New Methods for Computing Roots of Polynomials

Joab R. Winkler

Department of Computer Science

The University of Sheffield

United Kingdom

CONTENTS

1. DIFFICULTIES OF COMPUTING POLYNOMIAL ROOTS

There exist many algorithms for computing the roots of a polynomial:

- Bairstow, Graeffe, Jenkins-Traub, Laguerre, Müller, Newton, . . .

These methods yield satisfactory results if:

- The polynomial is of moderate degree
- The roots are simple and well-separated
- A good starting point in the iterative scheme is used

This heuristic has exceptions:

$$f(x) = \prod_{i=1}^{20}(x-i) = (x-1)(x-2)\cdots(x-20)$$

Example 1.1 Consider the polynomial

$$x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4$$

whose root is $x = 1$ with multiplicity 4. MATLAB returns the roots

1.0002, 1.0000 + 0.0002i, 1.0000 - 0.0002i, 0.9998 □

Example 1.2 The roots of the polynomial $(x-1)^{100}$ were computed by MATLAB.

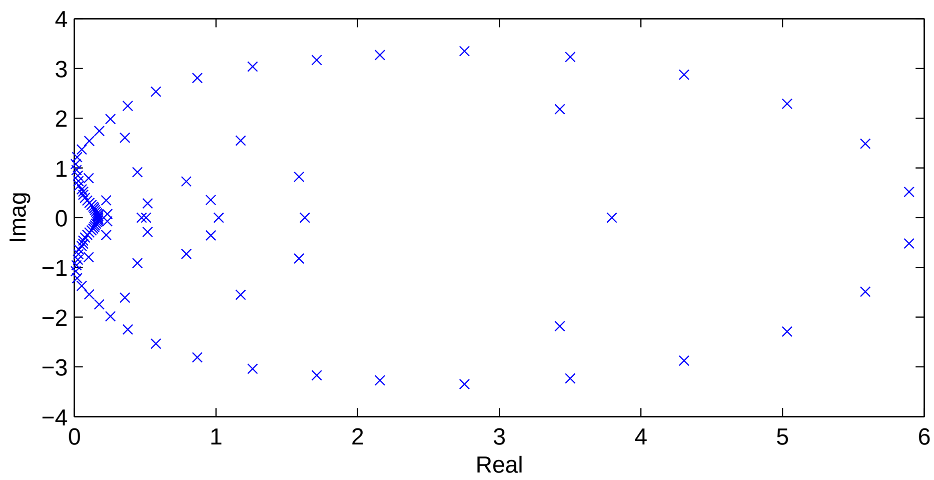


Figure 2.1: The computed roots of $(x-1)^{100}$.

