Coalgebras and combinatorial constructions

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I. Convolution

Suppose μ and ν are finite signed measures on the real line. Their *convolution* $\mu * \nu$ is defined by

$$\int_{-\infty}^{\infty} f(x)\, d(\mu * \nu)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x+y)\, d\mu(x)\, d\mu(y).$$

The convolution *identity* is given by the point mass δ at the origin:

$$\int_{-\infty}^{\infty} f(x)\, d\delta(x) = f(0).$$

Coproduct and product

There are two stages to the definition of convolution.

- Replace $f(x)$ by $f(x+y)$.
- Integrate with respect to the measure $d\mu(x)\,d\mu(y)$.

These two stages have names.

- The function $f(x)$ of one variable has *coproduct* $f(x+y)$ that is a function of two variables.
- The measure $d\mu(x)\,d\mu(y)$ is a *product* measure.

Thus the convolution $\mu * \nu$ is obtained by applying a coproduct and then a product.

II. Algebras and coalgebras

Let K be a field (real numbers or complex numbers). An *algebra* is a vector space with these scalars. It also has a *multiplication* $m : \mathcal{A} \bigotimes \mathcal{A} \to \mathcal{A}$ that is linear and satisfies the *associative law*

$$m(m \otimes I) = m(I \otimes m).$$

Apply this to a triple $a \otimes b \otimes c$. This gives

$$m((ab) \otimes c) = m(a \otimes (bc))$$

or

$$(ab)c = a(bc).$$

So it is the usual associative law.

Identity for an algebra

We also have an *identity* $e : K \to \mathcal{A}$ that maps the scalar 1 to the identity $e1$ of the algebra. It must satisfy a left identity condition and a right identity condition. The left identity condition is

$$m(e \otimes I) = I$$

where $I : K \otimes \mathcal{A} \to \mathcal{A}$ on the right is the identification given by scalar multiplication. Apply this to a pair $k \otimes a$. This gives

$$m(ek \otimes a) = (ek)a = ka.$$

In particular $(e1)a = a$, so $e1$ is a left identity in the usual sense.

Associative law for a coalgebra

Let K be a field (real numbers or complex numbers). An *coalgebra* is a vector space $\mathcal{C}$ with a *comultiplication* $\Delta : \mathcal{C} \to \mathcal{C} \bigotimes \mathcal{C}$ that is linear and satisfies the *associative law*

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta.$$

Apply this to an element c. This gives

$$(\Delta \otimes I) \sum c_1 \otimes c_2 = (I \otimes \Delta) \sum c_1 \otimes c_2$$

or

$$\sum \sum c_{11} \otimes c_{12} \otimes c_2 = \sum \sum c_1 \otimes c_{21} \otimes c_{22}.$$

So it says that the order of the decomposition does not matter.

Coidentity for a colgebra

We also have an *coidentity* $e : \mathcal{C} \to K$ that maps each element c of $\mathcal{C}$ to a scalar ϵc. It must satisfy a left coidentity condition and a right coidentity condition. The left coidentity condition is

$$(\epsilon \otimes I)\Delta = I$$

where $I : \mathcal{C} \to K \otimes \mathcal{C}$ on the right is the inverse to the identification given by scalar multiplication. Apply this to an element c of the coalgebra. This gives

$$(\epsilon \otimes I) \sum c_1 \otimes c_2 = \sum \epsilon c_1 \otimes c_2 = c.$$

This says that the coidentity undoes the effect of the decomposition.

Convolution for algebraists

Let $\mathcal{A}$ be an algebra, and let $\mathcal{C}$ be a coalgebra. Consider linear transformations L, M from $\mathcal{C}$ to $\mathcal{A}$. The *convolution* is defined by

$$L * M = m(L \otimes M)\Delta.$$

The order of operation is comultiply, apply transformations, multiply.

Apply this to an element c of the coalgebra. We get

$$(L * M)(c) = m(L \otimes M) \sum c_1 \otimes c_2 = \sum (Lc_1)(Mc_2).$$

The order is decompose, transform the parts, multiply.

The identity for convolution is just the linear transformation $e\epsilon : \mathcal{C} \to \mathcal{A}$. The order of operation is coidentity, identity.

Example: The polynomial coalgebra $\mathcal{B} = K[x]$.

