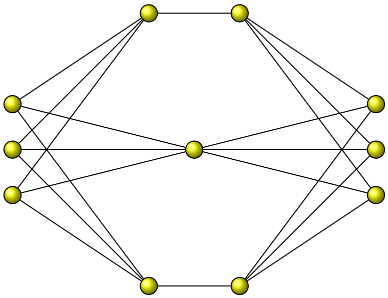
## Conjecture (Jackson)

A 3-connected graph (that is not bipartite of odd order) has no chromatic roots in the interval $(1, 2)$.

# Bipartite graphs of odd order



# A counterexample

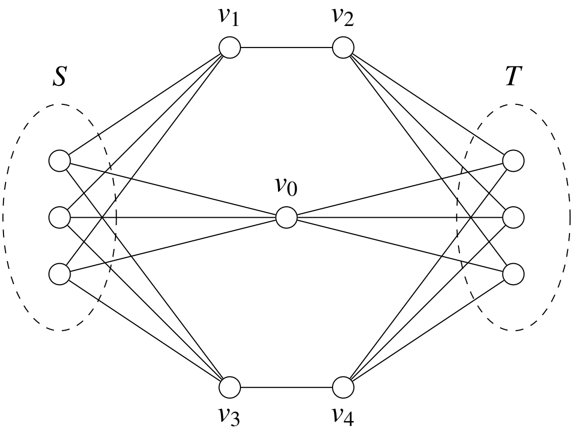


This 11-vertex graph is clearly 3-connected and not bipartite, yet it has real chromatic roots at

$$1.90263\ldots \qquad 2.42196\ldots$$

and so Jackson's conjecture is false.

# A FAMILY OF GRAPHS




The graph $X(s,t)$ has $s$ vertices in $S$ and $t$ vertices in $T$.

# Main result

**Theorem (Royle 2007)**

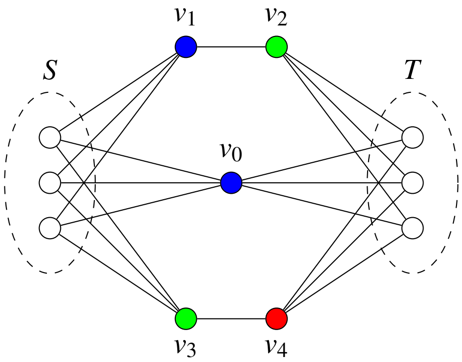
The graph $X(s,t)$ is 3-connected, not bipartite and has a chromatic root in $(1,2)$ whenever $s \geq 3$ and $t \geq 3$ are both odd.

Computation of the chromatic polynomial and calculation of roots is feasible for a particular $s$, $t$ but impossible in general, so we have to work round this problem.

Main aim of proof is to show that the *derivative*

$$P'_{X(s,t)}(2) < 0.$$

# Add up contributions




This “type” of colouring contributes

$$x(x-1)(x-2)(x-2)^s(x-3)^t$$

# Putting it together

There are 27 types of colouring, but any that use *exactly two* colours on $\{v_0, v_1, v_3\}$ and $\{v_0, v_2, v_4\}$ create a factor of $(x-2)^s$ or $(x-2)^t$, which contributes nothing to the *derivative*.

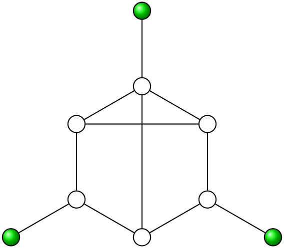
So just need to consider the ones that use 1 or 3 colours on $\{v_0, v_1, v_3\}$ and $\{v_0, v_2, v_4\}$, differentiate the terms, substitute $x = 2$ with the result that

$$P'_{X(s,t)}(2) = 2\left((-1)^s + (-1)^t + (-1)^{s+t}\right)$$

When $s$ and $t$ are both odd then $P'_{X(s,t)}(2) = -2$. □

# Many more examples

This construction can be varied in numerous ways, e.g. completely join any odd set of $s \geq 3$ vertices to the green vertices to produce another graph with a chromatic root in $(1, 2)$.



