Chromatic zeros for some recursively defined families of graphs

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Cambridge 2008

Introduction

Definition

The chromatic polynomial $P(G, \lambda)$ of a graph G in the variable λ counts for positive integers λ the proper vertex $\lambda-$colourings of G.

Problem:

In this paper we study the problem of the location of complex zeros of chromatic polynomials (chromatic complex zeros) for some families of graphs. The problem is related to some phenomenon in statistical mechanic (see Sokal [10, 11] with references).

A list of additional interesing results in this area one can find in [1], [3] -[6], [9] -[11]. In this paper we study the location of complex zeros for some medial graphs. The medial graphs play a significiant role in a study of Petrie walks [12].

Reduction Formulas:

In computing chromatic polynomials, we make use of Whitney's reduction formula given in [9]. The formula is

$$P(G,\lambda) = P(G_{-e},\lambda) - P(G/_e,\lambda) \quad (1)$$

or equivalently

$$P(G_{-e},\lambda) = P(G,\lambda) + P(G/_e,\lambda) \quad (2)$$

where G_{-e} is the graph obtained from G by deleting an edge e and $G/_e$ is the graph obtained from G by contracting the edge e.

We also make use of gluing formula given in [9]. The formula is

$$P(G,\lambda) = P(H,\lambda)P(F,\lambda)/P(K_p,\lambda) \quad (3)$$

where G is a gluing of two disjoint graphs H and F over their complete subgraph K_p for $p \geqslant 1$.

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Chromatic polynomial formula for some medial graphs

Definition

A medial graph Med(G) is defined for a plane embedding of a planar graph G as follows: $V(Med(G)) = E(G)$ and two vertices in Med(G) are adjacent if and only if the respective edges are incident and belong to the boundary of the same region of the embedding of G, and one edge per the rotation around each common end-vertex of the edges is setting for each common region. Thus medial graphs are 4-regular and they can have loops and multiple edges.

Definition

Let us define quasi-medial graph $\mathrm{M(G)}$ as the graph constructed from $\mathrm{Med(G)}$ by deleting loops and parallel edges.

We count the chromatic polynomial $\mathrm{P(Med(G)}, \lambda)$ as $\mathrm{P(M(G)}, \lambda)$.

An infinite family of hamiltonian medial graphs of sufficiently large order with chromatic complex zeros in the disk $|z - d/2| \leqslant r$ is presented, where d is the average degree of a graph and $r > r_0$ with r_0 as maximum positive root of some polynomial equation.

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Figure 1. The quasi-medial graph of a cartesian product of P_{n+1} and K_2 for $n \geqslant 1$.

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Moreover, F_n is the graph obtained from G_n by contracting the edge $\{0, 1\}$, and H_n is the graph obtained from G_n by extending the vertex 0 (see Figure 1).

Let $G_n(\lambda)$ denote the chromatic polynomial of the graph G_n, $F_n(\lambda)$ denote the chromatic polynomial of the graph F_n and $H_n(\lambda)$ denote the chromatic polynomial of the graph H_n.

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Note that

$$G_1(\lambda) = P(C_4, \lambda) = \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2) =$$

$$\lambda(\lambda-1)\left(\lambda^2-3\lambda+3\right)$$

$$F_1(\lambda) = P(K_3, \lambda) = \lambda(\lambda-1)(\lambda-2)$$

and

$$H_1(\lambda) = P(K_3,\lambda)P(K_4,\lambda)/P(K_2,\lambda) + P(K_3,\lambda)P(K_2,\lambda)/P(K_1,\lambda)$$

$$= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3) + \lambda(\lambda-1)^2(\lambda-2)$$

$$= \lambda(\lambda-1)(\lambda-2)\left(\lambda^2-4\lambda+5\right)$$

By Whitney's formula we get

$$\mathrm{G_n}(\lambda) = \mathrm{G_{n-1}}(\lambda)(\mathrm{P}(\mathrm{K_3},\lambda)/\mathrm{P}(\mathrm{K_2},\lambda))^2\mathrm{P}(\mathrm{K_2},\lambda)/\mathrm{P}(\mathrm{K_1},\lambda) - \mathrm{F_n}(\lambda),$$

$$\mathrm{F_n}(\lambda) = \mathrm{G_{n-1}}(\lambda)(\mathrm{P}(\mathrm{K_3},\lambda)/\mathrm{P}(\mathrm{K_2},\lambda))^2 - \mathrm{H_{n-1}}(\lambda).$$

and

$$\mathrm{H_n}(\lambda) = \mathrm{F_n}(\lambda)\mathrm{P}(\mathrm{K_4},\lambda)/\mathrm{P}(\mathrm{K_2},\lambda) + \mathrm{H_{n-1}}(\lambda)\mathrm{P}(\mathrm{K_3},\lambda)/\mathrm{P}(\mathrm{K_1},\lambda).$$

We can express this fact by the following matrix equation.

$$\begin{pmatrix} G_n(\lambda) \\ H_n(\lambda) \\ F_n(\lambda) \end{pmatrix} = A \begin{pmatrix} G_{n-1}(\lambda) \\ H_{n-1}(\lambda) \\ F_{n-1}(\lambda) \end{pmatrix}, \tag{4}$$

where $n \geqslant 2$ and the transfer matrix A is defined as follows:

$$A = \begin{pmatrix} (\lambda-2)^3 & 1 & 0 \\ (\lambda-2)^3(\lambda-3) & 2(\lambda-2) & 0 \\ (\lambda-2)^2 & -1 & 0 \end{pmatrix}.$$

Note that (4) presents the linear recurrence relation.

