The Geometry of Polynomials and Applications

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based on joint work with
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Rota’s philosophy

G.-C. Rota: ”The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems on location of the zeros of certain polynomials and in giving these zeros a combinatorial interpretation.”

- Phase transitions, Lee-Yang type theorems
- (Random) Matrix theory, representation theory, total positivity, number theory
- Interacting particle systems, probability theory, negative dependence

Polynomials galore

Stable Polynomial

$f \in \mathbb{C}[z_1, \ldots, z_n]$ is stable if

$\mathfrak{Im}(z_j) > 0\ \forall j \Longrightarrow f(z_1, \ldots, z_n) \neq 0$

Real stable if $f \in \mathbb{R}[z_1, \ldots, z_n]$. Related notions:

- Half-Plane Property (HPP)
  Y. Choe, J. Oxley, A. Sokal, D. Wagner
- (Gårding) Hyperbolic Polynomial
  (PDE theory, control theory, optimization)
  $f \in \mathbb{R}[z_1, \ldots, z_n]$ of degree d is real stable iff its homogenization
  $$f_H(z_1, \ldots, z_n, z_{n+1}) = z_{n+1}^d f(z_1 z_{n+1}^{-1}, \ldots, z_n z_{n+1}^{-1})$$
  is hyperbolic w.r.t. every $\mathbf{v} \in \mathbb{R}_+^n \times \{0\}$, i.e.,
  $$t \mapsto f_H(\mathbf{u} + t\mathbf{v})$$
  has all real zeros if $\mathbf{u} \in \mathbb{R}^{n+1}$, $\mathbf{v} \in \mathbb{R}_+^n \times \{0\}$.

Rayleigh polynomial (h-NLC+)

Multi-affine polynomial $f(z_1, \ldots, z_n)$ with (nonnegative) real coefficients satisfying

$$\frac{\partial f}{\partial z_i} \cdot \frac{\partial f}{\partial z_j} - \frac{\partial f}{\partial z_i \partial z_j} \cdot f \geq 0$$

for $1 \leq i, j \leq n$ and $z \in \mathbb{R}^n_+$.

(Electrical networks: Kirchhoff-Rayleigh monotonicity for relative effective conductances; cf. D. Wagner)

Strongly Rayleigh polynomial

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for $1 \leq i, j \leq n$ and $z \in \mathbb{R}^n$.

Theorem [P. Brändén]. A multi-affine polynomial $f(z_1, \ldots, z_n)$ is strongly Rayleigh iff it is stable.

Examples.
(i) $f \in \mathbb{R}[t]$ is stable iff all its zeros are real;
(ii) Spanning tree polynomials are stable.

Notation

- $\mathcal{P}_n$ is the set of multi-affine polynomials
  $f(z) = \sum_{S \subseteq [n]} a_S z^S$
  with $a_S \geq 0 \; \forall S \subseteq [n] = \{1, \ldots, n\}$ and $f(1, \ldots, 1) = 1$, where $z = (z_1, \ldots, z_n)$ and $z^S = \prod_{i \in S} z_i$.

- $\mathfrak{P}_n$ is the set of probability measures on $2^{[n]} = \{S : S \subseteq [n]\} \equiv \{0, 1\}^{[n]}$. The i-th coordinate function on $2^{[n]}$ is the binary r.v. $X_i(S) = 1$ if $i \in S$ and 0 otherwise.

- 1-1 correspondence $\mathfrak{P}_n \leftrightarrow \mathcal{P}_n$:
  if $\mu \in \mathfrak{P}_n$ its generating polynomial
  $g_\mu(z) = \int z^S d\mu = \sum_{S \subseteq [n]} \mu(S) z^S \in \mathcal{P}_n$.

- $\mu \in \mathfrak{P}_n$ is (strongly) Rayleigh if g_μ is (strongly) Rayleigh.

Positive dependence

Positive Association (PA)

$\mu \in \mathfrak{P}_n$ is PA if $\int FG d\mu \geq \int F d\mu \cdot \int G d\mu$ for all increasing $F, G : 2^{[n]} \to \mathbb{R}$.

Positive Lattice Condition (PLC)

$\mu \in \mathfrak{P}_n$ satisfies PLC if for all $S, T \subseteq [n]$
$\mu(S \cup T) \cdot \mu(S \cap T) \geq \mu(S) \cdot \mu(T)$.

FKG Theorem

PLC $\Longrightarrow$ PA.

Negative dependence

Negative Association (NA)

$\mu \in \mathfrak{P}_n$ is NA if $\int FG d\mu \leq \int F d\mu \cdot \int G d\mu$ for all increasing $F, G : 2^{[n]} \to \mathbb{R}$ depending on disjoint sets of variables.

R. Pemantle & others: We need a theory of Negative Dependence. Under what conditions do we have NA?

Closure properties and CNA+

Strongly Rayleigh polynomials are closed under

(1). taking limits

(2). partial differentiation (conditioning on $X_j = 1$)

(3). scaling of variables with positive numbers (external fields)

(4). setting variables equal to nonnegative numbers (projections and conditioning on $X_j = 0$)

CNA+ := NA + (2) + (3) + (4)

Theorem [J.B., P. Brändén, T. Liggett].
Strongly Rayleigh measures are CNA+.

