RESUMMATION OF CLUSTER EXPANSION SERIES

(hard repulsion case: $\mathrm{supp}\,\Gamma \cap \mathrm{supp}\,\Gamma' \neq \emptyset$)

M. Zahradník Prague

Keywords: generalized exponential function
inclusion exclusion principle
connected graphs, cycles, paths
exactly solvable models
(no trees)

References: Mayer 1920? → ----, many authors...
→ Scott–Sokal JSP 2003

$$Z = \sum_{\{\Gamma_i\}} \prod_i (-w_{\Gamma_i})$$

summation over all collections of mutually compatible polymers Γ_i

$w_\Gamma \in \mathbb{C}$ (complex) weight

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1,\dots,\Gamma_N)} \prod_i (-w_{\Gamma_i}) \prod_{i<j} (1 + c_{\Gamma_i \Gamma_j})$$

$c_{\Gamma\Gamma'} = 0$ compatible

$c_{\Gamma\Gamma'} = -1$ incompatible

$$= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(\Gamma_1,\dots,\Gamma_N)} \prod_i (-w_{\Gamma_i}) \sum_{\substack{G \text{ graph} \\ \text{on } (1,\dots,N)}} c_G$$

where

$$c_G = c_G(\Gamma_1,\dots,\Gamma_N) = \prod_{\{i,j\}\in G} c_{\Gamma_i \Gamma_j}$$

THEOREM 0

$$Z = \exp\left(\sum_{M=1}^{\infty} \frac{1}{M!} \sum_{(\Gamma_1,\dots,\Gamma_M)} \prod_i (-w_{\Gamma_i}) \sum_{G \text{ connected}} c_G \right)$$

Why $-w_\Gamma$?

0) $\log(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$

cluster expansion

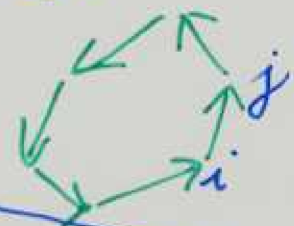
1) $\det(\mathbb{J}-W) = \sum_{\pi} \mathrm{sign}\,\pi \prod_{k:\pi(k)\neq k} (-w_{\pi(k),k})$

Laplacian

$w_{ii}=0$

$$= \sum_{\{C_\ell\}} \prod_\ell (-w_{C_\ell}) \qquad w_C = \prod_{(j,i)\in C} w_{ji}$$

collection of cycles in Λ (indexset)



2) Ising-Cramers-Wannier-Onsager

$$Z = \sum_{\{\Gamma_i\}} \prod_i (-w_{\Gamma_i}) \qquad w_\Gamma = \prod_{(\ell,\ell')\in\Gamma} t W_{\ell\ell'}$$

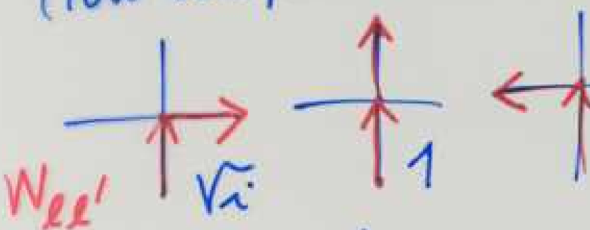
CYCLES!

contours

(low temperature or high temperature ones - duality)

$(\sqrt{i})^4 = -1$

$(\sqrt{-i})^4 = -1$



$\sqrt{-i} = \frac{1}{\sqrt{2}}(1-i)$

$$t < \frac{1}{\sqrt{2}+1}$$

CLUSTER EXPANSION

1) $\det(\mathbb{J}-W) = \exp\left(-\sum \frac{1}{m} \mathrm{Tr}\, W^m\right)$

2) Onsager $Z = \exp\left(-\sum w_P\right.$ PATHS!

closed paths in a dual lattice

GOAL:

FIND RESUMMATION METHOD IN THEOREM 0 SUCH THAT 1) 2) ARE SPECIAL CASES!