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Thus

$$\begin{pmatrix} G_n(\lambda) \\ H_n(\lambda) \\ F_n(\lambda) \end{pmatrix} = A^{n-1} \begin{pmatrix} G_1(\lambda) \\ H_1(\lambda) \\ F_1(\lambda) \end{pmatrix}, \; n \geqslant 2. \tag{5}$$

Therefore, by (5), we get the following formulas for $n \geqslant 2$

$$\begin{aligned} G_n(\lambda) &= g_1\lambda_1^{n-1} + g_2\lambda_2^{n-1}, \\ H_n(\lambda) &= h_1\lambda_1^{n-1} + h_2\lambda_2^{n-1}, \\ F_n(\lambda) &= f_1\lambda_1^{n-1} + f_2\lambda_2^{n-1}, \end{aligned} \tag{6}$$

where λ_1 and λ_2 are nonzero eigenvalues of the transfer matrix A and g_i, h_i, f_i, for $i = 1, 2$ are some coefficients in variable λ.

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The nonzero eigenvalues of A are exactly the eigenvalues of the submartix B of A, namely

$$\mathrm{B} = \begin{pmatrix} (\lambda-2)^3 & 1 \\ (\lambda-2)^3(\lambda-3) & 2(\lambda-2) \end{pmatrix}.$$

The eigenvalues of the transfer matrix B are as follows:

$$\lambda_1 = \tfrac{1}{2}\lambda^3 - 3\lambda^2 + 7\lambda - 6 + \tfrac{1}{2}\mathrm{u} = \tfrac{1}{2}(\lambda-2)(\lambda^2 - 4\lambda + 6 + \mathrm{u}_1),$$
$$\lambda_2 = \tfrac{1}{2}\lambda^3 - 3\lambda^2 + 7\lambda - 6 - \tfrac{1}{2}\mathrm{u} = \tfrac{1}{2}(\lambda-2)(\lambda^2 - 4\lambda + 6 - \mathrm{u}_1),$$

where

$$\mathrm{u}_1 = \sqrt{(\lambda^4 - 8\lambda^3 + 24\lambda^2 - 36\lambda + 28)}$$
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The eigenvalues are different outside the following roots:

$$\begin{pmatrix} .94778 - 1.4344i \\ .94778 + 1.4344i \\ 3.0522 - .39596i \\ 3.0522 + .39596i \\ 2.0 \\ 2.0 \end{pmatrix}$$

Evidently

$$\begin{pmatrix} \mathrm{G}_1(\lambda) \\ \mathrm{H}_1(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda(\lambda-1)(\lambda^2-3\lambda+3) \\ \lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5) \end{pmatrix} = \begin{pmatrix} \mathrm{g}_1+\mathrm{g}_2 \\ \mathrm{h}_1+\mathrm{h}_2 \end{pmatrix}.$$

By (5), in particular we get the formula

$$\begin{pmatrix} \mathrm{G}_2(\lambda) \\ \mathrm{H}_2(\lambda) \end{pmatrix} = \mathrm{B}\begin{pmatrix} \lambda(\lambda-1)(\lambda^2-3\lambda+3)) \\ \lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5) \end{pmatrix} = \begin{pmatrix} \mathrm{g}_1\lambda_1+\mathrm{g}_2\lambda_2 \\ \mathrm{h}_1\lambda_1+\mathrm{h}_2\lambda_2 \end{pmatrix}.$$

Thus

$$g_1 = \frac{1}{2}\lambda(\lambda-1)\left(\left(\lambda^2-3\lambda+3\right)+(\lambda-2)\left(\lambda^3-5\lambda^2+9\lambda-8\right)/u_1\right)$$

$$g_2 = \frac{1}{2}\lambda(\lambda-1)\left(\left(\lambda^2-3\lambda+3\right)-(\lambda-2)\left(\lambda^3-5\lambda^2+9\lambda-8\right)/u_1\right)$$

and

$$h_1 = \frac{1}{2}\lambda_{(3)}\left(\left(\lambda^2-4\lambda+5\right)+\left(\lambda^4-8\lambda^3+25\lambda^2-38\lambda+26\right)/u_1\right)$$

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where $\lambda_{(3)} = \lambda(\lambda-1)(\lambda-2)$.

Recall that $u_1 = \sqrt{\left(\lambda^4-8\lambda^3+24\lambda^2-36\lambda+28\right)}$

Similarly we get the coefficients f_1 and f_2

$$f_1 = \frac{1}{2}\lambda(\lambda-1)(\lambda-2)\left(1+\left(\lambda^3-6\lambda^2+12\lambda-10\right)/u\right)$$

$$f_2 = \frac{1}{2}\lambda(\lambda-1)(\lambda-2)\left(1-\left(\lambda^3-6\lambda^2+12\lambda-10\right)/u\right)$$

Recall that $u = (\lambda-2)\,u_1$, where

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Another family of medial graphs

Figure 1(a). The graph $L_{n,p}$ and the graph $B_{n,p}$

Let $\Theta_{2,2,p-1}$ be a theta graph with the paths of length 2,2 and $p-1$ between two vertices of degree 3, where $p \geqslant 2$.

For integer n, $n \geqslant 1$, let us denote by C_n a cycle of order n. C_1 is a graph with one vertex and one loop. C_2 is a graph with two vertices and two parallel edges.

Let $L_{n,p}$ denote the graph obtained from a cycle C_n by replacing each edge by the graph $\Theta_{2,2,p-1}$ as follows: delete an edge, say uv, and then glue the middle vertices of the paths of length 2 of $\Theta_{2,2,p-1}$ into the vertices u and v, as it is presented in Figure 1(a).

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Similarly, let $\mathrm{B}_{\mathrm{n,p}}$ denote the graph obtained from a cycle $\mathrm{C}_{\mathrm{n+1}}$ by replacing n edges by the graph $\Theta_{2,2,\mathrm{p}-1}$.

Let $\mathrm{C}_{\mathrm{2n,p}}$ be a graph obtained from a cycle C_{2n} by replacing each edge of a perfect matching of C_{2n} by p edges in parallel.

We can note, that $\mathrm{L}_{\mathrm{n,p}}$ is a medial graph for a plane $(\mathrm{p}+1)$-regular graph $\mathrm{C}_{\mathrm{2n,p}}$.