Thousands of 11–15 vertex examples, but all "the same".

# A revised conjecture

A graph is $\alpha$*-tough* if we cannot find $s$ independent vertices whose removal leaves more than $s$ connected components.

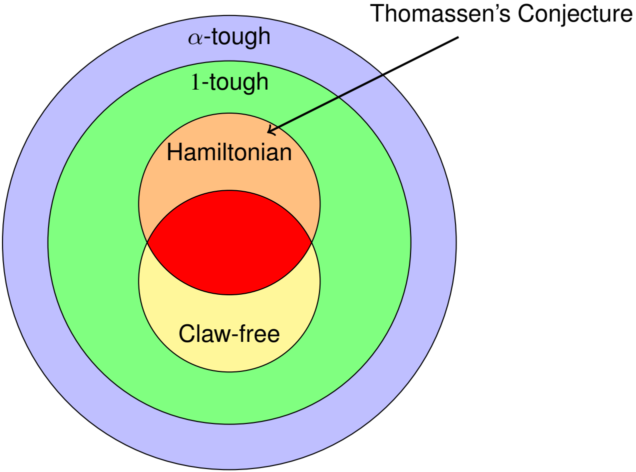
*All* of the thousands of examples have $3$-sets whose removal leaves more than 3 components.

Dong & Koh have produced a large family $\Gamma$ of graphs that can be shown *not* to have chromatic roots in $(1, 2)$ — all of these graphs are $\alpha$-tough.

**Conjecture (Dong & Koh)**

An $\alpha$-tough graph has no chromatic roots in $(1, 2)$.

# The big picture




To Beraha's Conjecture

# Beraha numbers

Sami Beraha was one of the pioneers of the study of chromatic roots, and the Beraha, Kahane & Weiss theorem on limiting curves of complex roots of recursively defined families of polynomials is still a fundamental tool.

When studying real chromatic roots, he first noted the role played by the *Beraha numbers* $B_n$ where

$$B_n = 2 + 2\cos\left(\frac{2\pi}{n}\right).$$

These form an increasing sequence converging to 4 incorporating various important points such as $B_5 = \tau + 1$ where $\tau = (1+\sqrt{5})/2$ is the Golden Ratio.

# Beraha's conjecture

Beraha conjectured that each Beraha number is a limit point of real chromatic roots of plane triangulations.

In particular, there should be planar triangulations with chromatic roots arbitrarily close to 4.

He proved that increasingly long width-4 strips of the triangular lattice with periodic boundary conditions have *complex* chromatic roots arbitrarily close to 4.

## Mini-Problem

No non-integer Beraha number can actually *be* a chromatic root, except possibly $B_{10}$! Resolve this.

# Upper root-free intervals

An *upper root-free interval* for a family $\mathcal{F}$ of graphs is a chromatic-root-free interval of the form

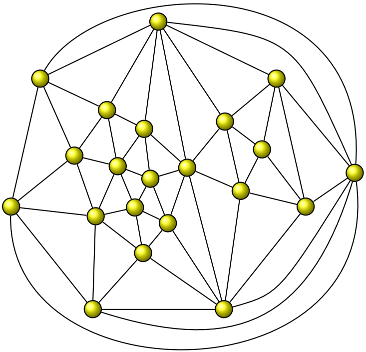
$$(r, \infty).$$

Any minor-closed class of graphs has an upper root-free interval, and the most important unresolved class is that of planar graphs.

## Question

What is the upper root-free interval for planar graphs?