LEMMA $\displaystyle\sum_{\substack{G \text{ connected on} \\ M=\{1,2,\dots,m\}}} (-1)^{|G|} = (-1)^{m-1} \underbrace{(m-1)!}_{\text{No of cycles on } M}$

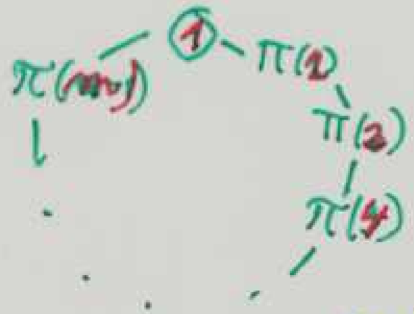
Proof $\displaystyle S(m) = \sum_G (-1)^{|G|} = \sum_{\{M_i\}} \prod_i \left(-S(m_i)\right)$

$\{M_i\}$ decomposition of $M \setminus \{1\}$

$-1 = (1-1)^k - 1$

compare $\displaystyle (m-1)! \equiv P(m) = \sum \prod_i P(m_i)$

$\{M_i\}$ decomposition of $M \setminus \{1\}$



cycle on M = collection of cycles on $M \setminus \{1\}$

Hence $S(m) = (-1)^{m-1} P(m)$

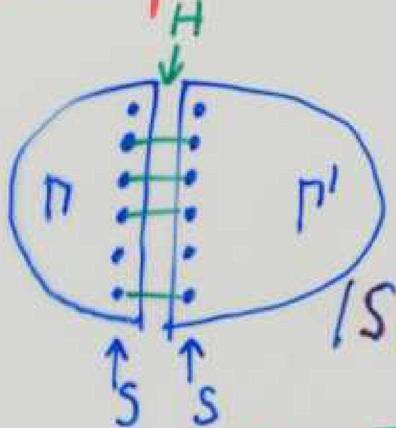
Problem: does there exist similarly nice formula for other classes of graphs (not only complete ones) e.g. bipartite?

$$Z = \exp\Big(-\sum_{\mathcal{C}} w_{\mathcal{C}}\Big) \qquad w_{\mathcal{C}} = \frac{1}{M!}\prod_{i=1}^{M}(-w_{\Gamma_i})\, c_G$$

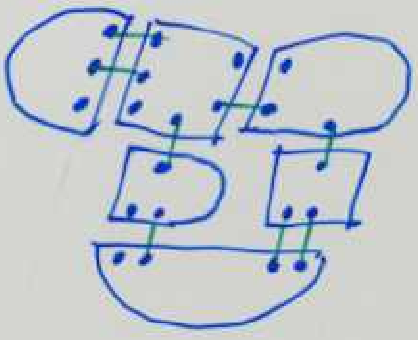
$\mathcal{C} = (\Gamma_1, \dots, \Gamma_M)$ & G connected

RESUMMATION, QUILTED CLUSTERS, CACTUSES...

quilted pair of polymers Γ, Γ' with $c_{\Gamma\Gamma'} = -1$
hence quilted cluster $\iff S = \operatorname{supp}\Gamma \cap \operatorname{supp}\Gamma' \neq \emptyset$



$|S| = s$



quilted cluster $\tilde{\mathcal{C}}$

$$-1 = (1-1)^s - 1 = \sum (-1)^{|H|}$$

H bipartite between copies of $t \in S$

| $\tilde{\mathcal{C}} = (\Gamma_1, \dots, \Gamma_M)$ & $\{H_{ij},\ \{i,j\} \in G\}$

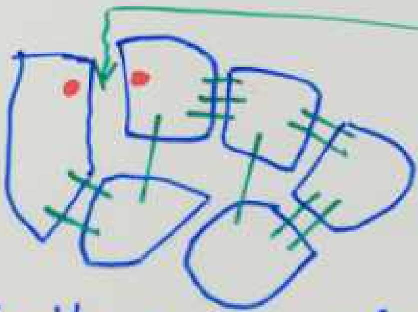
| $$Z = \exp\Big(-\sum_{\tilde{\mathcal{C}}} w_{\tilde{\mathcal{C}}}\Big) \qquad w_{\tilde{\mathcal{C}}} = \frac{1}{M!}\prod_{i=1}^{M}(-w_{\Gamma_i})(-1)^{\sum |H_{ij}|}$$