Thus $\mathrm{L}_{\mathrm{n,p}} = \mathrm{M}(\mathrm{C}_{\mathrm{2n,p}})$.

Let $\mathrm{l}_{\mathrm{n,p}}(\lambda)$ denote the chromatic polynomial of the graph $\mathrm{L}_{\mathrm{n,p}}$ and $\mathrm{b}_{\mathrm{n,p}}(\lambda)$ denote the chromatic polynomial of the graph $\mathrm{B}_{\mathrm{n,p}}$.

Set $l_{0,p}(\lambda) = \lambda,\ b_{0,p}(\lambda) = 0,\ s(p,\lambda) = l_{1,p}(\lambda)/\lambda$ and $h(p,\lambda) = b_{1,p}(\lambda)/(\lambda(\lambda-1))$.

Proposition (Bielak)

For each integer n, $n \geqslant 0$, the polynomials $l_{n,p}(\lambda)$ in integer variable λ can be expressed as follows:

$$l_{n,p}(\lambda) = (f(p,\lambda))^n + (\lambda-1)(g(p,\lambda))^n,$$

where

$$f(p,\lambda) = P(\Theta_{2,2,p-1},\lambda)/\lambda$$

and

$$g(p,\lambda) = s(p,\lambda) - h(p,\lambda) \ = $$

$$(P(C_{p+1},\lambda) + (\lambda-2)P(C_p,\lambda))/(\lambda(\lambda-1)).$$

Figure 1(b). Reduction of $L_{n,p}$ and $B_{n,p}$. The symbol $\odot$ denotes a vertex gluing of the graphs. A dotted line denotes a path.

Proof of Proposition.

For $\lambda = 0, 1, 2$ the result is evident. So we assume that $\lambda \neq 0, 1, 2$. We apply the reduction formula of Whitney (2) for the non edge $\mathrm{e} = \mathrm{uv}$ in $\mathrm{L_{n,p}}$ and for the non edge $\mathrm{e}' = \mathrm{uw}$ in $\mathrm{B_{n,p}}$, then we apply the gluing formula (3).

Note that $\mathrm{L_{n,p}}/_\mathrm{e}$ is obtained by gluing to $\mathrm{L_{n-1,p}}$ a copy of $\mathrm{C_{p+1}}$ into the vertex $\mathrm{u} = \mathrm{v}$. Similarly, $\mathrm{L_{n,p}} + \mathrm{e}$ is obtained by gluing to $\mathrm{B_{n-1,p}}$ a copy of $\mathrm{B_{1,p}}$ at the edge e with end vertices u, v (see Figure 1(b)).

Moreover, $\mathrm{B_{n,p}}/_{\mathrm{e}'}$ is obtained by gluing to $\mathrm{L_{n-1,p}}$ a copy of $\mathrm{B_{1,p}}$ into the vertex $\mathrm{u} = \mathrm{w}$. Similarly, $\mathrm{B_{n,p}} + \mathrm{e}'$ is obtained by gluing to $\mathrm{B_{n-1,p}}$ a copy of $\mathrm{B^+_{1,p}}$ at the edge $\mathrm{e}' = \mathrm{uw}$, where $\mathrm{B^+_{1,p}}$ is the graph obtained from $\mathrm{B_{1,p}}$ by an elementary subdivision of the edge uv (see Figures 1(a) and 1(b)).

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Proof of Proposition ....

Recall that $l_{0,p}(\lambda) = \lambda$ and $b_{0,p}(\lambda) = 0$.
Thus we get the following recurrence relation for $n \geqslant 1$:

$$\begin{pmatrix} l_{n,p}(\lambda) \\ \\ b_{n,p}(\lambda) \end{pmatrix} = A \begin{pmatrix} l_{n-1,p}(\lambda) \\ \\ b_{n-1,p}(\lambda) \end{pmatrix}, \tag{7}$$

where the transfer matrix A is defined as follows:

$$A = \begin{pmatrix} s(p,\lambda) & h(p,\lambda) \\ \\ (\lambda-1)h(p,\lambda) & f(p,\lambda) - h(p,\lambda) \end{pmatrix}.$$

Moreover, by the formula (1), we get $B_{n,p} = S_{n,p} - L_{n,p}$, where $S_{n,p} = B_{n,p} - \{uv\}$. Note that, by the formula (3), $P(S_{n,p}, \lambda) \ = \ (P(\Theta_{2,2,p-1}, \lambda))^n / \lambda^{n-1}$.

Proof of Proposition ....

Therefore, by (7), we get the following linear recurrence relation

$$l_{n,p}(\lambda) = g(p,\lambda) l_{n-1,p} \; + \; \lambda h(p,\lambda)(f(p,\lambda))^{n-1}, \quad n \geqslant 1.$$

Set $a = \lambda h(p,\lambda)/f(p,\lambda)$ and $d = \lambda(f(p,\lambda) - h(p,\lambda))/f(p,\lambda)$.
If $g(p,\lambda) \neq 0$ then we can write

$$l_{n,p}(\lambda) = g(p,\lambda) l_{n-1,p} \; + \; a(f(p,\lambda))^n + d[n=0], \quad n \geqslant 0.$$

We use generating functions for solving the above recursive relation. The rational generating function $F(z)$ in z for the above case is described as follows

$$F(z) = \frac{a \sum_n (f(p,\lambda))^n z^n}{1 - g(p,\lambda)z} + \frac{d}{1 - g(p,\lambda)z}.$$

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Proof of Proposition ....