# Woodall's record holder at 3.826785..



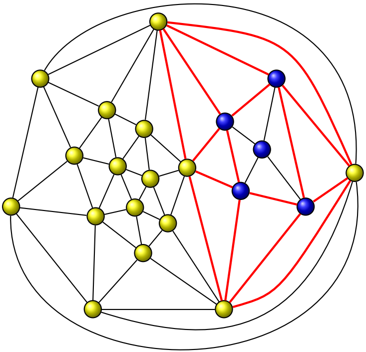
Why is *this* graph special?

# Initial strategy

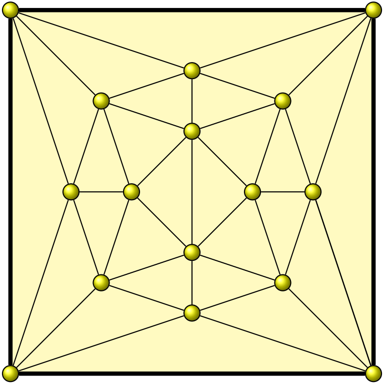
*And he puzzled and puzzled 'till his puzzler was sore. Then the Grinch thought of something he hadn't before.*

*"The Grinch Who Stole Christmas" by Dr Seuss*

# Decompose it



# Redraw the graph



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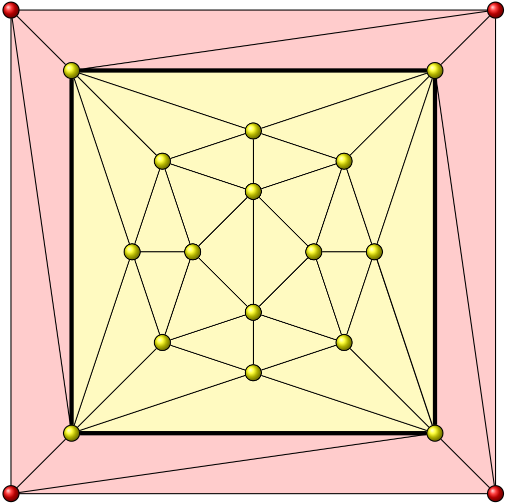

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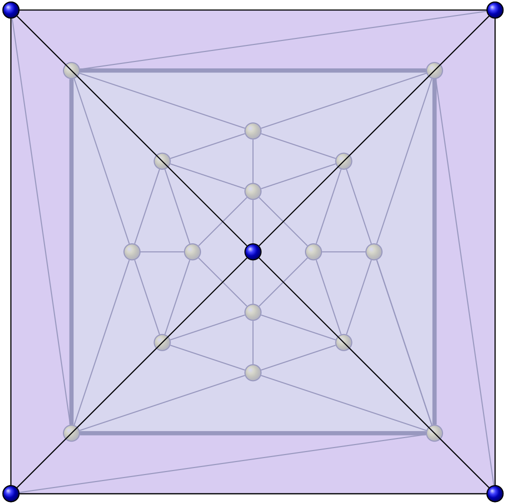
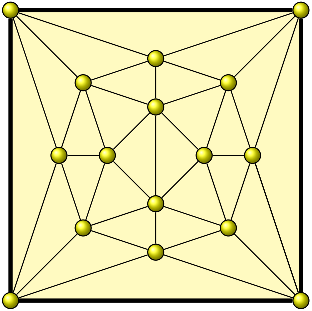


# Add a layer



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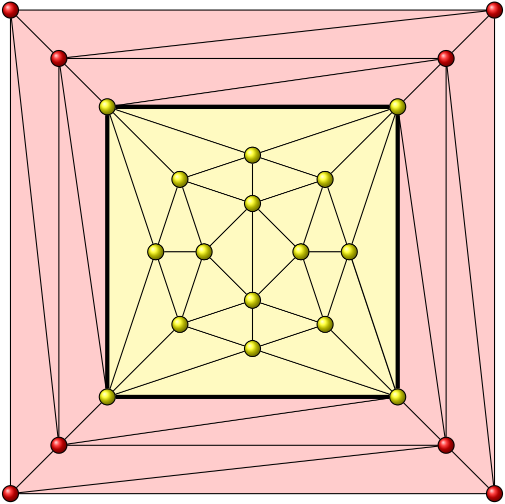

