# On graphs having no chromatic zero in (1,2)

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# 1912 ---

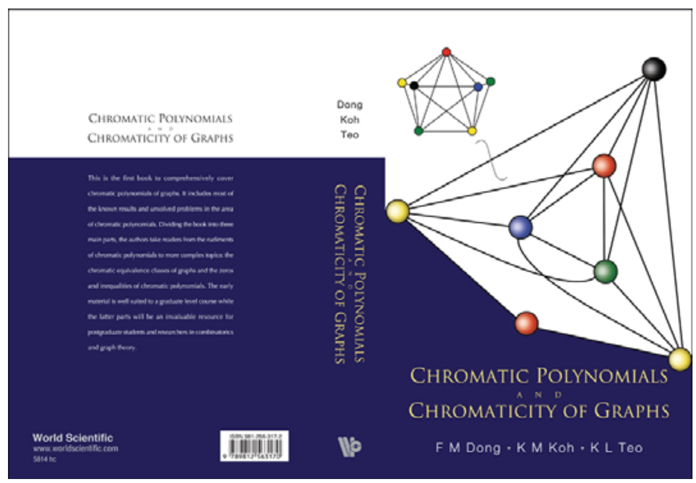
The chromatic polynomial was introduced by Birkhoff in 1912 as a way to attack the four-colour problem.



# Some References

- R.C. Read, An introduction to chromatic polynomials, *J. Combinatorial Theory* 4 (1968), 52-71.
- R.C. Read and W.T.Tutte, Chromatic polynomials, in: *Selected Topics in Graph Theory III* (eds. L.W.Beineke and R.J.Wilson), Academic Press, New York (1988), 15-42.






# Why polynomial

- For any graph G of order n,

$$P(G,\lambda)=\sum_{k=1}^{n}\alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1),$$

where $\alpha(G,k)$ is the number of partitions of V(G) into exactly k non-empty independent sets of G.



# Special case of Potts-model

- For any graph G =(V,E),

$$P(G,\lambda)=Z_G^{Potts}(\lambda,-1)=\sum_{A\subseteq E}(-1)^{|A|}\lambda^{c(A)},$$

where c(A) is the number of components of the spanning subgraph (V, A).

## Chromatic zero

- Let G be a graph.
- If $\lambda = \lambda_0$ is a root of the equation $P(G,\lambda)=0$, then $\lambda_0$ is called a chromatic zero of G.



## Family F

- Let F be the family of graphs which have no chromatic zero in (1, 2).
- (B. Jackson)
  every graph has no chromatic zero in (1,32/27].
- So F is the family of graphs which have no chromatic zero in (32/27, 2).



## Problems

- P1:
  For a given graph G, does G belong to F?
- P2:
  Can graphs in F be characterized by their structures?



## Conjecture 1

- By Jackson in 1993:
- If *G* is a 3-connected and non-bipartite graph, then *G* belongs to F.



## Conjecture 2

- By Thomassen in 1996:
- If *G* is a Hamiltonian graph, then *G* belongs to F.



## Motivation

- These problems are interesting and worth to study.
- to find useful tools to attack the following very famous conjectures:

(a) $P(G,\lambda) > 0 \ \forall \ \lambda \in (4,5)$ & planar graph G.
(b) $F(G,\lambda) > 0 \ \forall \ \lambda \in (4, \infty)$ & graph G.

# Basic properties

- If $G_1, G_2, \ldots, G_k$ are the components of G, then

$$P(G,\lambda) = P(G_1,\lambda)P(G_2,\lambda)\cdots P(G_k,\lambda)$$

- If $G_1, G_2, \ldots, G_k$ are the blocks of a connected graph G, then

$$P(G,\lambda) = \frac{1}{\lambda^{k-1}} P(G_1,\lambda)P(G_2,\lambda)\cdots P(G_k,\lambda)$$



# Non-separable graphs in F

A graph G belongs to F

⇓⇑

all blocks of all components of G belong to F.