So by theory of generating functions $l_{n,p}$ is the n−th coefficient of the formal power series for F(z). Thus we get

$$l_{n,p}(\lambda) = a(g(p,\lambda))^n \sum_{0 \leqslant j \leqslant n} (f(p,\lambda)/g(p,\lambda))^j + d(g(p,\lambda))^n.$$

Hence finally, if $g(p,\lambda) \neq 0$ and $f(p,\lambda) - g(p,\lambda) \neq 0$ then

$$l_{n,p}(\lambda) = (f(p,\lambda))^n + (\lambda - 1)(g(p,\lambda))^n.$$

(If λ is positive integer variable then the above condition holds. If λ is real or complex variable, we get some critical points for $g(p,\lambda) = 0$ and $g(p,\lambda) = f(p,\lambda)$.) □

Location of chromatic zeros for some medial graphs

Now let us consider $l_{n,p}(z)$ in complex variable z. We show the location of zeros of $l_{n,p}(z)$ for $p = 4, 3, 2$ for sufficiently large n.

By Proposition we have

$$f(p, \lambda) = (\lambda - 1)\left((\lambda - 3)(\lambda - 1)^{p-1} - 2(-1)^p + 3\frac{(\lambda-1)^{p-1} + (-1)^p}{\lambda}\right)$$

and

$$g(p, \lambda) = 2(\lambda - 1)^{p-1} + (-1)^p - 3\frac{(\lambda-1)^{p-1} + (-1)^p}{\lambda}.$$

Equivalently

$$f(p, \lambda) = (\lambda - 1)\left((\lambda - 1)^p - (2\lambda - 3)\frac{(\lambda-1)^{p-1} + (-1)^p}{\lambda}\right).$$

Find $|z - \frac{p+3}{p+1}| \leqslant r$ such that $|f(p, \lambda)| > |g(p, \lambda)|$.

Let us consider the case $p = 4$.

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and

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Equivalently

$$f(p, \lambda) = (\lambda - 1)\left((\lambda - 1)^{p} - (2\lambda - 3)\frac{(\lambda-1)^{p-1}+(-1)^{p}}{\lambda}\right).$$

Find $|z - \frac{p+3}{p+1}| \leqslant r$ such that $|\, f(p, \lambda)| > |\, g(p, \lambda)|$.

Let us consider the case $p = 4$.

Theorem (Bielak 2007)

For all sufficiently large n all chromatic zeros of $l_{n,4}(z)$ in complex variable z lie in the disk $|z - \frac{7}{5}| \leqslant r$, where $r > r_0$ and $r_0 - 2/5$ is maximum positive root of the equation $x^5 - \frac{6}{5}x^4 - \frac{23}{5}x^3 - \frac{166}{25}x^2 - \frac{202}{25}x - \frac{102}{25} = 0$.

Proof of Theorem 2:
The average degree of $L_{n,4}$ is equal to 14/5. Thus we consider the location of the complex zeros in the disk: $|z - 7/5| \leqslant r$ (Biggs [5]).

Considering the circle $|z - 7/5| = r$, we can apply well known Rouché's theorem for monomorphic functions (see Proposition 2.2.6 in [2]).

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Rouché's theorem for monomorphic functions

Let Ω be an open subset of the complex plane, and α be a Jordan curve in Ω.
Let f, g be meromorphic functions in Ω without poles on α such that

$$|f(z) - g(z)| < |g(z)| \quad (z \in \alpha).$$

Then

$$n_{Z(f)} - n_{P(f)} = n_{Z(g)} - n_{P(g)},$$

where

$n_{Z(h)}$ is the number of zeros of h in $\mathrm{Int}(\alpha)$,
$n_{P(h)}$ is the number of poles of h in $\mathrm{Int}(\alpha)$,
for $h = f,\ g$, counted according to their multiplicities.

Proof of Theorem 2 ....

By Proposition for $p = 4$, we get

$$f(4, z) = (z-1)(z-2)\left(z^3 - 4z^2 + 7z - 5\right)$$

and

$$g(4, z) = 2(z-1)^3 + 1 - 3\left(z^2 - 3z + 3\right).$$

Thus

$$\begin{aligned} f(4, z) &= (z-1)(z-2)\left((z-1)^3 - (z-2)^2\right) \\ &= (z-1)(z-2)\left((z-1)^3 - (z-1)^2 - 2(z-1) - 1\right) \end{aligned}$$

and

$$g(4, z) = (z-2)\left(2(z-1)^2 - (z-1) + 2\right).$$

Proof of Theorem 2 ....

Let $|z - \frac{p+3}{p+1}| = |z - \frac{7}{5}| = r$ and $x = r - \frac{2}{p+1} = r - \frac{2}{5}$.

So

$$|f(4,z)| \geqslant \left(x - \frac{1}{5}\right) x \left(x \left(x^2 - \left(x + \frac{4}{5}\right) - 2\right) - 1\right)$$
$$|g(4,z)| \leqslant (x+1)\left(2\left(x + \frac{4}{5}\right)^2 + \left(x + \frac{4}{5}\right) + 2\right).$$

Note that the inequality $|f(4,z)| \geqslant |g(4,z)|$ is satisfied if

$$x^5 - \frac{6}{5}x^4 - \frac{23}{5}x^3 - \frac{166}{25}x^2 - \frac{202}{25}x - \frac{102}{25} \geqslant 0.$$

Proof of Theorem 2 ....

Let x be the maximum positive root of the following equation

$$x^5 - \frac{6}{5}x^4 - \frac{23}{5}x^3 - \frac{166}{25}x^2 - \frac{202}{25}x - \frac{102}{25} = 0.$$

The numerical solution is $x = 3.3814$.

Set $r_0 = x + \frac{2}{5} = 3.3814 + \frac{2}{5} = 3.7814$

Recall that the center of the circle is: $\left(\frac{7}{5}, 0\right)$

$\frac{7}{5} + 3.7814 = 5.1814$ and $\frac{7}{5} - 3.7814 = -2.3814$

All the zeros of $f(4, z)$ lie in the circle $|z| \leqslant 2$. So they lie in the circle $|z - \frac{7}{5}| \leqslant r$, whenever $r > r_0$.

Thus all the zeros of $l_{n,4}(z)$ lie in the circle $|z - \frac{7}{5}| \leqslant r$ for some $r > r_0$ and for all sufficiently large n. $\Box$

Let $p = 3$.