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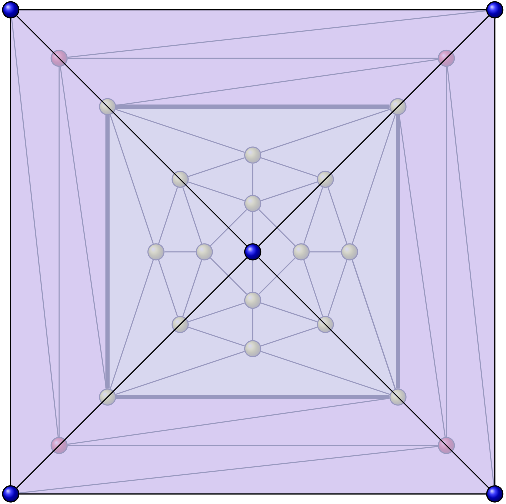

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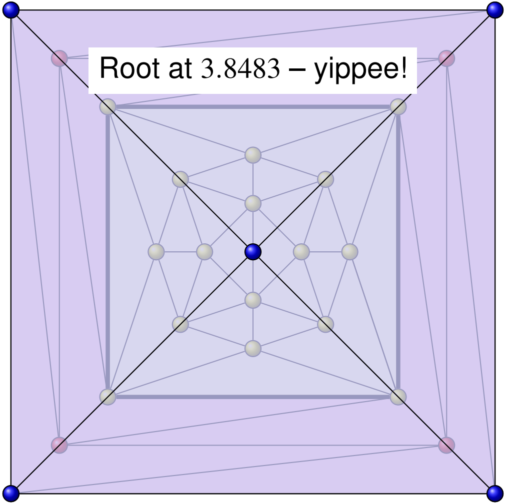
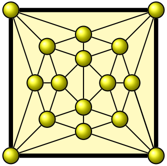


# Add *lots* of layers



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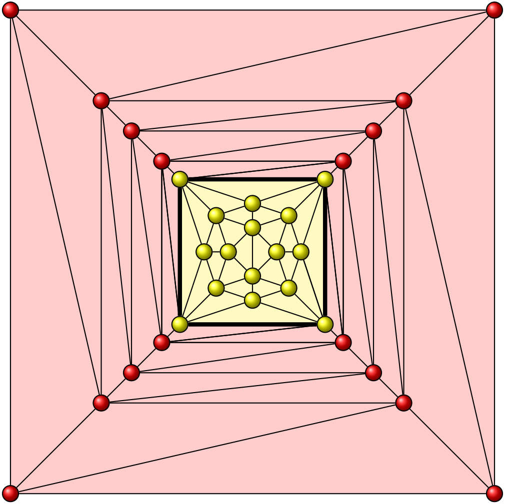

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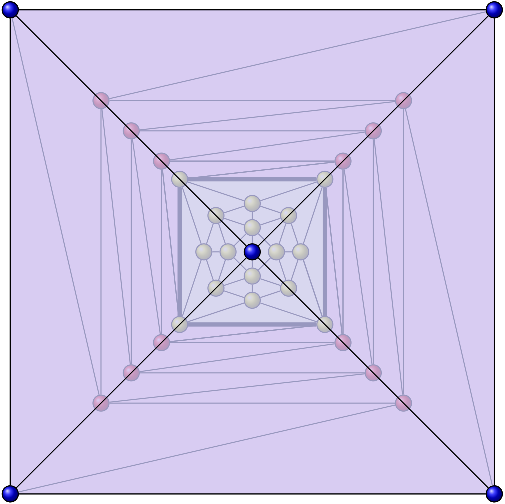