To determine F, it suffices to determine non-separable graphs in F.



# Non-separable graphs in F of Small orders

All graphs of order at most 5 belong to F, except $K_{2,3}$. (Draw graphs of order 2-4)

$K_{2,3}$:

$$P(K_{2,3},\lambda) = \lambda(\lambda\text{-}1)(\lambda^3\text{-}5\lambda^2\text{+}10\lambda\text{-}7)$$

Real zeros in (1,2): 1.43015…



# Well-known results

- The following graphs belong to F.

(i) (Trivial) Chordal graphs

(ii) (Birkhoff and Lewis 1946)

Plane Near-triangulations.

(iii) (Woodall) Outplanar graphs.



# How do I do?

- First I need to establish a fundamental result.
- Then apply this result to show a given set of non-separable graphs is a subset of F.

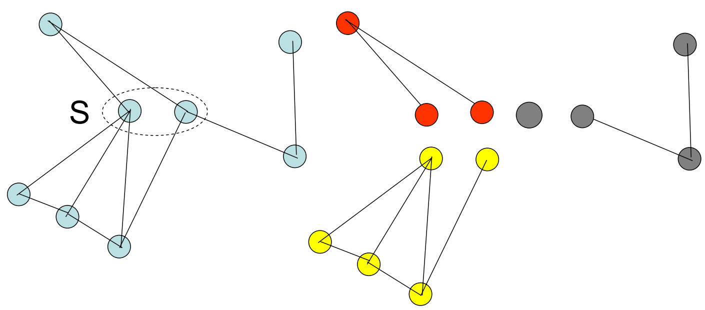


# S-bridges

- Let $G$ be a connected graph and $S \subseteq V(G)$.
- Let $V_1, V_2, \ldots, V_k$ be the vertex sets of the components of $G$-$S$.
- The subgraph of G induced by $S \cup V_i$ is called a $S$-bridge of G for all $i = 1,2,\ldots, k$.
- Thus, if G-S has exactly $k$ components, then G has exactly $k$ S-bridges.

# Example

- There are 3 S-bridges in the following graph:






# Theorem A

- A 2-connected graph G belongs to F if one of the following conditions holds:

(a) G has an edge uv such that both G-uv and G/uv are non-separable graphs in F;

(b) G has a complete cut-set S such that all S-bridges belong to F;

(show proof)



# Theorem A

- A 2-connected graph G belongs to F if one of the following conditions holds:

(c) G has a cut-set $S=\{x,y\}$ with $xy \notin E(G)$ such that c(G-S) is even and all S-bridges of G+xy and all blocks of $G \cdot xy$ belong to F.



# Theorem A

A 2-connected graph G belongs to F if

(d) (new) G has a cut-set $S=\{x,y\}$ with $xy \notin E(G)$ such that

(i) c(G-S) is odd;

(ii) all S-bridges of G+xy and all blocks of $G \cdot xy$ belong to F;

(iii) for every S-bridge $G_i$ of G, $b(G_i)$ is odd and all blocks of $G_i$ belong to F.

(see example in Maple program)



# Theorem B

- Let U be any family consisting of $K_2$, $K_3$ and some 2-connected graphs.

- Then $U \subseteq F$ if one of the following conditions holds for every $G \in U$ with $v(G) \geqslant 4$:

(a)G has an edge uv such that both G-uv and G/uv belong to U;



# Theorem B

(b) G has a complete cut-set S such that all S-bridges belong to U;

(C) G has a cut-set $S=\{x,y\}$ with $xy \notin E(G)$ such that c(G-S) is even, all S-bridges of G+xy and all blocks of $G \cdot xy$ belong to U.

## Thomassen's conjecture

- By Theorem B, taking U to be the family of $K_2$ and Hamiltonian graphs, Thomassen's conjecture holds if the following conjecture (also by Thomassen) is true:

- Conjecture:

  Any 3-connected Hamiltonial graph G contains an edge uv such that both G-uv and G · uv are Hamiltonial graphs.