□

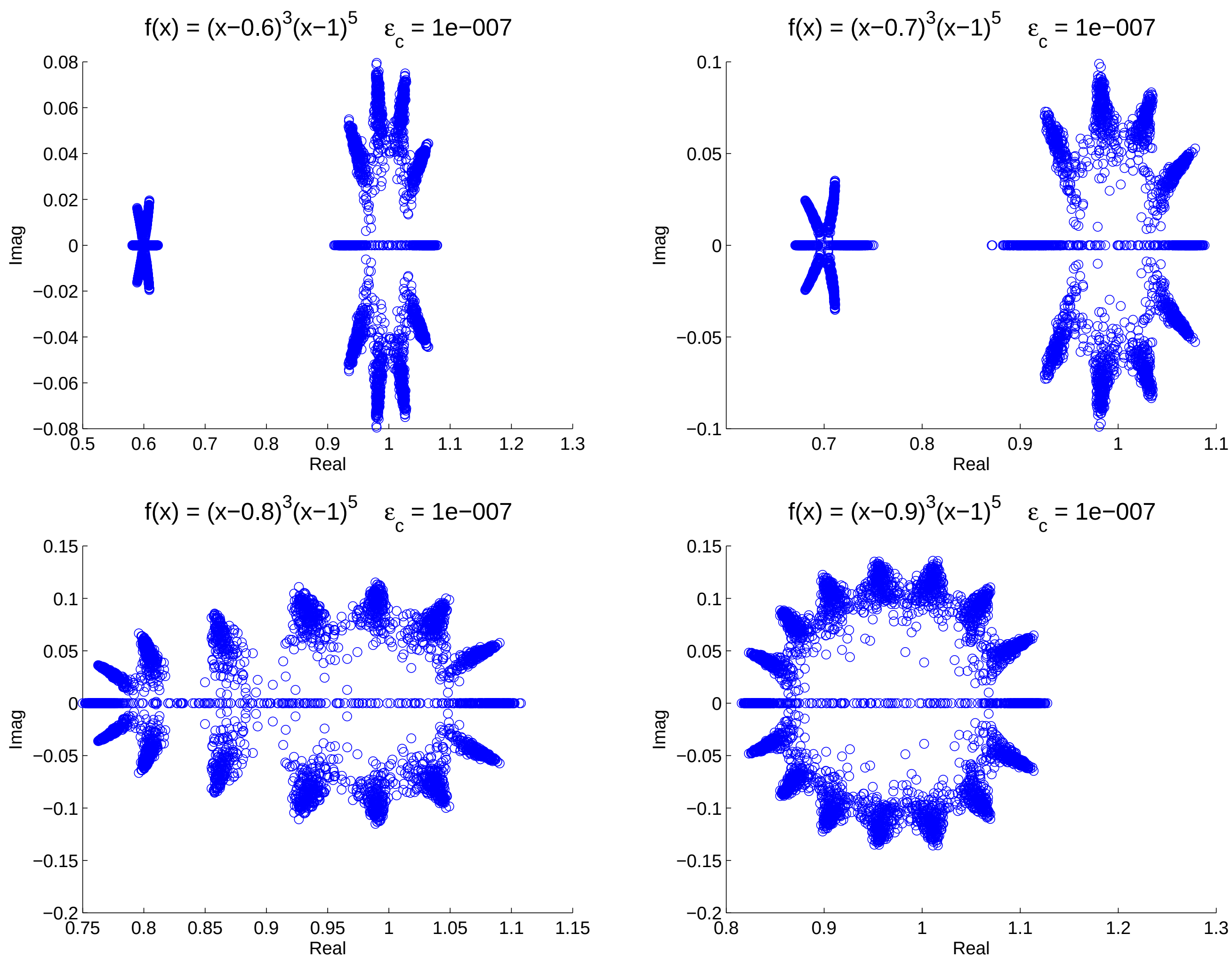


Figure 1.1: The root distribution of four perturbed polynomials.

2. THE GEOMETRY OF ILL-CONDITIONED POLYNOMIALS

- A root x_0 of multiplicity r introduces $(r-1)$ constraints on the coefficients.
- A monic polynomial of degree m has m degrees of freedom.
- The root x_0 lies on a manifold of dimension $(m-r+1)$ in a space of dimension m.
- This manifold is called a pejorative manifold because polynomials near this manifold are ill-conditioned.
- A polynomial that lies on a pejorative manifold is well-conditioned with respect to (the structured) perturbations that keep it on the manifold, which corresponds to the situation in which the multiplicity of the roots is preserved.
- A polynomial is ill-conditioned with respect to perturbations that move it off the manifold, which corresponds to the situation in which a multiple root breaks up into a cluster of simple roots.

Example 2.1 Consider a cubic polynomial $f(x)$ with real roots x_0, x_1 and x_2

$$\begin{aligned}(x - x_0)(x - x_1)(x - x_2) &= x^3 - (x_0 + x_1 + x_2)x^2 + \\ &\quad (x_0x_1 + x_1x_2 + x_2x_0)x - x_0x_1x_2\end{aligned}$$

- If $f(x)$ has one double root and one simple root, then $x_0 = x_1 \neq x_2$ and thus $f(x)$ can be written as

$$x^3 - (2x_1 + x_2)x^2 + (x_1^2 + 2x_1x_2)x - x_1^2x_2$$

The pejorative manifold of a cubic polynomial that has a double root is the surface defined by

$$\begin{pmatrix} -(2x_1 + x_2) & (x_1^2 + 2x_1x_2) & -x_1^2x_2 \end{pmatrix} \qquad x_1 \neq x_2$$