Given $\tilde{\mathcal{C}}$ erase all edges (t_1, t_1) from $\tilde{\mathcal{C}}$ where t_1 is the first point of Λ ($\equiv$ the lattice where all polymers live) in some auxiliary, fixed ordering $\prec$ of Λ

$\tilde{\mathcal{C}}$ DECOMPOSES INTO COMPONENTS

SIMPLE component of $\tilde{\mathcal{C}} \setminus \{\text{all } (t_1, t_1)\}$
contains only ONE polymer containing t_1

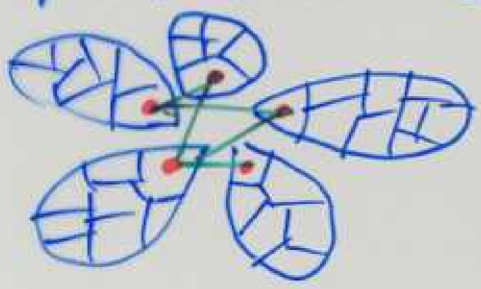
DUPLICATE components look like this:



existence of this edge does not affect the shape of components

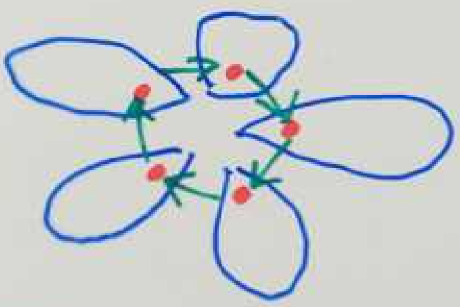
Contributions of these quilted clusters to $\sum_{\tilde{\mathcal{C}}=(\Gamma_1,\dots,\Gamma_M)\&\dots} w_{\tilde{\mathcal{C}}}$ CANCEL!)

SIMPLE (QUILTED) CLUSTER $\equiv$ all components simple. Looks as follows



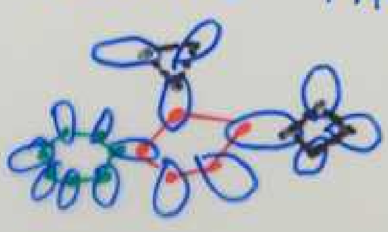
$$\sum_{G \text{ connected between all copies of } t_1 \text{ in } \tilde{\mathcal{C}}} (-1)^{|G|} =$$

$|T| = 5$



$$= \sum_{\text{all cycles on the set } T \text{ of all copies of } t_1 \text{ in } \tilde{\mathcal{C}}} (-1)^{|T-1|}$$

ITERATE THIS CONSTRUCTION



$t_1 < t_2$ etc.
$< t_3$

CACTUS of polymers
(recursively defined w.r. to $\prec$)

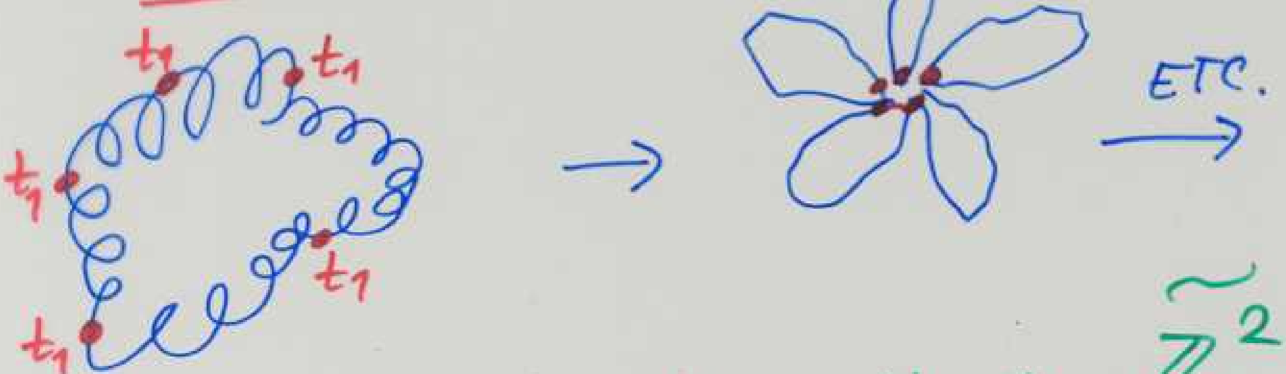
$$Z = \exp\left(-\sum_{\mathcal{K}} w_{\mathcal{K}}\right)$$