The coproduct of a polynomial $p(x)$ is $\Delta p(x) = p(x + y)$. In particular,

$$\Delta x^n = (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The coidentity ϵ is given by $\epsilon p(x) = p(0)$. In particular, $\epsilon x^n = 0$ unless $n = 0$, while $\epsilon x^0 = 1$.

Convolution in the dual of $\mathcal{B}$

Let L, M be linear transformations from $\mathcal{B} = K[x]$ to the scalars K. Then the convolution is given by

$$(L * M)(x^n) = \sum_{k=0}^{n} \binom{n}{k} (Lx^k)(Mx^{n-k}).$$

For each linear transformation L define a corresponding exponential generating function

$$f_L(z) = \sum_{k=0}^{\infty} (Lx^k) \frac{1}{k!} z^k.$$

Then

$$f_{L*M}(z) = f_L(z) f_M(z).$$

Convolution is just multiplication of exponential generating functions. So the dual algebra consists of formal power series $K[[z]]$.

Derivation and coderivation

A *derivative* is a linear map $d : \mathcal{A} \to \mathcal{A}$ such that

$$dm = m[d \otimes I + I \otimes d].$$

Thus

$$d(ab) = (da)b + a(db).$$

A *coderivative* is a linear map $\delta : \mathcal{C} \to \mathcal{C}$ such that

$$\Delta\delta = [\delta \otimes I + I \otimes \delta]\Delta.$$

Thus

$$\Delta\delta c = \sum \delta c_1 \otimes c_2 + \sum c_1 \otimes \delta c_2.$$

If δ is a coderivative of a coalgebra $\mathcal{C}$, then its dual d is a derivative of the dual algebra given by the convolution product.

Example of coderivative

Let $\delta p(x) = xp(x)$. Then δ is a coderivative. In fact

$$(x+y)p(x+y) = xp(x+y) + yp(x+y).$$

With $p(x) = x^n$ this is

$$\sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k} = \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1},$$

which reduces to Pascal's triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Add a point to an n element set. A k element subset of the new set either includes the point or does not include it.

The dual derivative

The corresponding derivative on the dual $K[[z]]$ is $\frac{d}{dz}$. In fact

$$f_{dL}(z) = \sum_{k=0}^{\infty} (Lx^{k+1}) \frac{1}{k!} z^k = \sum_{k=1}^{\infty} (Lx^k) \frac{1}{k!} k z^{k-1} = \frac{d}{dz} f_L(z).$$

III. The equilibrium lattice gas

$\mathcal{P}$ is a fixed (large) set of locations (or colors).
z_p is the *activity* at p (prior weight for finding a particle at location p).
$1 + t(p, q)$ is the (Boltzmann) factor for the *interaction* between a particle at p and a particle at q.

Let U_n be a finite set of particles (labels). A *particle configuration* (colored set) is a function $a : U_n \to \mathcal{P}$. The probability of a particle configuration is

$$\mathrm{prob}(a) = \frac{1}{\Xi(z)} \frac{1}{n!} \prod_{\{i,j\}} (1 + t(a(i), a(j))) \prod_i z_{a(i)}.$$

The expected number of particles at a point

$n_p(a) = \#\{i \mid a(i) = p\}$ is the number of particles at p.

$$\langle n_p \rangle = \sum_{n=0}^{\infty} \sum_{a:U_n \to \mathcal{P}} n_p(a)\mathrm{prob}(a).$$

Alternatively

$$\langle n_p \rangle = \frac{z_p \frac{\partial}{\partial z_p} \Xi(z)}{\Xi(z)} = z_p \frac{\partial}{\partial z_p} \log(\Xi(z)).$$

Graphs and a combinatorial logarithm

$$\Xi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{P}} \prod_{\{i,j\}} (1 + t(a(i), a(j))) \prod_{j} z_{a(j)}.$$

Let $|\mathcal{G}(a)| = \sum_{G=(U_n,E)} \prod_{\{i,j\}\in E} t(a(i), a(j))$. Then

$$\Xi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{P}} |\mathcal{G}(a)| \prod_{j\in U_n} z_{a(j)}.$$