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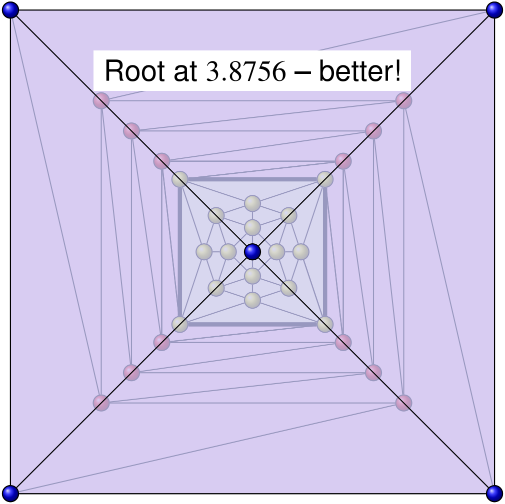


# The candidate family

Now we have a clear idea for a family of graphs — keep adding layers of lattice and see where the real roots go.

Let $X_n$ denote the graph obtained by taking the periodic triangular lattice of width 4 and height $n$ and gluing $W_4$ into the top face and $H$ into the bottom.

Therefore we now have two tasks:

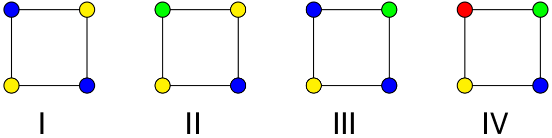
- Calculate the chromatic polynomial of $X_n$.
- Determine the behaviour of its real roots.

# Gluing at a square

Suppose $A$ and $B$ both have induced (ordered) 4-cycles, say $a_1a_2a_3a_4$ and $b_1b_2b_3b_4$.

What is the chromatic polynomial of the graph obtained by *gluing together $A$ and $B$ at the square* — that is, identifying $a_i$ with $b_i$?

We need to know the *number* of colourings of $A$ and $B$ that induce specific partitions on the distinguished 4-cycles.



# Partitioned Chromatic Polynomial

Express

$$P_A(x) = P_1(A,x) + P_2(A,x) + P_3(A,x) + P_4(A,x).$$

where $P_i(A,x)$ counts the colourings that induce partitions of Type $i$.

Then define the *partitioned chromatic polynomial* to be the vector

$$Q(A,x) = \begin{pmatrix} P_1(A,x) \\ P_2(A,x) \\ P_3(A,x) \\ P_4(A,x) \end{pmatrix}.$$

# Chromatic polynomial of a gluing

Then the chromatic polynomial of the gluing of $A$ and $B$ is the single entry of

$$Q(A)^T D Q(B)$$

where

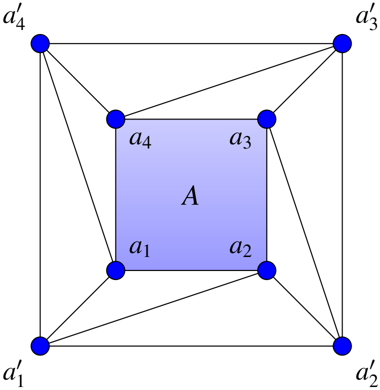
$$D = \begin{pmatrix} 1/\langle x\rangle_2 & 0 & 0 & 0 \\ 0 & 1/\langle x\rangle_3 & 0 & 0 \\ 0 & 0 & 1/\langle x\rangle_3 & 0 \\ 0 & 0 & 0 & 1/\langle x\rangle_4 \end{pmatrix},$$

and $\langle x\rangle_k$ denotes the $k$'th *falling factorial*

$$x(x-1)\cdots(x-k+1).$$

# Adding a layer

If $A$ is a graph with distinguished 4-cycle $a_1a_2a_3a_4$ then *adding a layer* of periodic triangular lattice creates a graph $A'$ with a new distinguished 4-cycle.



# THE TRANSFER MATRIX

Let $M$ be the $4 \times 4$ matrix with rows and columns indexed by the types, and where $M_{ij}$ is the polynomial counting the number of colourings of a $4_P \times 2_F$ triangular lattice strip that induce Type $i$ on the inner cycle, and Type $j$ on the outer cycle.

