The number of k-edge matchings in regular bipartite graphs

joint work with

S. Friedland } U. Illinois in Chicago
E. Krop }

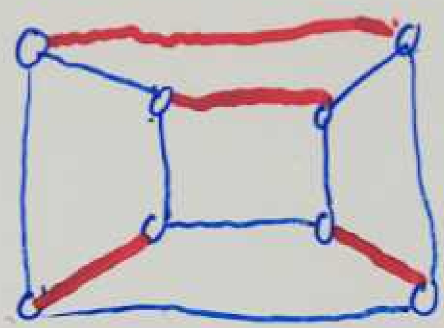
P.H. Lundow KTH - stockholm

Klas Markström
umeå, Sweden

A matching $M \subset E(G)$ is a set of disjoint edges

In physics this is called a monomer-dimer covering

If G has $2n$ vertices we call a matching with n edges a perfect matching



If M has k edges we call it a k-matching

Let $b_k(G)$ denote the number of k-edge matchings in G

$$\phi(G,x) = \sum_k b_k(G)\, x^k$$

The matching generating function

<u>Heilman and Lieb</u>

The zeros of $\phi(G,x)$ are real and non-positive

<u>Cor</u>

$$\left(b_k(G)\right)^2 \geq b_{k-1}(G)\, b_{k+1}(G)$$

The permanent

$$\mathrm{perm}(A) = \sum_{\sigma} \prod_{i=1}^{n} A_{i\sigma(i)}$$

Perm(A) counts the number of weighted perfect matchings in G.

The k-subpermanent sum

$$\mathrm{perm}_k(A) = \sum_{\substack{B \subset A \\ k \times k}} \mathrm{perm}(B)$$

$\mathrm{perm}_k(A)$ counts the number of k-edge matchings

---

perm(A) is #P-hard to compute

Ryser : $n 2^n$

Given an r-regular bipartite on $2n$ vertices what can we say about the number of matchings?

<u>Minc-Bregman</u>

$$\operatorname{perm}(A) \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}$$

$$r_i = \sum_{j=1}^{n} a_{ij}$$

<u>Cor</u>

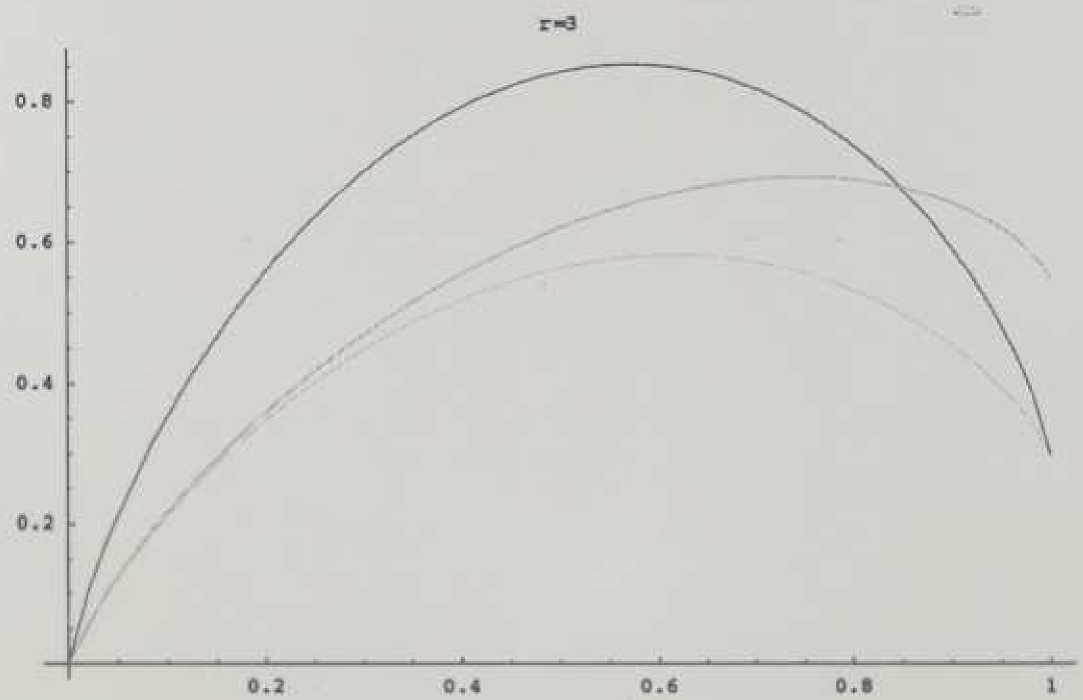
$$b_n(G) \leq (r!)^{\frac{n}{r}}$$

When $r|n$ equality holds for $G = \bigcup K_{r,r}$

<u>Conjecture 1</u>

$$b_k(G) \leq b_k\left(\tfrac{n}{2r} K_{r,r}\right)$$

<u>True</u> for $r=2$.


r=3
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1

Lower bounds?