The strongly Rayleigh class includes e.g. uniform random spanning tree measures, product meaures, balls-and-bins measures, determinantal measures.

Example: determinantal measures

Let A be an $n \times n$ matrix with $A(S) \geq 0$, $S \subseteq [n]$, $A(S) :=$ principal minor of A indexed by $S' = [n] \setminus S$.

Determinantal measure $\mu = \mu_A \in \mathfrak{P}_n$ defined by

$$\mu(S) = A(S) / \det(I + A), \quad S \subseteq [n].$$

Occurences:

Number theory: Montgomery, Conrey,...

Rep. theory and random permutations: Johansson, Borodin-Okounkov-Olshanski-Reshetikhin, ...

Probability theory: Lyons-Steif, Peres, ...

Mathematical physics: Daley, Vere-Jones, ...

The generating polynomial of μ_A is

$$g_{\mu_A}(z) = \det(I + A)^{-1} \cdot \det(Z + A),$$

where $Z = \mathrm{diag}(z_1, \ldots, z_n)$.

Hadamard-Fischer-Kotelyansky inequalities

Let $A = (a_{ij})$ be positive semidefinite ($A \geq 0$).

Hadamard: $\det(A) \leq a_{11} \cdots a_{nn}$

Fischer: $\det(A) \leq A(S) \cdot A(S')$

Kotelyansky: $A(S \cup T) \cdot A(S \cap T) \leq A(S) \cdot A(T)$

$\Uparrow$

Theorem [R. Lyons]. If $A \geq 0$ then μ_A is CNA+.

$\Uparrow$

Theorem [J.B., P. Brändén, T. Liggett].
Strongly Rayleigh measures are CNA+.

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Theorem [J.B., P. Brändén]. If B is Hermitian and $A_1, \ldots, A_n \geq 0$ (all of the same order) then

$$f(z_1, \ldots, z_n) = \det(z_1 A_1 + \ldots + z_n A_n + B)$$

is stable. In particular, if $A \geq 0$ then $g_{\mu_A}(z)$ is stable (= strongly Rayleigh).

Example: graphs, Laplacians, spanning trees

Let $G = (V, E)$, $V = [n]$, be a graph, and e an edge connecting $i < j$. Let A_e be the $n \times n$ matrix with nonzero entries $(A_e)_{ii} = (A_e)_{jj} = 1$, $(A_e)_{ij} = (A_e)_{ji} = -1$ ($\Rightarrow A_e \geq 0$). The Laplacian of G is

$$L(G) = \sum_{e \in E} w_e A_e.$$

Let $f_G(z, w) = \det(L(G) + Z)$, where $w = (w_e)_{e \in E}$, $z = (z_1, \ldots, z_n)$ and $Z = \mathrm{diag}(z_1, \ldots, z_n)$.

Kirchhoff's Matrix-Tree Theorem:

If G is connected then

$$\det(L(G)_{ii}) = \frac{\partial f_G}{\partial z_i}(z, w)\Big|_{z=0} = \sum_T w^T$$

where the sum is over all spanning trees.

All Minors Matrix-Tree Theorem:

$$f_G(z, w) = \sum_{\mathcal{F}} z^{\mathrm{roots}(\mathcal{F})} w^{\mathrm{edges}(\mathcal{F})}$$

where the sum is over all rooted spanning forests.

A positive answer to a conjecture of Wagner

For each spanning forest $\mathcal{F}$ let

$$\rho(\mathcal{F}) = \#\text{ of ways to root } \mathcal{F}$$

and define $\mu \in \mathfrak{P}_{|E|}$ by

$$\mu(S) = \begin{cases} \rho(\mathcal{F})/R \text{ if } S \text{ is a spanning forest,} \\ 0 \text{ otherwise,} \end{cases}$$

where $R = \sum_{\mathcal{F}} \rho(\mathcal{F})$.

Conjecture [D. Wagner]. μ is Rayleigh.

Proof. The generating polynomial of μ is

$$g_\mu(w) = \sum_{S \subseteq E} \mu(S) w^S = R^{-1} \det(L(G) + I)$$

$$= R^{-1} f_G(z, w)\Big|_{z_1 = \cdots = z_n = 1}$$

so g_μ is in fact stable/strongly Rayleigh.

Example: the symmetric exclusion process

Let $\tau = (ij)$ be a transposition and for $S \subseteq [n]$ let $\tau(S) = \{\tau(s) : s \in S\}$. Given $\mu \in \mathfrak{P}_n$ and $p \in [0, 1]$ define

$$\mu^{\tau,p}(S) := p\mu(S) + (1 - p)\mu(\tau(S)).$$

Theorem [J.B., P. Brändén, T. Liggett].

μ is strongly Rayleigh $\Longrightarrow$ $\mu^{\tau,p}$ is strongly Rayleigh.

In particular, this proves the following

Conjecture [R. Pemantle, T. Liggett]. If the initial configuration of a symmetric exclusion process is deterministic then the distribution at time t is NA $\forall\, t \geq 0$.