Theorem (Bielak)

For all sufficiently large n all chromatic zeros of $l_{n,3}(z)$ in complex variable z lie in the disk $|z - 3/2| \leqslant r$, where $r > r_0$ and $r_0 - 1/2$ is maximum positive root of the equation $x^4 - 2x^3 - 5x^2 - 6x - 3 = 0$.

Let $p = 2$.

Theorem (Bielak 2007)

For all sufficiently large n all chromatic zeros of $l_{n,2}(z)$ in complex variable z lie in the disk $|z - 5/3| \leqslant r$, where $r > r_0$ and $r_0 - 1/3$ is maximum positive root of the equation $3x^3 - x^2 - 6x - 4 = 0$.

Let $p = 3$.

Theorem (Bielak)

For all sufficiently large n all chromatic zeros of $l_{n,3}(z)$ in complex variable z lie in the disk $|z - 3/2| \leqslant r$, where $r > r_0$ and $r_0 - 1/2$ is maximum positive root of the equation $x^4 - 2x^3 - 5x^2 - 6x - 3 = 0$.

Let $p = 2$.

Theorem (Bielak 2007)

For all sufficiently large n all chromatic zeros of $l_{n,2}(z)$ in complex variable z lie in the disk $|z - 5/3| \leqslant r$, where $r > r_0$ and $r_0 - 1/3$ is maximum positive root of the equation $3x^3 - x^2 - 6x - 4 = 0$.

Proof of Theorem 3:

The average degree of $L_{n,3}$ is equal to 3.
Thus we consider the location of the complex zeros in the disk: $|z - 3/2| \leqslant r$ (see[5]).
Considering the circle $|z - 3/2| = r$, we can apply well known Rouché's theorem for monomorphic functions (see Proposition 2.2.6 in [2]). So, by Proposition for $p = 3$, we get

$$|f(3, z)| = |z - 1||(z - 1)^3 - (z - 2)(2z - 3)| \geqslant$$

$$(r - 1/2)^4 - 2r(r + 1/2)(r - 1/2)$$

and

$$|(z - 1)(g(3, z))^n| = |z - 1||2(z - 2)^2 + (z - 1)|^n \leqslant$$

$$(r + 1/2)(2(r + 1/2)^2 + r + 1/2)^n.$$

Proof of Theorem 3 ....

Thus for all sufficiently large n we have $|f(3,z))|^n > |g(3,z)|^n$ on $|z-3/2| = r$. Set $x = r - 1/2$ and solve the eguation (8)

$$x^4 - 2x^3 - 5x^2 - 6x - 3 = 0. \tag{8}$$

Set $r > r_0$, where $r_0 - 1/2$ is maximum positive root of the above equation (8).
Moreover, note that zeros of $f(3,z)$ are in the disk $|z| < 2.75$.
It follows by the fact that the polynomial
$f(3,z)/(z-1) + 53/512 = z^3 - 5z^2 + 10z - 7 + 53/512$ has all zeros in the disk $|z| < 2.25$ (see theorem 1.4.21 [8]). Hence for all sufficiently large n all zeros of $l_{n,3}(z)$ also lie in the disk $|z-3/2| \leqslant r$, whenever $r > r_0$. Note that $r_0 \in (4.25, 4.375)$. $\square$

Proof of Theorem 4:

The average degree of $L_{n,2}$ is equal to 10/3. Thus we consider the location of the complex zeros in the disk: $|z - 5/3| \leqslant r$ (see[5]).
Considering the circle $|z - 5/3| = r$, we can apply well known Rouché's theorem for monomorphic functions (see Proposition 2.2.6 in [2]). So by Proposition for $p = 2$, we get

$$|f(2, z)| = |z - 1||z - 2|^2 \geqslant (r - 2/3)(r - 1/3)^2$$

and

$$|(z - 1)(g(2, z))^n| = \ |z - 1||2(z - 2)|^n \leqslant (r + 1/3)(2r + 2/3)^n.$$

Proof of Theorem 4 ....

Thus for all sufficiently large n we have $|f(2,z))|^n > |g(2,z)|^n$ on $|z-5/3| = r$. Set $x = r - 1/3$ and solve the eguation (9)

$$3x^3 - x^2 - 6x - 4 = 0. \tag{9}$$

Set $r > r_0$, where $r_0 - 1/3$ is maximum positive root of the above equation (9).
Moreover, note that zeros of $f(2,z)$ are in the disk $|z-1| \leqslant 1$.
Hence for all sufficiently large n all zeros of $l_{n,2}(z)$ also lie in the disk $|z-5/3| \leqslant r$, whenever $r > r_0$. Note that $r_0 \in (2, 2+1/6)$.
$\square$

More detailed information about the zeros for sufficiently large n follows from the theorem of Beracha et al. [1], Biggs [5] and from Sokal [11].

Their result says that the limit points of the zeros of a sequence of the polynomials of the form $l_{n,p}$ are the points on the curves where $f(p, n)$ and $g(p, n)$ are of equal modulus (together with some isolated points).

For the case $p = 2$, we get the limit points of the zeros in the set

$$\{\,1, 2\} \cup \{z : |z-1||z-2| = 2\}.$$

For the case $p = 3$, we get the limit points of the zeros in the set

$$\{\,1\} \cup \{z : |z-1||(z-1)^3-(z-2)(2z-3)| = |2(z-2)^2+(z-1)|\}.$$

For the case $p = 4$, we get the limit points of the zeros in the set

$$\{\,1, 2\} \cup \{z : |z-1||z-2||z^3-4z^2+7z-5| = |2\left(z-1\right)^3+1-3\left(z^2-3z+3\right)|\}.$$

Problem 1. Find the location of the chromatic zeros for the medial graph of a plane (p+1)-regular graph $C_{2n,p}$, $p > 3$. Note that average degree of $L_{n,p}$ is equal to $(6+2p)/(p+1)$. Thus we can consider the location of the complex zeros in the disk: $|z-(p+3)/(p+1)| \leqslant r$, for some real constant r.