- If $f(x)$ has a triple root, then $x_0 = x_1 = x_2$ and thus $f(x)$ can be written as

$$x^3 - 3x_0 x^2 + 3x_0^2 x - x_0^3$$

The pejorative manifold of a cubic polynomial that has a triple root is the curve defined by

$$\begin{pmatrix} -3x_0 & 3x_0^2 & -x_0^3 \end{pmatrix}$$

$\square$

Theorem 2.1 The condition number of the real root x_0 of multiplicity r of the polynomial $f(x) = (x - x_0)^r$, such that the perturbed polynomial also has a root of multiplicity r, is

$$\rho(x_0) := \frac{\Delta x_0}{\Delta f} = \frac{1}{r\,|x_0|}\frac{\|(x-x_0)^r\|}{\|(x-x_0)^{r-1}\|} = \frac{1}{r\,|x_0|}\left(\frac{\sum_{i=0}^{r}\binom{r}{i}^2 (x_0)^{2i}}{\sum_{i=0}^{r-1}\binom{r-1}{i}^2 (x_0)^{2i}}\right)^{\frac{1}{2}}$$

where $\|\cdot\| = \|\cdot\|_2$ and

$$\Delta f = \frac{\|\delta f\|}{\|f\|} \qquad \text{and} \qquad \Delta x_0 = \frac{|\delta x_0|}{|x_0|}$$

Example 2.2 The condition number $\rho(1)$ of the root $x_0 = 1$ of $(x-1)^r$ is

$$\rho(1) = \frac{1}{r}\left(\frac{\sum_{i=0}^{r}\binom{r}{i}^2}{\sum_{i=0}^{r-1}\binom{r-1}{i}^2}\right)^{\frac{1}{2}}$$

This expression reduces to

$$\rho(1) = \frac{1}{r}\sqrt{\frac{\binom{2r}{r}}{\binom{2(r-1)}{r-1}}} = \frac{1}{r}\sqrt{\frac{2(2r-1)}{r}} \approx \frac{2}{r} \quad \text{for large } r$$

Compare with the componentwise and normwise condition numbers

$$\kappa_c(1) \approx \frac{|\delta x_0|}{\varepsilon_c} \qquad \text{and} \qquad \kappa_n(1) \approx \frac{|\delta x_0|}{\varepsilon_n}$$

- $\rho(1)$ is independent of the the noise level (assumed to be small)
- $\rho(1)$ decreases as the multiplicity r of the root $x_0 = 1$ increases □

3. A SIMPLE POLYNOMIAL ROOT FINDER

Let $w_i(x)$ be the product of all factors of degree i of $f(x)$

$$f(x) = w_1(x) w_2^2(x) w_3^3(x) \cdots w_{r_{\max}}^{r_{\max}}(x)$$

Perform a sequence of GCD computations

$$\begin{aligned}
q_1(x) &= \text{GCD}\left(f(x), f^{(1)}(x)\right) &&= w_2(x) w_3^2(x) w_4^3(x) \cdots w_{r_{\max}}^{r_{\max}-1}(x) \\
q_2(x) &= \text{GCD}\left(q_1(x), q_1^{(1)}(x)\right) &&= w_3(x) w_4^2(x) w_5^3(x) \cdots w_{r_{\max}}^{r_{\max}-2}(x) \\
q_3(x) &= \text{GCD}\left(q_2(x), q_2^{(1)}(x)\right) &&= w_4(x) w_5^2(x) w_6^3(x) \cdots w_{r_{\max}}^{r_{\max}-3}(x) \\
q_4(x) &= \text{GCD}\left(q_3(x), q_3^{(1)}(x)\right) &&= w_5(x) w_6^2(x) w_7^3(x) \cdots w_{r_{\max}}^{r_{\max}-4}(x) \\
&\vdots
\end{aligned}$$

The sequence terminates at $q_{r_{\max}}(x)$, which is a constant.