## A recent result

- Theorem (Bielak 2006)

Let G be a graph of minimum degree at least 3 such that G has a Hamiltonian cycle containing at most one edge of every claw in G.

Then G contains an edge e such that both G-e and G/e are Hamiltonian.



## Corollary

Every Hamiltonian claw-free graph G of minimum degree at least 3 contains an edge e such that both G-e and G/e are Hamiltonian.



## An ordering

- Let G be a graph G of order $n$, $n \geq 2$, and $x_1, x_2, \ldots, x_n$ a vertex ordering of G.

- For every $i$: $2 \leq i \leq n$, let

$$V_i = N(x_i) \cap \{x_j : j = 1, 2, \ldots, i-1\}.$$



## $\gamma$ -ordering

- $x_1, x_2, \ldots, x_n$ is called a $\gamma$-ordering of G if $x_1 x_2 \in E(G)$ and for every non-empty $I \subseteq \{3, \ldots, n\}$, either

  (a) $V_i$ is not independent for some $i \in I$, or

  (b)
$$\left| \bigcup_{i \in I} V_i \right| \geq |I| + 1 .$$

- (show example)



- If G is of order $n$, $n \geq 3$, and $x_1, x_2, \ldots, x_n$ is a $\gamma$-ordering of G, then $x_1, x_2, \ldots, x_{n-1}$ is a $\gamma$ -ordering of G-$x_n$.

# Family Γ

- Let Γ be the family of graphs which a contain γ -ordering.



# Properties of Family Γ

- For any G in Γ, if G is not $K_2$, then
  - (a) G is 2-connected;
  - (b) G contains triangles;
  - (c) G has a vertex $x$ such that $G\text{-}x \in \Gamma$;
  - (d) $G + xy$ is in Γ for any non-adjacent vertices x,y.

(It is not true in general that for any $H \in F$ and $e \notin E(H) \Rightarrow H+e \in F$)



# Minimal graphs in Γ

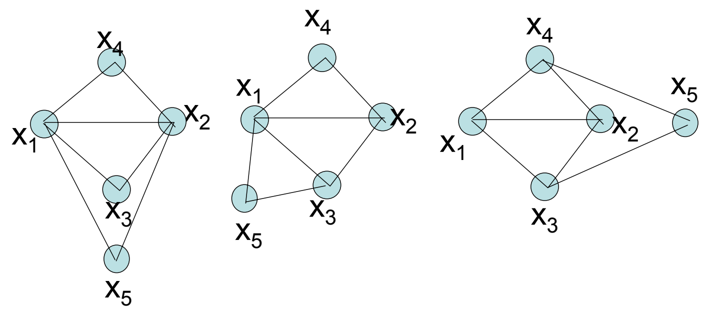
- A minimal graph G in Γ means that removing any edge in G produces a graph not in Γ.

- Order 2: $K_2$;
- Order 3: $K_3$;
- Order 4: $K_4$-e.



# Small minimal graphs in Γ

Order 5:






# Produce graphs in Γ Recursively

- Every graph G, except $K_2$, in Γ can be produced from a graph H in Γ by adding a new vertex $w$ and some edges joining $w$ to some properly selected vertices in H.



# Theorem C

- Dong and Koh in 2006:

$$\Gamma \subseteq F,$$

i.e.,

every graph in Γ contains no zeros in (1,2).

## Corollary 1

- Thomassen showed that graphs with a Hamiltonian path have no chromatic zero in (1, 1.29559…), but may have in (1.29559…, 2).
- By Theorem C,

  If G has a Hamiltonian path $x_1\, x_2 \ldots x_n$ such that $|N(x_i)\cap\{x_j: j=1,2,\ldots,i-1\}| \geq 2$ holds for all $i$: $3 \leq i \leq n$, then G is a graph in Γ and thus it contains no zeros in (1,2).