Van der Waerden's conjecture

Falikman 81
Egorichev

$$b_n(G) \geq \left(\frac{r}{e}\right)^n$$

Schrijver

$$b_n(G) \geq \left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$$

The Tverberg conjecture (Friedland 82)

$$b_k(G) \geq \binom{n}{k}^2 \frac{k!\, r^k}{n^k}$$

Conjecture 2

$$b_k(G) \geq \left(1+\frac{1}{rn}\right)^{rn-1}\left(1-\frac{k}{rn}\right)^{rn-m}\left(\frac{kr}{n}\right)^k\binom{n}{k}^2$$

Conjecture 3

$p \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{2n}\log\left(b_{p\cdot n}(G)\right) \leq$$

$$\frac{1}{2}\left(p\log r - p\log p - 2(1-p)\log(1-p) + (r-p)\log\left(1-\frac{p}{r}\right)\right)$$

We look at $k=4$

$$b_4(G) = p_4(n,r) + \underset{\substack{\uparrow \\ \# \, C_4 \text{ in } G}}{a_4(G)}$$

$$0 \le a_4(G) \le a_4(K_{r,r}) \cdot \frac{n}{r}$$

For a random r-regular graph

a_4 converges to

$$Po\left(\frac{(r-1)^4}{4}\right)$$

Conjecture 3 corresponds to the expected number of matchings in a random regular graph

Def

$p(\bar{x}) = p(x_1, x_2, \dots x_n)$

a polynomial with real coefficients is positive hyperbolic if

1. p is homogeneous with degree $n \geq 0$
2. $p(\bar{x}) > 0$ when $\bar{x} > 0$
3. $p(\bar{x} + t \cdot \bar{u})$

   has n real roots in t for all $\bar{u} > 0$ and $x \in \mathbb{R}^n$

$$\mathrm{Cap}(p) = \inf_{\substack{\bar{x} > 0 \\ x_1 x_2 \dots x_n = 1}} p(\bar{x})$$

Gurvits

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, 0 \dots 0) \geq \frac{n!}{n^n} \mathrm{Cap}(p)$$

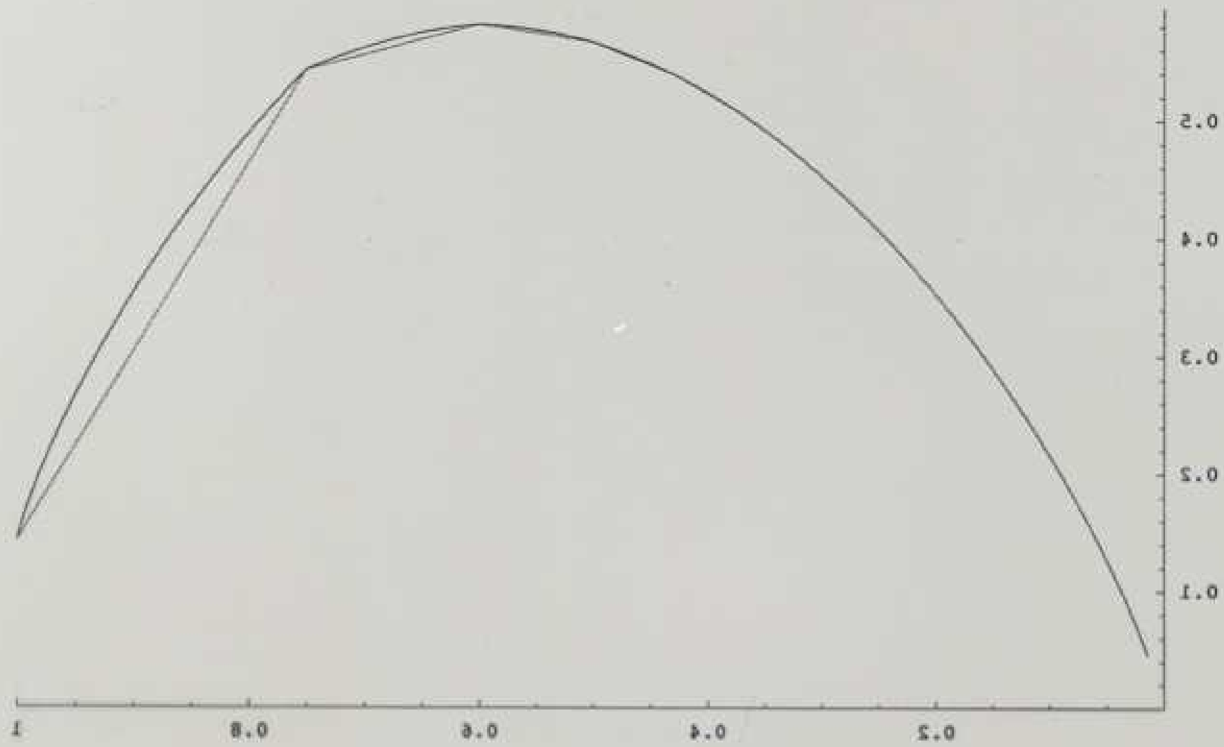
Using the new bounds for stable hyperbolic polynomials Gurvits improved the lower bounds for perm(A)