A set of polynomials $h_i(x), i = 1, \ldots, r_{\max}$, is defined such that

$$\begin{aligned} h_1(x) &= \frac{f(x)}{q_1(x)} &= w_1(x)w_2(x)w_3(x)\cdots \\ h_2(x) &= \frac{q_1(x)}{q_2(x)} &= w_2(x)w_3(x)\cdots \\ h_3(x) &= \frac{q_2(x)}{q_3(x)} &= w_3(x)\cdots \\ &\vdots \\ h_{r_{\max}}(x) &= \frac{q_{r_{\max}-2}}{q_{r_{\max}-1}} &= w_{r_{\max}}(x) \end{aligned}$$

The functions, $w_1(x), w_2(x), \cdots, w_{r_{\max}}(x)$, are determined from

$$w_1(x) = \frac{h_1(x)}{h_2(x)}, \quad w_2(x) = \frac{h_2(x)}{h_3(x)}, \quad \cdots \quad , w_{r_{\max}-1}(x) = \frac{h_{r_{\max}-1}(x)}{h_{r_{\max}}(x)}$$

until

$$w_{r_{\max}}(x) = h_{r_{\max}}(x)$$

The equations

$$w_1(x) = 0, \quad w_2(x) = 0, \quad \cdots \quad , w_{r_{\max}}(x) = 0$$

contain only simple roots, and they yield the simple, double, triple, etc., roots of $f(x)$.

- If x_0 is a root of $w_i(x)$, then it is a root of multiplicity i of $f(x)$.

□

Mathematical operations performed in this root finder:

- GCD computations
- Polynomial division
- Solution of simple polynomial equations

3.1 Discussion of method

- The computation of the GCD of two polynomials is an ill-posed problem because it is not a continuous function of their coefficients:
  - The polynomials $f(x)$ and $g(x)$ may have a non-constant GCD, but the perturbed polynomials $f(x) + \delta f(x)$ and $g(x) + \delta g(x)$ may be coprime.
- The determination of the degree of the GCD of two polynomials reduces to the determination of the rank of a resultant matrix, but the rank of a matrix is not defined in a floating point environment. The determination of the rank of a noisy matrix is a challenging problem that arises in many applications.
- Polynomial division, which reduces to the deconvolution of their coefficients, is an ill-conditioned problem that must be implemented with care in order to obtain a computationally reliable solution.

4. APPROXIMATE GREATEST COMMON DIVISORS

If $f(x)$ is exact and all computations are performed in a symbolic environment, the GCD of $f(x)$ and its derivative $f^{(1)}(x)$ can be computed by the Sylvester resultant matrix $S(f, f^{(1)})$.

The polynomial $f(x)$ is rarely known exactly, and so the given data is

$$\tilde{f}(x) = f(x) + \delta f(x)$$

and $\tilde{f}(x)$ and $\tilde{f}^{(1)}(x)$ are (with probability almost 1) coprime.

- The polynomials $\tilde{f}(x)$ and $\tilde{f}^{(1)}(x)$ have an approximate greatest common divisor.
- Use the method of structured total least norm applied to $S(\tilde{f}, \tilde{f}^{(1)})$ to compute the smallest perturbation of $S(\tilde{f}, \tilde{f}^{(1)})$ such that its perturbed form is singular, which implies that the perturbed form of $\tilde{f}(x)$ has a multiple root.

Example 4.1 If $m = 4$ and $n = 3$, then

$$f(x) = \sum_{i=0}^{4} a_i x^i$$

the Sylvester resultant matrix $S(f, f^{(1)})$ is

$$S(f, f^{(1)}) = \begin{bmatrix} a_4 & & & 4a_4 & & & \\ a_3 & a_4 & & 3a_3 & 4a_4 & & \\ a_2 & a_3 & a_4 & 2a_2 & 3a_3 & 4a_4 & \\ a_1 & a_2 & a_3 & a_1 & 2a_2 & 3a_3 & 4a_4 \\ a_0 & a_1 & a_2 & & a_1 & 2a_2 & 3a_3 \\ & a_0 & a_1 & & & a_1 & 2a_2 \\ & & a_0 & & & & a_1 \end{bmatrix}$$

4.1 The non-uniqueness of the Sylvester resultant matrix

- An approximate GCD of $f(x)$ and $g(x)$ is equal to, up to a scalar multiplier, an approximate GCD of $f(x)$ and $\alpha g(x)$, where α is an arbitrary non-zero constant.
- The resultant matrix $S(f, \alpha g)$ should be used when it is desired to compute an approximate GCD of $f(x)$ and $g(x)$.
- Since $S(f, \alpha g) \neq \alpha S(f, g)$, the inclusion of α permits a family of approximate GCDs, rather than only one approximate GCD, to be computed.