Evidently $(p+3)/(p+1) = 1+2/(p+1)$. So the center of the circle tends to (1,0) if $p \to \infty$.

Chromatic polynomial formula for some planar quasi-quadriangulation

Figures 2-3. The graphs M_n, U_n, W_n, J_n, V_n, X_n.

Let $\mathrm{M_n},\ \mathrm{U_n},\ \mathrm{W_n},\ \mathrm{J_n},\ \mathrm{V_n}, \mathrm{X_n}$ be the graphs presented in Figures 2 and 3. Let $\mathrm{Z_n}$ be the graph obtained from $\mathrm{U_n}$ by deleting the vertex 0.

Note that $\mathrm{X_n}$ be obtained from $\mathrm{M_n}$ by adding the edge $\{\mathrm{a}, \mathrm{b}\}$.

Now we consider the chromatic polynomial for the graphs.

Let the symbol $\mathrm{W_{1,4}}$ denote the wheel of order 5, i.e., $\mathrm{W_{1,4}} = \mathrm{K_1} + \mathrm{C_4}$.
We need also the graphs $\mathrm{W_1},\ \mathrm{Z_1},\ \mathrm{U_1},\ \mathrm{V_1},\ \mathrm{L_1}$ and $\mathrm{J_1}$ presented in Figure 4.

Let $\mathrm{M_n},\ \mathrm{U_n},\ \mathrm{W_n},\ \mathrm{J_n},\ \mathrm{V_n}, \mathrm{X_n}$ be the graphs presented in Figures 2 and 3. Let $\mathrm{Z_n}$ be the graph obtained from $\mathrm{U_n}$ by deleting the vertex 0.

Note that $\mathrm{X_n}$ be obtained from $\mathrm{M_n}$ by adding the edge $\{a, b\}$.

Now we consider the chromatic polynomial for the graphs.

Let the symbol $W_{1,4}$ denote the wheel of order 5, i.e.,
$W_{1,4} = K_1 + C_4$.
We need also the graphs $W_1,\ Z_1,\ U_1,\ V_1,\ L_1$ and J_1 presented in Figure 4.

Let us define $\mathrm{V}_0 = \mathrm{K}_3,\ \mathrm{J}_0 = \mathrm{K}_2,\ \mathrm{W}_0 = \mathrm{K}_2$ and $\mathrm{Z}_0 = \mathrm{K}_2$.

Then by formula (2) applied to the $\{\mathrm{a}, 1''\}$ and/or to the $\{\mathrm{b}, 1''\}$ and by formula (3) we get the following system of equations, for $\mathrm{n} \geqslant 1$,

$$\mathrm{M_n}(\lambda) = \mathrm{P}(\mathrm{M_n} + \{\mathrm{ab}\}, \lambda) + \mathrm{U_n}(\lambda) = \mathrm{X_n}(\lambda) + \mathrm{U_n}(\lambda)$$

$$\mathrm{U_n}(\lambda) = \frac{\mathrm{W_1}(\lambda)}{\mathrm{K_2}(\lambda)} \mathrm{W_{n-1}}(\lambda) + \frac{\mathrm{C_4}(\lambda)}{\mathrm{K_1}(\lambda)} \mathrm{Z_{n-1}}(\lambda)$$

$$W_n(\lambda) = W_{n-1}(\lambda)\frac{W_{1,4}(\lambda)}{K_2(\lambda)} + Z_{n-1}(\lambda)\frac{K_3(\lambda)K_3(\lambda)}{K_1(\lambda)K_2(\lambda)}$$

$$X_n(\lambda) = \frac{V_1(\lambda)}{K_3(\lambda)}V_{n-1}(\lambda) + 2\frac{C_4(\lambda)K_3(\lambda)}{K_2(\lambda)K_2(\lambda)} J_{n-1}(\lambda)$$

$$Z_n(\lambda) = W_{n-1}(\lambda)\frac{K_3(\lambda)K_3(\lambda)}{K_2(\lambda)K_2(\lambda)} + Z_{n-1}(\lambda)\frac{K_2(\lambda)K_2(\lambda)}{K_1(\lambda)K_1(\lambda)}$$

$$V_n(\lambda) = \frac{L_1(\lambda)}{K_3(\lambda)}V_{n-1}(\lambda) + 2\frac{K_4(\lambda)K_3(\lambda)}{K_2(\lambda)K_2(\lambda)} J_{n-1}(\lambda)$$

$$J_n(\lambda) = \frac{K_3(\lambda)K_4(\lambda)}{K_3(\lambda)K_2(\lambda)}V_{n-1}(\lambda) + \left(\frac{K_2(\lambda)K_3(\lambda)}{K_2(\lambda)K_1(\lambda)} + \frac{K_3(\lambda)K_3(\lambda)}{K_2(\lambda)K_2(\lambda)}\right) J_{n-1}(\lambda)$$

Thus we can write the following linear relation for $\mathrm{V_n}$ and $\mathrm{J_n}$ for $\mathrm{n} \geqslant 1$.

$$\begin{pmatrix} \mathrm{V_n}(\lambda) \\ \\ \mathrm{J_n}(\lambda) \end{pmatrix} = \mathrm{T} \begin{pmatrix} \mathrm{V_{n-1}}(\lambda) \\ \\ \mathrm{J_{n-1}}(\lambda) \end{pmatrix} = \mathrm{T^{n-1}} \begin{pmatrix} \mathrm{V_1}(\lambda) \\ \\ \mathrm{J_1}(\lambda) \end{pmatrix} = \mathrm{T^n} \begin{pmatrix} \mathrm{V_0}(\lambda) \\ \\ \mathrm{J_0}(\lambda) \end{pmatrix}, \tag{10}$$

where

$$\mathrm{T} = \begin{pmatrix} (\lambda-3)^3 + (\lambda-2)^2 & 2(\lambda-2)^2(\lambda-3) \\ \\ (\lambda-2)(\lambda-3) & (\lambda-2)(2\lambda-3) \end{pmatrix}. \tag{11}$$