Gurvits + Friedland proved

thm

the asymptotic lower bound conjecture holds for $p = \frac{r}{r+s}$ $s = 0, 1, 2, \ldots$

using the convexity of $b_n(6)$ we get a bound for all p



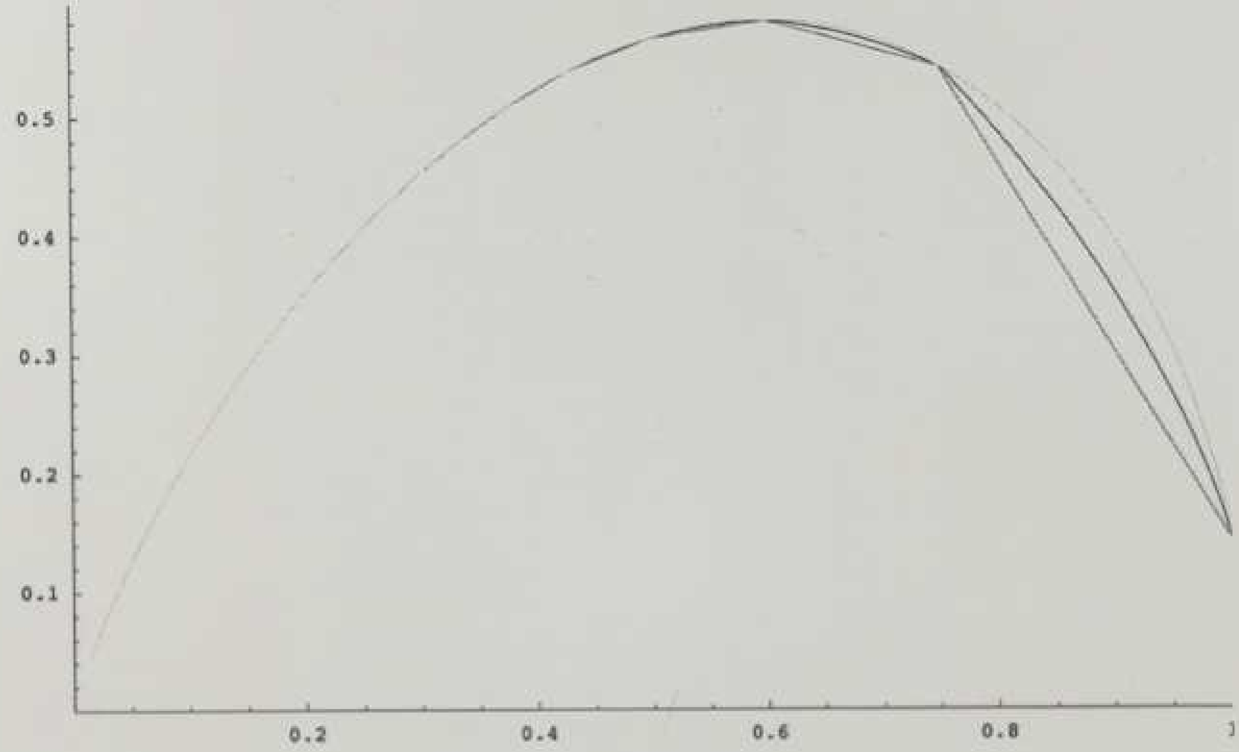
0.5
0.4
0.3
0.2
0.1
1
0.8
0.6
0.4
0.2

Newton

If all zeros of $c_0 + c_1 x + \ldots c_n x^n$ are real ~~and~~ ~~[illegible]~~ then

$$\left(\frac{c_i}{\binom{n}{i}}\right)^2 \geq \left(\frac{c_{i-1}}{\binom{n}{i-1}}\right)\left(\frac{c_{i+1}}{\binom{n}{i+1}}\right)$$

Using this we get an improved bound


0.5
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1