## Corollary 2

- If G has a vertex ordering $x_1, x_2, \ldots, x_n$ such that $N(x_i)\cap\{x_j: j=1,2,\ldots,i-1\}$ is not independent in G for all $i$: $3 \leq i \leq n$, then G is a graph in Γ and thus it contains no zeros in (1,2).
- For example,

(a) graphs containing a 2-tree as spanning subgraph;

(b) 2-connected plane near-triangulations;

(c) complete t-partite graph for $t \geq 3$.



## Corollary 3

- All (n-Δ+1)-connected graph are contained in Γ and thus it contains no zeros in (1,2), where n is the order of G and Δ is the maximal degree of G.
- If $\Delta \leq n-2$, then "n-Δ+1" cannot be replaced by "n-Δ".
- For example, take $K_{2,3}$.



## Corollary 4

- Suppose that (A, B) is the bipartition of the complete bipartite graph K(p,q) with |A|=p and |B|=q.
- Let G be the graph obtained from K(p,q) by adding an edge joining any two vertices in B.
- If $2 \leq q \leq p+1$, then G is a graph in Γ and thus it contains no zeros in (1,2).



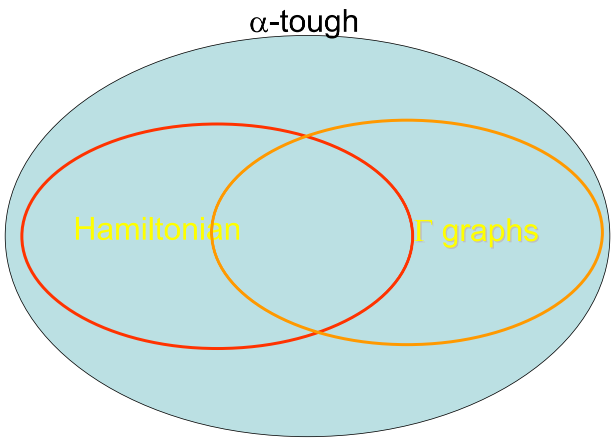
## α-tough graphs

- A graph *G* is said to be tough if G-S contains at most |S| components for every non-empty subset S of V(G).
- A graph *G* is said to be α-tough if G-S contains at most |S| components for every non-empty independent set S in G.
- A tough graph must be 2-connected if its order is at least 3.
- It was shown that all graphs in Γ are α-tough.



## Another conjecture

- Conjecture (Dong and Koh, 2006)
  Every α-tough graph contains no zero in (1,2).
- Hamiltonian cycle → α-tough.
- This conjecture has a weaker condition than Thomassen's conjecture.
- As I know, this conjecture is not proven yet.






## Special case

- Is Dong and Koh's conjecture holds for planar graphs?



## An old result

- Let G be a 2-connected simple plane graph and $\rho(G)$ the number of faces which are not bounded by $C_3$.
- Plane near-triangulations: $\rho(G) \le 1$.
- Birkhoff and Lewis

Every plane near-triangulation belongs to F.



## Extension

- Let G be a 2-connected plane graph.
- Dong and Koh

(1) $\rho(G) \le 2 \Rightarrow G \in F$;

(2) $3 \le \rho(G) \le 4$ & G is $\alpha$-tough $\Rightarrow G \in F$.



## Minor-closed class

- Let $\Omega$ be a set of graphs.
- $\Omega$ is called a minor-closed class if all minors of G also belong to $\Omega$ for every $G \in \Omega$.



## Minor-closed class

Theorem D

If $\Omega$ is a minor-closed class, then

$$\Omega \subseteq F \quad \Leftrightarrow \quad K_{2,3} \notin \Omega.$$

Theorem E:

$K_{2,3}$ is not a minor of G $\Rightarrow$ $G \in F$.

# Minor-closed class

Theorem E ⇒ Theorem D

$K_{2,3} \notin \Omega$ and $\Omega$ is minor-closed

⇒ $\Omega \subseteq$ {G: $K_{2,3}$ is not a minor of G} $\subseteq$ F.