This is called the *transfer matrix* because it encodes information about how colourings of the distinguished 4-cycle are transferred to the new distinguished cycle.

If $A'$ is the graph obtained by adding a layer to $A$, then

$$Q(A') = MDQ(A).$$

# The actual matrix

$$\begin{pmatrix} \langle x\rangle_4 & \langle x\rangle_5 & \langle x\rangle_5 & \langle x\rangle_6 \\ \langle x\rangle_5 & \langle x\rangle_4 + 2\langle x\rangle_5 + \langle x\rangle_6 & \langle x\rangle_4 + 2\langle x\rangle_5 + \langle x\rangle_6 & 4\langle x\rangle_5 + 4\langle x\rangle_6 + \langle x\rangle_7 \\ \langle x\rangle_5 & \langle x\rangle_4 + 2\langle x\rangle_5 + \langle x\rangle_6 & \langle x\rangle_4 + 2\langle x\rangle_5 + \langle x\rangle_6 & 4\langle x\rangle_5 + 4\langle x\rangle_6 + \langle x\rangle_7 \\ \langle x\rangle_6 & 4\langle x\rangle_5 + 4\langle x\rangle_6 + \langle x\rangle_7 & 4\langle x\rangle_5 + 4\langle x\rangle_6 + \langle x\rangle_7 & M_{44} \end{pmatrix}$$

where

$$M_{44} = 2\langle x\rangle_4 + 16\langle x\rangle_5 + 20\langle x\rangle_6 + 8\langle x\rangle_7 + \langle x\rangle_8.$$

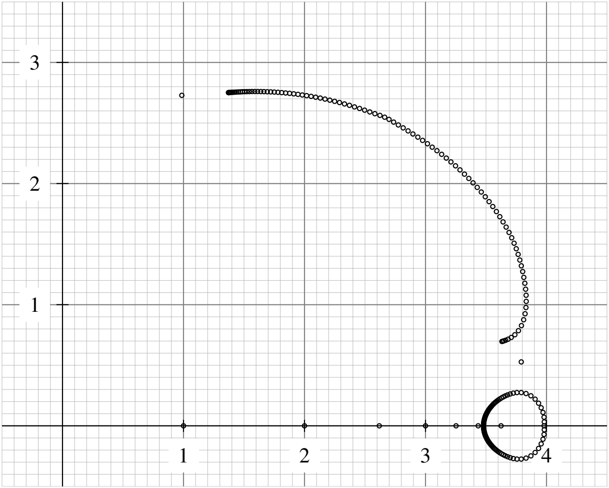
# Putting it together

Let $A$ and $B$ be graphs with distinguished 4-cycles. Then the graph obtained by inserting $A$ into the top face and $B$ into the bottom face of a $4_P \times n_F$ triangular lattice strip has chromatic polynomial

$$Q(A)^T D(MD)^{n-1} Q(B).$$

Thus we can symbolically (i.e. with Maple) compute the chromatic polynomial of these double-ended lattice graphs of any reasonable length and with any particular end-graphs.

# The roots for $X_{100}$



# The real roots

| $n$ | Value |
|---|---|
| 2 | 3.848343257 |
| 4 | 3.875604098 |
| 8 | 3.905148525 |
| 16 | 3.932717391 |
| 32 | 3.955237394 |
| 64 | 3.971732100 |
| 128 | 3.982848013 |
| 256 | 3.989898687 |
| 512 | 3.994181944 |

*Surely* these chromatic roots are tending to 4.

# RIGOROUS PROOF

The proof can be made rigorous by observing that in the limit, it is the *spectral properties* of $MD$ that determine the ultimate behaviour of its powers.