Let $|\mathcal{G}_c(a)| = \sum_{G_c=(U_n,E)} \prod_{\{i,j\}\in E} t(a(i), a(j))$. Then

$$\log \Xi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{P}} |\mathcal{G}_c(a)| \prod_{j\in U_n} z_{a(j)}.$$

Graph example

Example: $\mathcal{P}$ is the vertex set of a graph $\mathcal{H} = (\mathcal{P}, \mathcal{E})$ with loops.
$t(p, q) = -1$ if $\langle p, q \rangle \in \mathcal{E}$ and $t(p, q) = 0$ otherwise.

$$\Xi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{G=(U_n,E)} (-1)^{|E|} \sum_{a:(U_n,E)\to(\mathcal{P},\mathcal{E})} \prod_{j \in U_n} z_{a(j)}.$$

Special case: $\mathcal{H} = (\mathcal{P}, \mathcal{E})$ is a complete graph on k vertices. Then the graph homomorphisms $a : G = (U_n, E) \to \mathcal{H} = (\mathcal{P}, \mathcal{E})$ are proper colorings of G with k colors.

IV. Combinatorial Constructions

Reference: F. Bergeron, G. Labelle, and P. Leroux, Combinatorial Species and Tree-like Structures, Cambridge University Press, 1998.
This book presents a systematic account of constructions in enumerative combinatorics.

Goal: Motivate the form of these constructions by analogy with coalgebra concepts.

Particle configurations

Fix a set $\mathcal{P}$ of locations. Consider a finite set U of particles (or labels). A *particle configuration* (or colored set) is a function $a : U \to \mathcal{P}$. If $V \subseteq U$, then a_V is the restriction of a to V.

The *combinatorial coproduct* of a particle configuration $a : U \to \mathcal{P}$ is

$$\Delta a = \{\langle a_V, a_W \rangle \mid U = V \sqcup W\}.$$

For each location p in $\mathcal{P}$, the *combinatorial coderivative* is given for $a : U \to \mathcal{P}$ by

$$\delta_p a : U^* = U \sqcup \{*\} \to \mathcal{P}$$

with $\delta_p a$ assigning location p to the extra particle $*$.
The coproduct rule comes from

$$\{\langle \bar{U}, \bar{V} \rangle \mid U^* = \bar{V} \sqcup \bar{W}\} = \{\langle V^*, W \rangle | U = V \sqcup W\} \sqcup \{\langle V, W^* \rangle \mid U = V \sqcup W\}.$$

The combinatorial coexponential

The *combinatorial coexponential* $\exists$ is given for $a : U \to \mathcal{P}$ by

$$\exists a = \{\{a_V \mid V \in \Gamma\} \mid \Gamma \in \mathrm{Par}[U]\}.$$

Coexponential rule: If S denotes distributed sum, then

$$S\exists a = \langle \exists, \exists \rangle \Delta a.$$

The coexponential rule comes from

$$\{\langle \Gamma, \langle \Delta, \Sigma \rangle \rangle \mid \Gamma \in \mathrm{Par}[U], \Gamma = \Delta \sqcup \Sigma\}$$

$$= \{\langle \langle V, \Delta \rangle, \langle W, \Sigma \rangle \rangle \mid U = V \sqcup W, \Delta \in \mathrm{Par}[V], \Sigma \in \mathrm{Par}[W]\}.$$

$$\sum_{m=0}^{n} S(n,m) 2^m = \sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k}.$$

Weighted sets

Fix a commutative ring R. (This could be the real numbers). A *weighted set* is a function $\alpha : A \to R$, where A is some underlying finite set. The objects in A are to be counted, and α is the method of counting.

The *disjoint union* of weighted sets is obtained by taking the disjoint union of the underlying sets and defining the function in the natural way. The *cartesian product* of weighted sets is $\alpha \times \beta : A \times B \to R$ given by

$$(\alpha \times \beta)(a, b) = \alpha(a)\beta(b)$$

for a in A, b in B.

The *total weight* $|\alpha|$ of a weighted set is

$$|\alpha| = \sum_{a \in A} \alpha(a).$$

This is the final result of the count.

Combinatorial constructions

A *combinatorial construction* $\mathcal{L}$ assigns to each particle configuration a a weighted set $\mathcal{L}(a)$.