(go to multiplicity?)



# Proof of Theorem E

- Theorem E:
  $K_{2,3}$ is not a minor of G ⇒ G $\in$ F
- Let U be the set of non-separable G such that $K_{2,3}$ is not a minor of G.

Apply Theorem B to prove Theorem E.



# Theorem B

- Let U be any family consisting of $K_2$, $K_3$ and some 2-connected graphs.
- Then U$\subseteq$F if one of the following conditions holds for every G $\in$ U with v(G) $\geqslant$ 4:

(a)G has an edge uv such that both G-uv and G/uv belong to U;

(b)G has a complete cut-set S such that all S-bridges belong to U;

(c)G has a cut-set S={x,y} with xy$\notin$ E(G) such that c(G-S) is even and all S-bridges of G+xy and G ·xy belong to U.



# Proof

Let G $\in$ U with v(G) $\geqslant$ 4. So G is 2-connected.

(a) If G is 3-connected, then both G-e and G/e belong to U for any edge e in G, as $K_{2,3}$ is not a minor of G.



# Proof

(b) Now assume that G is 2-connected but not 3-connected.

Let {x,y} be any vertex-cut of G.

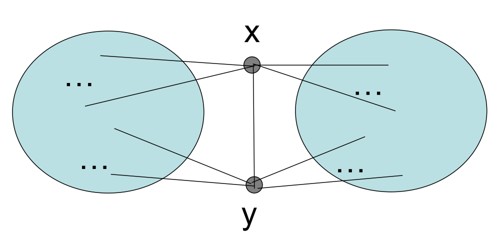
So c(G-{x,y}) ≥ 2.

Since $K_{2,3}$ is not a minor of G, we have c(G-{x,y}) ≤ 2.

Thus c(G-{x,y}) = 2.

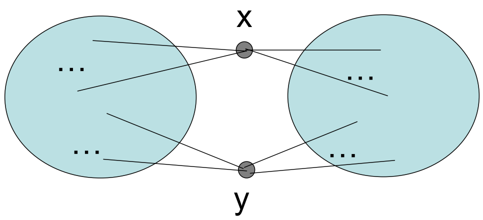


# Case 1: x, y are adjacent




Each {x,y}-bridge is a minor of G and non-separable and thus belongs to U.

# Case 2: x, y are not adjacent




All {x,y}-bridges of G+xy and all blocks of G•xy are minors of G and non-separable.

Thus they belong to U.



# Proof

Hence, by Theorem B, $U \subseteq F$.



# Multiplicity of chromatic zeros

- Known results:

(i) G is connected $\Rightarrow$ multiplicity of root 0 of $P(G,\lambda)$ is 1;

(ii) G is 2-connected $\Rightarrow$ multiplicity of root 1 of $P(G,\lambda)$ is 1.

Question:

G is 3-connected $\Rightarrow$ multiplicity of root 2 of $P(G,\lambda)$ is 1 ?



# A family of graphs

- For any integer k≥ 1, let $H_k$ be the graph obtained from $K_{3,k}$ by adding a new vertex x and three new edges joining x to $x_2$, $x_3$.
- The chromatic polynomial of $H_k$ is

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3)^{k+1} + (3\lambda-8)\lambda(\lambda-1)(\lambda-2)^{k} + \lambda(\lambda-1)^{k}(\lambda-2)$$



# Properties of $H_k$

(i) For any $k \geq 3$, $H_k$ is 3-connected;

(ii) For any even $k \geq 2$, $(\lambda-2)^2 \mid P(H_k, \lambda)$.

(iii) $H_k \in F$.



# Future consideration

1. The conjecture on α-tough graphs.
2. Try to find other conditions which are weaker than α-tough but make graphs still in F.
3. Try to apply the methods used here to study the two famous conjectures on flow polynomial and chromatic polynomials.