Similarly, we can write the following linear relation for $\mathrm{W_n}$ and $\mathrm{Z_n}$ for $\mathrm{n} \geqslant 1$.

$$\begin{pmatrix} \mathrm{W_n}(\lambda) \\ \mathrm{Z_n}(\lambda) \end{pmatrix} = \mathrm{N} \begin{pmatrix} \mathrm{W_{n-1}}(\lambda) \\ \mathrm{Z_{n-1}}(\lambda) \end{pmatrix} = \mathrm{N}^{\mathrm{n}-1} \begin{pmatrix} \mathrm{W_1}(\lambda) \\ \mathrm{Z_1}(\lambda) \end{pmatrix} = \mathrm{N}^{\mathrm{n}} \begin{pmatrix} \mathrm{W_0}(\lambda) \\ \mathrm{Z_0}(\lambda) \end{pmatrix}, \tag{12}$$

where

$$\mathrm{N} = \begin{pmatrix} (\lambda-2)((\lambda-2)^2-(\lambda-3)) & (\lambda-2)^2(\lambda-1) \\ (\lambda-2)^2 & (\lambda-1)^2 \end{pmatrix}. \tag{13}$$

Moreover, for $\mathrm{n} \geqslant 1$,

$$\begin{aligned} \mathrm{U_n}(\lambda) = & \ ((\lambda-1)(\lambda-2)^2 - (\lambda-2))\ \mathrm{W_{n-1}}(\lambda) \\ & + ((\lambda-1)^3 - (\lambda-1)(\lambda-2))\ \mathrm{Z_{n-1}}(\lambda) \end{aligned}$$

and

$$\begin{aligned} \mathrm{X_n}(\lambda) = & \ ((\lambda-2)^3 + (\lambda-3))\ \mathrm{V_{n-1}}(\lambda) \\ & + 2(\lambda-2)((\lambda-1)^2 - (\lambda-2))\ \mathrm{J_{n-1}}(\lambda). \end{aligned}$$

Let $x = \sqrt{(\lambda^6 - 12\lambda^5 + 64\lambda^4 - 202\lambda^3 + 408\lambda^2 - 492\lambda + 265)}$.

The eigenvalues of T are the following

$$\lambda_1 = \frac{1}{2}\left(\lambda^3 - 6\lambda^2 + 16\lambda - 17 + x\right),$$

$$\lambda_2 = \frac{1}{2}\left(\lambda^3 - 6\lambda^2 + 16\lambda - 17 - x\right),$$

and they are different outside the following roots:

$$\begin{array}{c} .9384 - 2.3074i \\ .9384 + 2.3074i \\ 2.1756 - .29473i \\ 2.1756 + .29473i \\ 2.886 - .72942i \\ 2.886 + .72942i \end{array}$$

Thus $\mathrm{V_n}(\lambda) = \mathrm{v}_1\lambda_1^{\mathrm{n}} + \mathrm{v}_2\lambda_2^{\mathrm{n}}$ and $\mathrm{J_n}(\lambda) = \mathrm{j}_1\lambda_1^{\mathrm{n}} + \mathrm{j}_2\lambda_2^{\mathrm{n}}$.

We count the coefficients $\mathrm{v}_1,\ \mathrm{v}_2$ and $\mathrm{j}_1,\ \mathrm{j}_2$ using the relation

$$\begin{pmatrix} \mathrm{V}_1(\lambda) \\ \mathrm{J}_1(\lambda) \end{pmatrix} =$$

$$\begin{pmatrix} (\lambda-3)^3+(\lambda-2)^2 & 2(\lambda-2)^2(\lambda-3) \\ (\lambda-2)(\lambda-3) & (\lambda-2)(2\lambda-3) \end{pmatrix} \begin{pmatrix} \lambda(\lambda-1)(\lambda-2) \\ \lambda(\lambda-1) \end{pmatrix} =$$

$$\begin{pmatrix} \lambda^6-9\lambda^5+33\lambda^4-62\lambda^3+59\lambda^2-22\lambda \\ \lambda^5-6\lambda^4+14\lambda^3-15\lambda^2+6\lambda \end{pmatrix}.$$

Similarly we use the value V_0 and J_0.

So we have

$$v_1 = \lambda\left(\lambda - 1\right)\left(\lambda - 2\right)\left(1 + \left(-3\lambda + 6\right)/x\right)$$

$$v_2 = \lambda\left(\lambda - 1\right)\left(\lambda - 2\right)\left(1 - \left(-3\lambda + 6\right)/x\right)$$

and

$$j_1 = \lambda\left(\lambda - 1\right)\left(1 + \left(\lambda^2 - 7\lambda + 11\right)/x\right)$$

$$j_2 = \lambda\left(\lambda - 1\right)\left(1 - \left(\lambda^2 - 7\lambda + 11\right)/x\right)$$

Thus

$$V_n(\lambda) =$$

$$\lambda(\lambda-1)(\lambda-2)(1-3(\lambda-2)/x)\left(\frac{1}{2}(\lambda^3-6\lambda^2+16\lambda-17+x)\right)^n$$
$$+(\lambda-1)(\lambda-2)(1+3(\lambda-2)/x)\left(\frac{1}{2}(\lambda^3-6\lambda^2+16\lambda-17-x)\right)^n$$

and

$$J_n(\lambda) =$$

$$\lambda(\lambda-1)(1+(\lambda^2-7\lambda+11)/x)\left(\frac{1}{2}(\lambda^3-6\lambda^2+16\lambda-17+x)\right)^n$$
$$+\lambda(\lambda-1)(1-(\lambda^2-7\lambda+11)/x)\left(\frac{1}{2}(\lambda^3-6\lambda^2+16\lambda-17-x)\right)^n$$

Let $y = \sqrt{(\lambda^6 - 12\lambda^5 + 66\lambda^4 - 206\lambda^3 + 377\lambda^2 - 378\lambda + 161)}$,

The eigenvalues of N are as follows:

$\lambda_3 = \frac{1}{2}\lambda^3 - 3\lambda^2 + \frac{15}{2}\lambda - \frac{13}{2} + \frac{1}{2}y$

$\lambda_4 = \frac{1}{2}\lambda^3 - 3\lambda^2 + \frac{15}{2}\lambda - \frac{13}{2} - \frac{1}{2}y$

and they are different outside the following roots:

$1.728 - .40595i$,
$1.728 + .40595i$,
$2.0829 - .89467i$,
$2.0829 + .89467i$,
$2.1891 - 2.2697i$,
$2.1891 + 2.2697i$

Thus $W_n(\lambda) = w_1\lambda_3^n + w_2\lambda_4^n$ and $Z_n(\lambda) = z_1\lambda_3^n + z_2\lambda_4^n$.