Therefore we will fix $x = 4 - \epsilon$ and determine the eigenvalues and eigenvectors of $MD$. The eigenvectors are given by the following expressions:

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 2 - 5\epsilon + 10\epsilon^2/3 + O(\epsilon^3) \\
\lambda_3 &= 2 - 8\epsilon + 26\epsilon^2/3 + O(\epsilon^3) \\
\lambda_4 &= 0
\end{aligned}$$

# Eigenvectors

The corresponding eigenvectors are $v_1 = (1, -1, -1, 1)^T$,

$$v_2 = \begin{pmatrix} 3/2 + 35\epsilon/12 + 1103\epsilon^2/216 + O(\epsilon^3) \\ 1 + 5\epsilon/3 + 85\epsilon^2/27 + O(\epsilon^3) \\ 1 + 5\epsilon/3 + 85\epsilon^2/27 + O(\epsilon^3) \\ -1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} -\epsilon/3 - 14\epsilon^2/27 + O(\epsilon^3) \\ 1/2 + \epsilon/6 + 4\epsilon^2/27 + O(\epsilon^3) \\ 1/2 + \epsilon/6 + 4\epsilon^2/27 + O(\epsilon^3) \\ 1 \end{pmatrix}$$

and $v_4 = (0, 1, -1, 0)^T$.

Moreover these are “orthogonal” in that

$$v_i^T D v_j = 0 \qquad i \neq j.$$

# THE END-GRAPHS DETERMINE WHAT HAPPENS

Now the limiting behaviour of the chromatic polynomial depends only on the coordinates of $Q(A)$ and $Q(B)$ with respect to the basis $\{v_1, v_2, v_3, v_4\}$. If

$$Q(A) = \alpha_1 v_1 + \ldots + \alpha_4 v_4$$

$$Q(B) = \beta_1 v_1 + \ldots + \beta_4 v_4$$

then the value of $Q(A)^T D(MD)^{n-1} Q(B)$ is given by

$$\sum_{i=1}^{i=4} \alpha_i \beta_i \lambda_i^{n-1} \|v_i\|^2$$

This is dominated by the largest eigenvalue for which $\alpha_i$ and $\beta_i$ are both non-zero.

# The Woodall-based family

For the two graphs $H$ and $W_4$ found as the end-graphs of Woodall's example we have

$$\begin{aligned}
\alpha_1 \|v_1\|^2 &= 0, \\
\beta_1 \|v_1\|^2 &= 0, \\
\alpha_2 \|v_2\|^2 &= -50\epsilon + O(\epsilon^2), \\
\beta_2 \|v_2\|^2 &= 5 + 20\epsilon/3 + O(\epsilon^2).
\end{aligned}$$

and hence $\lambda_2$ dominates. Most importantly

$$\alpha_2 \beta_2 = -250\epsilon + O(\epsilon^2)$$

so the chromatic polynomial is *negative* at $4 - \epsilon$.

# Many other pairs

It can be shown that if $A$ is planar with a distinguished 4-cycle, then $Q(A)$ never has any $v_1$-component.

Call such a graph *positive* if the series expansion at $x = 4 - \epsilon$ of its $v_2$-component has a positive leading term and *negative* if it has a negative leading term.

**Theorem (Royle 2006)**

If $A$ and $B$ have different signs, then the double-ended lattice graph on a periodic triangular lattice of width 4 with ends $A$ and $B$ has real chromatic roots tending to 4 as its length tends to $\infty$.

Therefore the upper root-free interval for planar graphs is *at most* $[4, \infty)$.

# Comments

- The smallest planar *negative* graph has 10 vertices.
- This general family of graphs (double-ended lattice graphs) had previously been studied by the statistical physicists Roček, Shrock & Tsai.
- This result now eliminates almost any (faint) hope of an analytic proof of the 4-colour theorem.

**Conjecture[2] (Salas & Sokal)**

No *bipartite* planar graph has a chromatic root larger than $1+\tau$ where $\tau$ is the golden ratio.

To Jackson's Conjecture

[2] Added in proof: This conjecture was withdrawn at the CSM meeting — the true value is likely to be 3