Strongly Rayleigh $\Rightarrow$ CNA+: main ideas

1. Symmetric homogenization: let $\mu \in \mathfrak{P}_n$ and define $\mu_H \in \mathfrak{P}_{2n}$ by

$$\mu_H(S) = \begin{cases} \frac{\mu\left(S \cap [n]\right)}{\binom{n}{|S\cap[n]|}} & \text{if } |S| = n, \\ 0 \text{ otherwise,} \end{cases}$$

for $S \subseteq [2n]$. The g.p. $g_{\mu_H} = g_{\mu_H}(z_1, \ldots, z_{2n})$ is

$$g_{\mu_H} = \sum_{S \subseteq [n]} \frac{\mu(S)}{\binom{n}{|S|}} z^S e_{n-|S|}(z_{n+1}, \ldots, z_{2n}),$$

where $z = (z_1, \ldots, z_n)$ and

$$e_j(w_1, \ldots, w_m) = \sum_{1 \le i_1 < \cdots < i_j \le m} w_{i_1} \cdots w_{i_j}$$

is the j-th elementary symmetric polynomial.

Theorem [J.B., P. Brändén, T. Liggett].
μ strongly Rayleigh $\Longleftrightarrow$ μ_H strongly Rayleigh.

Sketch of proof:

$\sum_{S\subseteq[n]} \mu(S)z^S$ is stable $\Longrightarrow$

{Using Gårding's results on hyperbolicity cones}

$\sum_{S\subseteq[n]} \mu(S)z^S y^{n-|S|}$ is stable $\Longrightarrow$

{Grace-Walsh-Szegö Coincidence Theorem}

$\sum_{S\subseteq[n]} \mu(S)\binom{n}{|S|}^{-1} z^S e_{n-|S|}(z_{n+1},\dots,z_{2n})$
is stable.

2. Feder-Mihail Theorem: Let $\mathcal{S}$ be a class of probability measures such that

- $\mathcal{S}$ is closed under conditioning,
- $\mu(X_iX_j) \leq \mu(X_i)\cdot\mu(X_j)$ for all $i \neq j$, $\mu \in \mathcal{S}$,
- any $\mu \in \mathcal{S}$ has a homogeneous gen. polynomial.

Then every $\mu \in \mathcal{S}$ is NA.

3. Facts:
(I) (Strongly) Rayleigh measures with homogeneous gen. polynomials meet the requirements.

(II) NA is closed under projections.

More (positive) results on negative dependence

- Proof of Pemantle's conjecture on stochastic domination for truncations of strongly Rayleigh measures and counterexamples for other classes of CNA+ measures (J.B., P. Brändén, T. Liggett).

- Extensions of Lyons' results on NA, the Löwner order and stochastic domination for determinantal measures (J.B., P. Brändén, T. Liggett).

- **Conjecture** [Pemantle, Wagner].
  Rayleigh = CNA+.

  True for e.g. exchangeable measures (Pemantle).

  Proof for the class of almost exchangeable measures (J.B., P. Brändén, T. Liggett).

- Distributional limits for the symmetric exclusion process (T. Liggett), the HPP for certain matroids studied by A. Sokal, D. Wagner (J.B.).

Negative results on negative dependence

Let $(a_k)_{k=0}^n$ be a sequence of nonnegative numbers.

Log-Concave (LC)

$$a_k^2 \geq a_{k-1}a_{k+1}$$

Ultra-Log-Concave (ULC)

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}$$

$$i < j < k, a_i a_k \neq 0 \Longrightarrow a_j \neq 0$$

The rank sequence of $\mu \in \mathfrak{P}_n$ is $(a_k)_{k=0}^n$:

$$a_k = \mu\left(\sum_{i=1}^n X_i = k\right), \text{ i.e., } g_\mu(t, \ldots, t) = \sum_{k=0}^n a_k t^k,$$

where $g_\mu(z_1, \ldots, z_n) = \sum_{S \subseteq [n]} \mu(S) z^S$ is the generating polynomial of μ.

μ is ULC if its rank sequence is ULC.

Fact

$g(z_1, \ldots, z_n)$ is stable $\Rightarrow g(t, \ldots, t)$ is real-rooted.

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Newton's Inequalities

If $\sum_{k=0}^{m} a_k t^k$ is real-rooted then $(a_k)_{k=0}^{m}$ is ULC.

$\Downarrow$

If $\mu \in \mathfrak{P}_n$ is strongly Rayleigh then μ is ULC.

"**Big Conjecture**" [R. Pemantle, Y. Peres, D. Wagner].
If $\mu \in \mathfrak{P}_n$ is Rayleigh then μ is ULC.

(would $\Rightarrow$ Mason's conjecture for Rayleigh matroids)

Conjecture [R. Pemantle, Y. Peres].
If $\mu \in \mathfrak{P}_n$ is CNA+ then μ is ULC.

Counterexamples [J. B., P. Brändén, T. Liggett].
For any $n \geq 20$ $\exists$ almost exchangeable $\mu \in \mathfrak{P}_n$ for which both the "Big Conjecture" and CNA+ $\Rightarrow$ ULC fail.