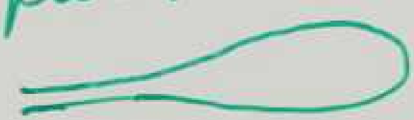
FINAL RESULT

$$w_{\mathcal{K}} = \frac{1}{M!} \prod_{i=1}^{M} (w_{\Pi_i})$$

EXAMPLES

(1) Each closed path in $\mathbb{Z}^d$ is uniquely represented as a cactus of cycles



(2) Each closed path in $\tilde{\mathbb{Z}}^2$ not containing  is uniquely represented as a cactus of Cramers-Wannier contours

UP TO NOW, ONLY NEW LOOK AT CLASSICAL RESULTS. POSSIBLE NEW DEVELOPMENTS:

1) analysis of the cluster expansion of the Ising models in region where sum over closed paths is only nonabsolutely summable

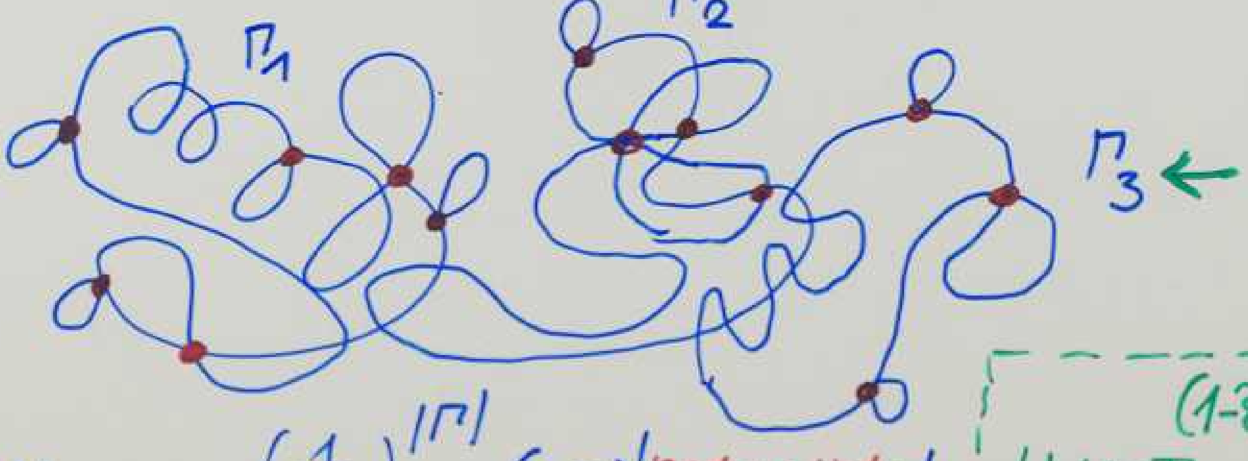
2) MASSIVE small perturbations of massless gaussians yield polymer models of the following type: BROWNIAN EVEN GRAPHS IN $\mathbb{Z}^d$ $(d \geqslant 3)$

$V \subset \mathbb{Z}^d$ set of vertices

E edges: walks from $\underline{i}$ t $\cancel{j}$

walks must not intersect other vertices

THIS MODEL APPEARS e.g. for $\sum(x_i - x_j)^2 + \varepsilon \sum x_i^4$



$$W_\Gamma = \left(\frac{1}{2d}\right)^{|\Gamma|} \cdot (-\varepsilon)^{|\text{red points}|}$$

$(1-\tilde{\varepsilon})^S$

HINT:

$S = \mathrm{supp}\,\Gamma$

SHEAVES, NOT CACTUSES....