We count the coefficients $w_1,\ w_2$ and $z_1,\ z_2$ using the relation

$$\begin{pmatrix} W_1 \\ Z_1 \end{pmatrix} =$$

$$\begin{pmatrix} (\lambda-2)\left((\lambda-2)^2-(\lambda-3)\right) & (\lambda-2)^2(\lambda-1) \\ (\lambda-2)^2 & (\lambda-1)^2 \end{pmatrix} \begin{pmatrix} \lambda(\lambda-1) \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5) \\ \lambda(\lambda-1)(\lambda^2-3\lambda+3) \end{pmatrix}.$$

Similarly we use the value W_0, Z_0 and we get

$$w_1 = \frac{1}{2}\lambda(\lambda-1) + \frac{1}{2}\lambda(\lambda-1)\left(\lambda^3-6\lambda^2+11\lambda-7\right)/y$$

$$w_2 = \frac{1}{2}\lambda(\lambda-1) - \frac{1}{2}\lambda(\lambda-1)\left(\lambda^3-6\lambda^2+11\lambda-7\right)/y$$

$$z_1 = \frac{1}{2}\lambda\left(1+\left(\lambda^3-2\lambda^2-3\lambda+7\right)/y\right)$$

$$z_2 = \frac{1}{2}\lambda\left(1-\left(\lambda^3-2\lambda^2-3\lambda+7\right)/y\right).$$

Thus

$$W_n = w_1\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}+\frac{1}{2}y\right)^n + w_2\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}-\frac{1}{2}y\right)^n$$

$$Z_n = z_1\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}+\frac{1}{2}y\right)^n + z_2\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}-\frac{1}{2}y\right)^n.$$

$$\mathrm{w}_1 = \frac{1}{2}\lambda\left(\lambda-1\right)+\frac{1}{2}\lambda\left(\lambda-1\right)\left(\lambda^3-6\lambda^2+11\lambda-7\right)/\mathrm{y}$$

$$\mathrm{w}_2 = \frac{1}{2}\lambda\left(\lambda-1\right)-\frac{1}{2}\lambda\left(\lambda-1\right)\left(\lambda^3-6\lambda^2+11\lambda-7\right)/\mathrm{y}$$

$$\mathrm{z}_1 = \frac{1}{2}\lambda\left(1+\left(\lambda^3-2\lambda^2-3\lambda+7\right)/\mathrm{y}\right)$$

$$\mathrm{z}_2 = \frac{1}{2}\lambda\left(1-\left(\lambda^3-2\lambda^2-3\lambda+7\right)/\mathrm{y}\right).$$

Thus

$$\mathrm{W}_\mathrm{n} = \mathrm{w}_1\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}+\frac{1}{2}\mathrm{y}\right)^\mathrm{n}+ \mathrm{w}_2\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}-\frac{1}{2}\mathrm{y}\right)^\mathrm{n}$$

$$\mathrm{Z}_\mathrm{n} = \mathrm{z}_1\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}+\frac{1}{2}\mathrm{y}\right)^\mathrm{n}+ \mathrm{z}_2\left(\frac{1}{2}\lambda^3-3\lambda^2+\frac{15}{2}\lambda-\frac{13}{2}-\frac{1}{2}\mathrm{y}\right)^\mathrm{n}.$$

Remarks

Theorem (Gauss and Lucas [7])

Any closed half-plane that contains all roots of a polynomial $g(x)$ in a complex variable x also contains all roots of its derivative $g'(x)$.

By the theorem w get the following results.

Theorem (Gauss and Lucas [7])

Any closed half-plane that contains all roots of a polynomial $g(x)$ in a complex variable x also contains all roots of its derivative $g'(x)$.

By the theorem w get the following results.

Theorem (Bielak 2001)

Let G be $\chi-$chromatic graph of order $n > \chi + 1$, size m with t_1 triangles. Let $B_\chi = \left(m - \binom{\chi}{2}\right)/(n-\chi)$,

$$s_{2,\chi} = \binom{m}{2} - t_1 - \binom{\chi}{2}m + \sum_{j=1}^{\chi-1}\binom{j+1}{2}j,$$

$$D_\chi = (n-\chi-1)^2\left(\binom{\chi}{2} - m\right)^2 - 2(n-\chi-1)(n-\chi)s_{2,\chi}.$$

If $D_\chi \geqslant 0$, then $P(G,\lambda)$ has a zero r such that
$re(r) \geqslant B_\chi + \frac{\sqrt{D_\chi}}{(n-\chi)(n-\chi-1)}$.

If $D_\chi < 0$, then $P(G,\lambda)$ has a zeros r_1 i r_2 such that
$re(r_1) \geqslant B_\chi$ and $im(r_2) \geqslant \frac{\sqrt{-D_\chi}}{(n-\chi)(n-\chi-1)}$.

Theorem (Bielak 2001)

If a graph G is of size m, order n and chromatic number χ, where $n > \chi + 1$, then $(m - \binom{\chi}{2})/(n - \chi)$ is the upper bound for the modulus of all chromatic zeros for the graph G if and only if G is a $(\chi - 1)$-tree.

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