Example from physics. The *connected graph* construction $\mathcal{G}_c$ assigns to the particle configuration $a : U \to \mathcal{P}$ the weighted set $\mathcal{G}_c(a)$.
Its underlying set consists of all connected graphs $G_c = (U, E)$ with vertex set consisting of the set U of particles.
The weight of such a graph is

$$\mathcal{G}_c(a)(G_c) = \prod_{\{i,j\} \in E} t(a(i), a(j)).$$

Combinatorial operations

The *combinatorial convolution* of two combinatorial constructions is given for a particle configuration $a : U \to \mathcal{P}$ as the weighted set

$$(\mathcal{L} * \mathcal{M})(a) = \sum_{\langle V,W \rangle} \mathcal{L}(a_V) \times \mathcal{M}(a_W),$$

where V and W are disjoint sets with union U. Thus it is a combinatorial coproduct followed by the separate action of the two constructions followed by the cartesian product.

The *combinatorial derivative* d_p is defined by

$$d_p \mathcal{L}(a) = \mathcal{L}(\delta_p a).$$

Example of combinatorial operations

Let $\mathcal{E}_1^p$ be the construction that assigns to a particle configuration $a : U \to \mathcal{P}$ exactly one point with weight one if U has one point and a locates that point at p. Otherwise it assigns the empty set.

Let $\mathcal{G}_c$ be the connected graph construction.
Consider the construction $\mathcal{E}_1^p * d_p\mathcal{G}_c$. We have

$$(\mathcal{E}_1^p * d_p\mathcal{G}_c)(a) = \sum_{i \in U : a(i) = p} (d_p\mathcal{G}_c)(a_{U \setminus \{i\}}) = \sum_{i \in U : a(i) = p} \mathcal{G}_c(a).$$

In other words, it produces all pairs consisting of a weighted graph with the particles as vertex set and a distinguished particle located at p.

The combinatorial exponential

The combinatorial exponential is given for $a : U \to \mathcal{P}$ by

$$(\mathcal{E} \circ \mathcal{L})(a) = \sum_{\Gamma} \prod_{V \in \Gamma} \mathcal{L}(a_V)$$

where Γ ranges over partitions of U. The order of operation is coexponential, construction, product.

Example: $\mathcal{E} \circ \mathcal{G}_c = \mathcal{G}$.

The exponential of the sum is the product of the exponentials:

$$\mathcal{E} \circ (\mathcal{L} + \mathcal{M}) = (\mathcal{E} \circ \mathcal{L}) * (\mathcal{E} \circ \mathcal{M}).$$

The exponential generating function

If $\mathcal{L}$ is a combinatorial construction, then its *exponential generating function* is defined for z in $K^{\mathcal{P}}$ by

$$f_{\mathcal{L}}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a:U_n \to \mathcal{P}} |\mathcal{L}(a)| \prod_{j \in U_n} z_{a(j)}.$$

The product property is

$$f_{\mathcal{L}*\mathcal{M}}(z) = f_{\mathcal{L}}(z) f_{\mathcal{M}}(z).$$

The derivative property is

$$f_{d_p \mathcal{L}}(z) = \frac{\partial}{\partial z_p} f_{\mathcal{L}}(z).$$

The exponential of an exponential generating function

The exponential property is

$$f_{\mathcal{E}\circ\mathcal{L}}(z) = \exp(f_{\mathcal{L}}(z)).$$

Example: Since $\mathcal{E} \circ \mathcal{G}_c = \mathcal{G}$, we have

$$\exp(f_{\mathcal{G}_c}(z)) = f_{\mathcal{G}}(z) = \Xi(z).$$

So $f_{\mathcal{G}_c}(z) = \log(\Xi(z))$.

Conclusion

Combinatorial construction: Each element of the weighted set is a connected graph on the set of particles, together with a distinguished particle at location p.
Physical interpretation: The expected number of particles at p is the exponential generating function of this construction.

$$\langle n_p \rangle = f_{\mathcal{E}_1^p * d_p \mathcal{G}_c}(z) = z_p \frac{\partial}{\partial z_p} f_{\mathcal{G}_c}(z).$$

Conclusion: The statistical mechanical formulas are identical to the combinatorial formulas. Their ultimate origin is related to concepts of coproduct and coexponential and coderivation.