Example 4.2 Consider the exact polynomials

$$\begin{aligned}\hat{f}_1(x) &= (x-0.25)^8(x-0.5)^9(x-0.75)^{10}(x-1)^{11}(x-1.25)^{12}\\ \hat{g}_1(x) &= (x+0.25)^4(x-0.25)^5(x-0.5)^6\end{aligned}$$

with signal-to-noise ratio $\mu = 10^8$.

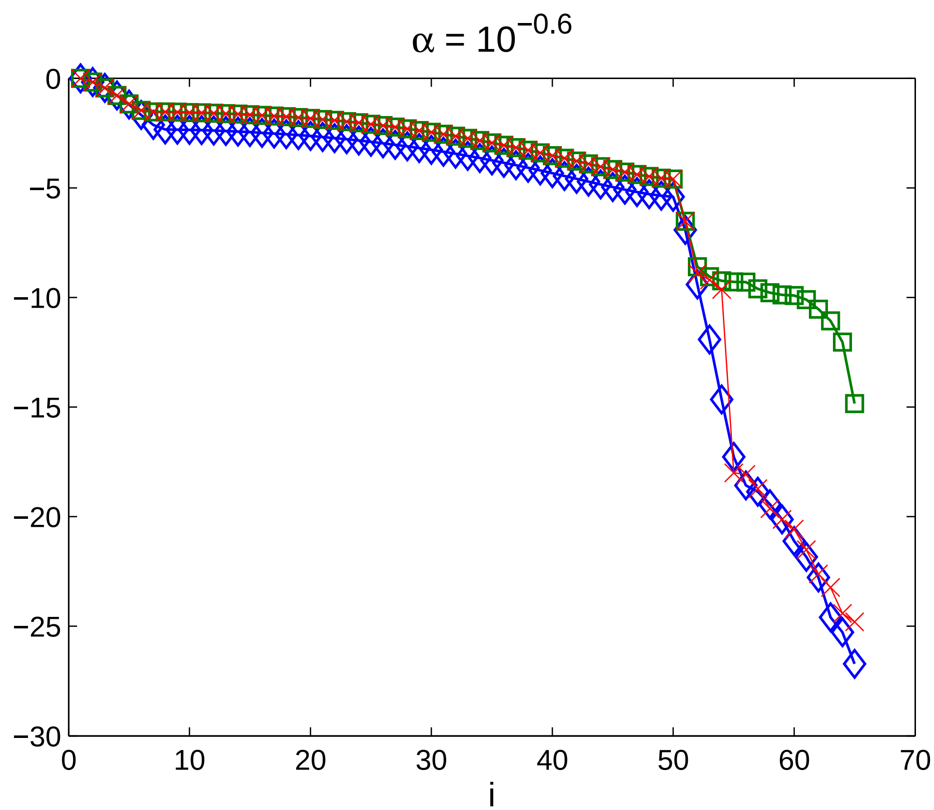


Figure 4.1: The normalised singular values of the Sylvester resultant matrix, on a logarithmic scale, for (i) the theoretically exact data $S(\hat{f}_1, \hat{g}_1)$, $\diamondsuit$; (ii) the given inexact data $S(f_1, g_1)$, $\Box$; (iii) the computed data $S(\tilde{f}_{1,0}, \tilde{g}_{1,0})$, $\times$, for $\alpha = 10^{-0.6}$. $\Box$

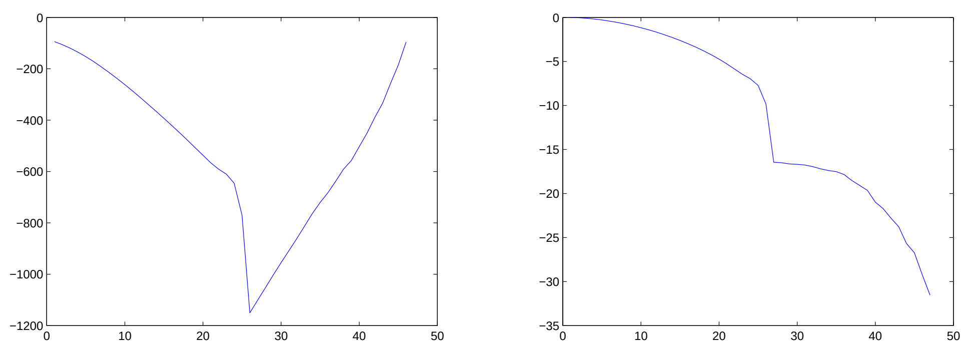
5. THE CALCULATION OF THE POLYNOMIAL ROOTS

- Use the method of least squares to perform the polynomial division
- Obtain initial estimates of the roots of the polynomial by solving a set of polynomial equations, each of whose roots is simple.
- Refine these estimates by using the method of non-linear least squares.

6. EXAMPLES

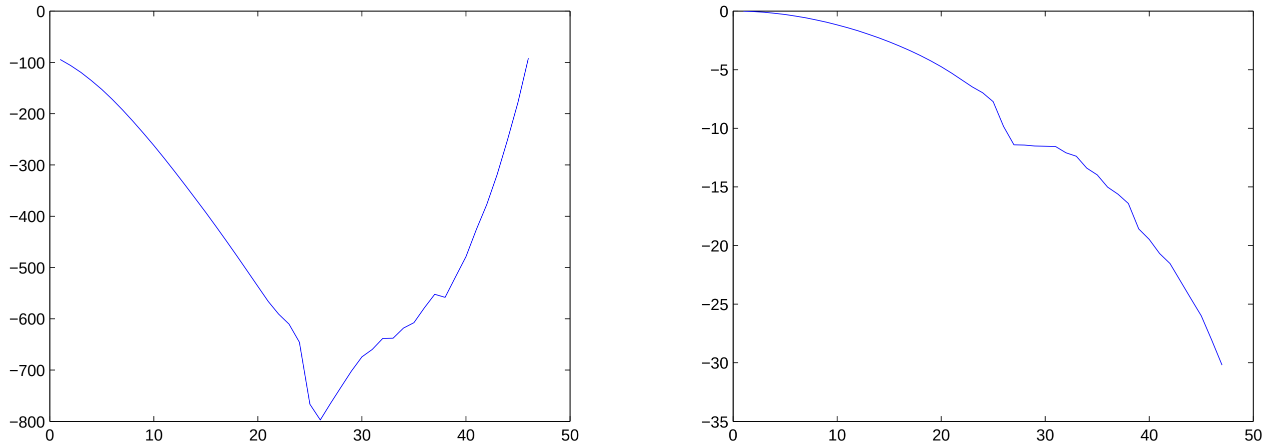
Example 6.1 Consider the Sylvester resultant matrix of $f(x)$ and its derivative $f^{(1)}(x)$, which has rank 26.

$$f(x) = (x - 0.1)^8 (x - 0.5)^8 (x - 0.9)^8$$



(i) (ii)

Figure 6.1: (i) The rank estimate of $S(f, f^{(1)})$ from the principle of maximum likelihood, and (ii) the singular values of $S(f, f^{(1)})$, in the absence of noise.



(i) (ii)

Figure 6.2: (i) The rank estimate of $S(f, f^{(1)})$ from the principle of maximum likelihood, and (ii) the singular values of $S(f, f^{(1)})$, for a signal-to-noise ratio of 10^9.

□

Example 6.2 Consider the polynomial

$$f(x) = (x-1)^{20}(x-2)^{15}(x-3)^{10}(x-4)^5$$

The computed roots are

Multiplicity	Root
1	20
2	15
3	10
4	5

7. FURTHER RESEARCH

- Use the displacement structure of matrices to optimise the computational efficiency of the method.
- Optimise the calculation of the scale parameter α in the Sylvester resultant matrix.
- Investigate the use of threshold independent methods, for example, the principle of maximum likelihood, for the estimation of the rank of a matrix.
- Consider the problem that occurs when there are bounds on the displacement of each coefficient.
- Extend to bivariate and trivariate polynomials.

8. SUMMARY

- A radically new method of solving polynomial equations has been described.
- This method first calculates the multiplicities of the roots, and then computes their values.
- The method is a computational implementation of the theory of pejorative manifolds.
- Computational experiments show that it is very successful.